Decomposing the hypercube $Q_n$ into $n$ isomorphic edge-disjoint trees

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Abstract
We show that the edge set of the $n$-dimensional hypercube $Q_n$ is the disjoint union of the edge sets of $n$ isomorphic trees.

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1. Introduction

The problem of finding edge-disjoint trees in a hypercube arises for example in the context of parallel computing [3]. Independent of applications it is of high aesthetic appeal. The hypercube of dimension $n$, denoted by $Q_n$, comprises $2^n$ vertices each corresponding to a distinct binary string of length $n$. Two vertices are adjacent if and only if their corresponding binary strings differ in exactly one position. Since each vertex of $Q_n$ has degree $n$, the number of edges is $n2^{n-1}$. A variety of decomposability options derive from this fact. In the remainder of the introduction we focus on three of them. The first two have been dealt with before in the literature, the third is the topic of this article.

1. Roskind and Tarjan [7] obtained a polynomial-time algorithm to find the maximum number of edge-disjoint spanning trees in a connected
graph $G$. For $Q_n$, edge-counting yields $\left\lfloor \frac{n^2}{2} \right\rfloor$ as an upper bound; in fact equality holds and the answer is $\left\lfloor \frac{n^2}{2} \right\rfloor$ for many $n$-regular graphs (see [1]). For even $n$, an explicit construction of $n/2$ edge-disjoint spanning trees in $Q_n$ appears in [1]. These trees are not isomorphic. There are $n/2$ leftover edges, which form a path. For odd $n$, an explicit construction has not yet been found.

2. So decomposing the edge set of $Q_n$ into spanning trees is impossible. What about a decomposition into isomorphic trees? Surprisingly, for every tree $T$ with $n$ edges, the edge set of $Q_n$ can by [2] be covered by $2^{n-1}$ trees isomorphic to $T$. This has been extended in [3, 4], and will be followed up in Section 4.

3. As opposed to decomposing $Q_n$ into $2^{n-1}$ isomorphic trees of size $n$, in this article we decompose $Q_n$ into $n$ trees of size $2^{n-1}$ isomorphic to some tree $T_n$. Our tree $T_n$ needs to have a very specific shape.

In Section 2, we present two equivalent definitions of $T_n$: a direct one (used in the proof of the main result in Section 3) and a recursive one.

2. Construction of $T_n$

Rather than dealing with 0,1-strings, it will be more convenient to let the vertex set $V(Q_n)$ be the family of subsets of $[n]$, where $[n] := \{1, 2, \ldots, n\}$, with $X, X' \in V(Q_n)$ being adjacent if and only if their symmetric difference has size 1. Before tackling the formal definition of $T_n$, let us show some small cases (Figure 1); of course, we must have $|V(T_n)| = 2^{n-1} + 1$ for every $n$.

The following notation shall be employed throughout the article. Permutations $\pi$ of $[n]$ will be written in cycle notation and to the right of the argument, separated by a dot. Thus if $\pi = (3,4)(2,5,7)$ (where $n \geq 7$), then $3 \cdot \pi = 4$ and $7 \cdot \pi = 2$. Each permutation of $[n]$ induces, in the obvious way, a permutation of the vertex set $V(Q_n)$. Using this notation, applying the permutation $\pi$ above to the vertex (say) $X = \{1,2,4\}$ of $Q_n$ yields $X \cdot \pi = \{1,5,3\}$. Furthermore, for $\{X_1, X_2, \ldots\} \subseteq V(Q_n)$ we set $\{X_1, X_2, \ldots\} \cdot \pi := \{X_1 \cdot \pi, X_2 \cdot \pi, \ldots\}$. If $T$ is a subgraph of $Q_n$ and $\pi$ a permutation, then $T \cdot \pi$ is defined as the subgraph of $Q_n$ that has vertex set $V(T) \cdot \pi$ and edge set $\{X \cdot \pi, Y \cdot \pi\} : \{X, Y\} \in E(T)$.

Let us resume the discussion of our trees $T_n$ and put $n = 3$ and $\pi = (1,2,3)$. See Figure 2, where for clarity we drop brackets and commas in naming sets.
$T_1 = \{1\}$

$T_2 = \{1\}$

$T_3 = \{1, 2\}$

$T_4 = \{1, 2, 3, 4\}$

Figure 1: The trees $T_1$ to $T_4$.

$T_3 \cdot \pi = \{123\}$

$T_3 \cdot \pi^2 = \{123\}$

Figure 2: Application of permutations to a tree.
The edge-disjoint decomposition of $Q_3$ into the isomorphic rooted trees $T_3$, $T_3 \cdot \pi$, and $T_3 \cdot \pi^2$ is now apparent. The corresponding decomposition of $Q_4$ into the trees $T_4 \cdot \pi^i$ ($0 \leq i < 4$, $\pi := (1, 2, 3, 4)$) is already more surprising (see Figure 3).

![Figure 3: Decomposition of the four-dimensional hypercube.](image-url)

Let us now define $T_n$ in general. For $n \in \mathbb{N}$ let $V(T_n)$ consist of all subsets of $[n - 1]$ plus the root vertex $[n]$. Furthermore, we assign a parent vertex $p(v)$ to every vertex $v$ other than $[n]$, namely the set $v \cup \{x(v)\}$, where

$$x(v) = \min(\mathbb{N} \setminus v).$$

In other words, to obtain the parent vertex of $v$, we add the smallest positive integer that is not yet contained in the set $v$. If $v \neq [n - 1]$, then this is an element of $[n - 1]$; if $v = [n - 1]$, then $x(v) = n$ and $p(v) = [n]$. Now the edge set $E(T_n)$ consists of all pairs $(v, p(v))$.

The fact that every vertex other than $[n]$ is adjacent to a unique superset implies that the resulting graph $T_n$ has no cycles; since it explicitly has $2^{n-1} + 1$ vertices and $2^{n-1}$ edges, it therefore is a tree.

Let us show that $T_n$ can be constructed in a recursive manner as well. For starters, observe that the subtrees $S_{5,1}, \ldots, S_{5,4}$ of $T_5$ in Figure □ are isomorphic to $T_1, \ldots, T_4$ respectively. Generally for $n \geq 2$ and $1 \leq i \leq n - 1$, define $S_{n,i}$ as the tree isomorphic to $T_i$ obtained by adding the elements $i + 1, \ldots, n - 1$ to the names of all the vertices. In particular, all vertices of $S_{n,i}$ belong to $V(Q_{n-1})$ and $S_{n,i}$ has the root $[n - 1]$.

**Proposition 1.** The tree $T_n$ ($n \geq 2$) is isomorphic to the tree obtained by gluing $T_1, \ldots, T_{n-1}$ at their roots and then attaching a new root.
Proof. Because $T_i \cong S_{n,i}$ ($1 \leq i \leq n-1$) it suffices to show the following:

Disregarding the common root $[n-1]$, the vertex sets of $S_{n,1}, \ldots, S_{n,n-1}$ are mutually disjoint and have union $V(Q_{n-1}) \setminus \{[n-1]\}$.

Indeed, disjointness follows from the fact that each vertex $v \neq [n-1]$ of $S_{n,i}$ has the property $\{i+1, i+2, \ldots, n-1\} \subseteq v$ but $i \notin v$. That the union is $V(Q_{n-1}) \setminus \{[n-1]\}$ follows from disjointness together with

$$
(|V(S_{n,1})| - 1) + \cdots + (|V(S_{n,n-1})| - 1) = 1 + 2 + \cdots + 2^{n-2} = 2^{n-1} - 1.
$$

3. Main result

Our main result states that the edges of the $n$-dimensional hypercube $Q_n$ can be decomposed into isomorphic copies of $T_n$.

**Theorem 2.** Let $\pi$ be the cycle $(1, 2, \ldots, n)$. If $\{X, Y\} \in E(T_n)$, then $\{X, Y\} \notin E(T_n \cdot \pi^i)$ for all $0 < i < n$. In particular, the trees $T_n \cdot \pi^i$ ($0 < i < n$) decompose $Q_n$.

Proof. Obviously, if the edge $\{X, Y\}$ occurs in several trees, the larger of the two sets $X$ and $Y$ is the parent vertex in all these trees. Hence it suffices to show the following: if a set $X$ belongs to the vertex sets of both $T_n$ and
Then the parent vertices of $X$ in $T_n$ and $T_n \cdot \pi_i$ are distinct. This will imply that $T_n$ and $T_n \cdot \pi_i$ have no common edges.

Since the vertex set of $T_n$ consists of $[n]$ plus all subsets of $[n]$ that omit $n$, the vertex set of $T_n \cdot \pi_i$ consists of $[n]$ plus all subsets of $[n]$ that omit $i$.

Hence, a set $X$ can only belong to the vertex sets of both $T_n$ and $T_n \cdot \pi_i$ if $i, n \not\in X$ or if $X = [n]$. Since $X = [n]$ has no parent vertex, it suffices to consider the case that $i, n \not\in X$. Let $X \cup \{x\}$ and $X \cup \{y\}$ be the parent vertices of $X$ in $T_n$ and $T_n \cdot \pi_i$, respectively. Since $i \not\in X$, we have $1 \leq x \leq i$ by the definition of $T_n$. Let $Z$ be the vertex of $T_n$ such that $X = Z \cdot \pi_i$. From $n \not\in X$ follows $(n - i) \not\in Z$, and so the parent vertex $Z \cup \{z\}$ of $Z$ satisfies $z \in \{i + 1, \ldots, n-i\}$. This forces $y = z \cdot \pi_i \in \{i+1, \ldots, n\}$. Altogether, this shows that $x \neq y$, and so the parents $X \cup \{x\}$ and $X \cup \{y\}$ are distinct, as claimed.

Therefore, the edge sets of $T_n$ and $T_n \cdot \pi_i$ are disjoint for all $0 < i < n$. This also implies that the edge sets of $T_n \cdot \pi_i$ and $T_n \cdot \pi_j$ are disjoint for all $0 \leq i < j < n$: if they were not disjoint, the edge sets of $T_n$ and $T_n \cdot \pi_{j-i}$ would not be disjoint either, a contradiction.

Since $T_n$ has $2^{n-1}$ edges and the specified trees are pairwise edge-disjoint, it follows immediately that they decompose $Q_n$. $\square$

4. Three related matters

After discovering $T_n$ in its recursive guise (Proposition 1), we noticed that $T_n$ coincides with the set enumeration tree introduced in [8], except for the dual labeling of nodes (in [8] the root is $\phi$, not $[n]$). Not the theoretical properties of $T_n$ are considered in [8], but its usefulness as (to cite from the abstract) “a vehicle for representing sets and/or enumerating them in a best-first fashion”. Many subsequent uses of the SE-tree, e.g. in data mining, can be surveyed with Google-Scholar.

Second, let $F$ be a subset of $E(Q_n)$. Following Ramras, call $F$ a fundamental set for $Q_n$ with respect to group $G$ if $G$ is a subgroup of $\text{Aut}(Q_n)$ such that $\{g(E) : g \in G\}$ form an edge decomposition of $Q_n$. It is shown in [1] that if $|F| = n$ and the graph induced by $F$ is connected with at most one cycle (e.g. a tree), then $F$ is a fundamental set for $Q_n$ with respect to some group. Results about fundamental $2n$-element sets are contained in [3]. Our tree $T_n$ fits into this framework in that $E(T_n)$ is a fundamental size $2^{n-1}$ set for the group $G \subseteq \text{Aut}(Q_n)$ that is induced by the cyclic group $\langle \pi \rangle \subseteq S_n$. 
Third, every rooted tree $T$ (e.g. $T = T_n$) becomes a unique partially ordered set $(T, \leq)$ when the root is postulated as largest element. It is an open problem to find necessary or sufficient conditions for a rooted tree $(T, \leq)$ to be cover-preserving order-embeddable into $(Q_n, \subseteq)$. That is, we want a map $\phi : T \rightarrow Q_n$ that satisfies

1. $\forall x, y \in T \ x \leq y \iff \phi(x) \subseteq \phi(y)$,
2. $\forall x, y \in T \ x \prec y \Rightarrow \phi(x) \prec \phi(y)$.

Here $x \prec y$ means that $x < y$ and $(x < z \leq y \Rightarrow z = y)$. We mention that for posets $(P, \leq)$ with universal bounds 0 and 1 the problem is settled in [4] in terms of the chromatic number of some auxiliary graph. Notice that $(T_4, \leq)$ is not order-embedded in $Q_4$: while $\{2\} \subseteq \{2, 3\}$ in $Q_4$, the corresponding vertices are not comparable in $(T_4, \leq)$.

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References


