The many benefits of putting stack filters into disjunctive or conjunctive normal form

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Abstract

Stack filters are nonlinear filters used for image processing (examples: median filters, order statistics). In the translation-invariant case a stack filter is determined by a positive Boolean function \( b \). Many important properties of stack filters (idempotency, co-idempotency, order relations) can be tested in polynomial time if the DNF and/or CNF of \( b \) are known.

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1. Introduction

Let us go into medias res. One simple example of a stack filter would be the operator \( \Phi : \mathbb{R}^\mathbb{Z} \rightarrow \mathbb{R}^\mathbb{Z} \) which maps a series \( f = \{ f_i \mid i \in \mathbb{Z} \} \) to the series \( \Phi f \) whose \( i \)th component is defined by \( [\Phi f]_i := (f_{i-2} \land f_i) \lor f_{i+1} \). Hereby \( f_i \land f_j \) and \( f_i \lor f_j \) are defined as the minimum and maximum, of the real numbers \( f_i \) and \( f_j \), respectively. Not surprisingly, the behaviour of \( \Phi \) is determined by the underlying positive Boolean function \( b : \{ 0, 1 \}^4 \rightarrow \{ 0, 1 \} \) that maps \( (x_{-2}, x_{-1}, x_0, x_1) \) to \( (x_{-2} \land x_0) \lor x_1 \).

In Section 2 we review the conjunctive (CNF) and disjunctive (DNF) normal forms of positive Boolean functions and, for later purposes, explicitly derive one from the other for some nontrivial \( b_n : \{ 0, 1 \}^n \rightarrow \{ 0, 1 \} \).

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In Section 3 it is indicated how stack filters \( \Psi : \mathbb{R}^Z \to \mathbb{R}^Z \) arise in nonlinear image processing. Interestingly, \( \Psi \) need not originally be defined in terms of \( \land \) and \( \lor \). We then proceed to the computation of the DNF and CNF of some concrete stack filters (i.e., of their underlying positive Boolean functions). In particular the \( b_n \) of Section 2 corresponds to the stack filter \( \Psi := L_n \circ U_n \) where \( L_n \) and \( U_n \) are the thoroughly investigated stack filters of [6].

In Section 4 we discuss four “benefits” of normal forms of stack filters. As to the first benefit, when both the CNF and DNF of \( \Psi \) are known, there is a polynomial algorithm [10] to decide whether or not \( \Psi \) is idempotent, i.e. whether \( \Psi \circ \Psi = \Psi \). Second, the co-idempotency of \( \Psi \), i.e. \((I - \Psi) \circ (I - \Psi) = I - \Psi\), where \( I \) is the identity map, can also be tested in polynomial time. We further expand upon the related computation of all noise series \( g := f - \Psi f \) of \( \Psi \), in particular for \( \Psi := L_n \circ U_n \). Third, two stack filters \( \Phi \) and \( \Psi \) are said to be comparable, say \( \Phi \leq \Psi \), if \( \Phi f \leq \Psi f \) for all series \( f \in \mathbb{R}^Z \). Given their DNF (or CNF) it can be tested in polynomial time whether or not \( \Phi \leq \Psi \). Fourth, a stack filter \( \Phi \) is neighbourly trend preserving if \( f_i \leq f_{i+1} \) implies \( [\Phi f]_i \leq [\Phi f]_{i+1} \), and \( f_i \geq f_{i+1} \) implies \( [\Phi f]_i \geq [\Phi f]_{i+1} \). If \( \Phi \) is given in normal form this property can be checked in polynomial time.

2. Prerequisites about positive Boolean functions

Let us review some well-known facts from Boolean logic which shall be crucial in later sections. For \( x, y \in \{0, 1\}^n \) write \( x \leq y \) if \( x_i \leq y_i \) for all \( 1 \leq i \leq n \). Any function \( b : \{0, 1\}^n \rightarrow \{0, 1\} \) is called a Boolean function. It is positive (or monotone) if for all \( x, y \in \{0, 1\}^n \) it follows from \( x \leq y \) that \( b(x) \leq b(y) \). As opposed to the general case, a positive \( b \) admits a unique minimal disjunctive normal form (the DNF), and dually a unique minimal conjunctive normal form (the CNF).

Namely, for all \( x = (x_1, \ldots, x_n) \in \{0, 1\}^n \) put \( \text{One}(x) := \{i \mid x_i = 1\} \) and \( \text{Zero}(x) := \{i \mid x_i = 0\} \). A subset \( C \subseteq \{1, \ldots, n\} \) is a 1-set of \( b \) if \( b(x) = 1 \) for the unique \( x \) with \( \text{One}(x) = C \). Dually call \( D \subseteq \{1, \ldots, n\} \) a 0-set of \( b \) if \( b(y) = 0 \) for the unique \( y \) with \( \text{Zero}(y) = D \). Let \( \mathcal{C} = \mathcal{C}(b) \) be the set of all nonvoid minimal 1-sets and let \( \mathcal{D} = \mathcal{D}(b) \) be the set of all nonvoid minimal 0-sets. If \( b(x) = 1 \) for all \( x \in \{0, 1\}^n \) then \( \mathcal{D} = \emptyset \). Dually, if \( b(x) = 0 \) for all \( x \in \{0, 1\}^n \) then \( \mathcal{C} = \emptyset \). But for a nonconstant positive Boolean function \( b \) both clusters \( \mathcal{C} \) and \( \mathcal{D} \) are nonvoid antichains. (A family of sets is an antichain if no member properly contains another member of that family.) The DNF (respectively the CNF) of \( b \) is then defined as

\[
\bigvee_{C \in \mathcal{C}} \left( \bigwedge_{j \in C} x_j \right) \quad \text{respectively} \quad \bigwedge_{D \in \mathcal{D}} \left( \bigvee_{j \in D} x_j \right).
\]  

Other authors speak of \( T \)-sets and \( F \)-sets, rather than of 1-sets and 0-sets of a Boolean function. While their \( T \)-sets coincide with our 1-sets, their \( F \)-sets are usually defined to be the complements within \([1, n]\) of our 0-sets.
Example 1. Consider the positive Boolean function $b : \{0, 1\}^3 \to \{0, 1\}$ defined by

\[
\begin{align*}
    b(0, 0, 0) &= b(0, 1, 0) = b(0, 0, 1) := 0 \\
    b(1, 0, 0) &= b(1, 1, 0) = b(1, 0, 1) = b(0, 1, 1) = b(1, 1, 1) := 1
\end{align*}
\]

The minimal 1-sets are $\{1\}$ and $\{2, 3\}$, whereas the minimal 0-sets are $\{1, 2\}$ and $\{1, 3\}$. Hence $x_1 \lor (x_2 \land x_3)$ is the DNF and $(x_1 \lor x_2) \land (x_1 \lor x_3)$ is the CNF of $b(x_1, x_2, x_3)$. Check that both expressions indeed yield $b(x_1, x_2, x_3)$ for all $(x_1, x_2, x_3)$ in $\{0, 1\}^3$.

Note that being positive amounts to the fact that $b(x) = 1$ and $x \leq y$ imply $b(y) = 1$ (equivalently: $b(x) = 0$ and $y \leq x$ imply $b(y) = 0$). Generally a subset $\mathcal{F}$ of a partially ordered set is an order filter if $x \in \mathcal{F}$ and $x \leq y$ imply $y \in \mathcal{F}$. The $\leq$-minimal members of $\mathcal{F}$ are the unique generators of $\mathcal{F}$. Dually a subset $\mathcal{J}$ is an order ideal if $x \in \mathcal{J}$ and $y \leq x$ imply $y \in \mathcal{J}$. The $\leq$-maximal members of $\mathcal{J}$ are the unique generators of $\mathcal{J}$.

One normal form of a positive Boolean function can be obtained from the other one by “multiplying out”. For instance in Example 1

\[
(x_1 \lor x_2) \land (x_1 \lor x_3) = (x_1 \land x_1) \lor (x_1 \land x_3) \lor (x_2 \land x_1) \lor (x_2 \land x_3)
\]

by distributivity. Furthermore $x_1 \land x_1$ and $x_1 \land x_3$ and $x_2 \land x_1$ are all $\leq x_1$, so the right-hand side equals $x_1 \lor (x_2 \land x_3)$. The relation between DNF and CNF can be characterized in another way:

**Theorem 1 (Folklore).** Let $\mathcal{C}$ and $\mathcal{D}$ be nonvoid antichains of subsets of a finite set $A$. The following are equivalent:

(a) $\mathcal{C}$ and $\mathcal{D}$ are “coupled” in the sense that for some nonconstant positive Boolean function $b : \{0, 1\}^A \to \{0, 1\}$ one has $\mathcal{C} = \mathcal{C}(b)$ and $\mathcal{D} = \mathcal{D}(b)$;

(b) $\mathcal{D}$ is the family of all minimal transversals of $\mathcal{C}$;

(c) $\mathcal{C}$ is the family of all minimal transversals of $\mathcal{D}$.

**Proof.** Assuming (a), identify $\{0, 1\}^A$ with the powerset $\mathcal{P}(A)$ and think of the family $\mathcal{F} \subseteq \mathcal{P}(A)$ of all 1-sets of $b$ as an order filter generated by the sets $C$ in $\mathcal{C}$. By the definition of “1-set” and “0-set” the family $\mathcal{J} := \mathcal{P}(A) - \mathcal{F}$ is the order ideal of all complements of 0-sets of $b$. By definition of $\mathcal{D}$ the generators of $\mathcal{J}$ are the sets $D := A - D$ ($D \in \mathcal{D}$). As to the equivalence (a) $\iff$ (b), a set $D \subseteq A$ is minimal with the property of intersecting all members of $\mathcal{C}$ iff $\overline{D}$ is maximal with the property of not containing any member of $\mathcal{C}$, i.e. iff $\overline{D}$ is a maximal member of $\mathcal{J}$. Dually, one may consider the order filter $\mathcal{J}$ of all 0-sets, and the corresponding order ideal $\mathcal{F} := \mathcal{P}(A) - \mathcal{J}$ of all complements of 1-sets.

This yields the equivalence (a) $\iff$ (c). \qed

**Corollary 2.** Let $\mathcal{C}$ and $\mathcal{D}$ be coupled. If also $\mathcal{C}' \supseteq \mathcal{C}$ and $\mathcal{D}' \supseteq \mathcal{D}$ are coupled then necessarily $\mathcal{C}' = \mathcal{C}$ and $\mathcal{D}' = \mathcal{D}$.

**Example 2.** The word “all” in (b) and (c) is essential. Say $\mathcal{C} := \{\{-2, -1, 0\}, \{0, 1, 2\}\}$ and $\mathcal{D} := \{\{0\}, \{-2, 1\}, \{-1, 2\}\}$. 
Although each $C \in \mathcal{C}$ is a minimal transversal of $\mathcal{D}$ and each $D \in \mathcal{D}$ is a minimal transversal of $\mathcal{C}$, neither does $\mathcal{C}$ consist of all minimal transversals of $\mathcal{D}$ (consider $\{-1, 0, 1\}$), nor does $\mathcal{D}$ consist of all minimal transversals of $\mathcal{C}$ (consider $\{-1, 1\}$).

Henceforth, we shall denote a set $\{m, m+1, \ldots, n\}$ of consecutive integers by $[m, n]$. The relevance of the following coupled pair of clusters $\mathcal{C}_n$ and $\mathcal{D}_n$ will be clear later. Consider the cluster $\mathcal{C}_n$ of all subsets of $[-2n, 2n]$ of the two types:

(i) $[k - n, k]$ \quad ($0 \leq k \leq n$),
(ii) $[j - n, j] \cup [i, i + n]$ \quad ($1 \leq i \leq n$) ($i - 1 - n \leq j \leq -1$),

as well as the cluster $\mathcal{D}_n$ of all subsets of $[-2n, 2n]$ of types

(A) $\{a, c\}$ \quad ($-n \leq a < c \leq n$ and $c - a = n$),
(B) $\{a, c\}$ \quad ($-(n + 1) \leq a < c \leq n + 1$ and $c - a = n + 1$),
(C) $\{a, b, c\}$ \quad ($a < b < c$ and $c - a \geq n + 2$),

such that $\{a, b, c\}$ moreover satisfies

(C1) $a \leq 0 < b < c$ \quad $\Rightarrow$ \quad $b - a \leq n - 1$, $c - b \leq n + 1$,
(C2) $a < b \leq 0 \leq c$ \quad $\Rightarrow$ \quad $b - a \leq n + 1$, $c - b \leq n - 1$,
(C3) $a < b = 0 < c$ \quad $\Rightarrow$ \quad $b - a \leq n - 1$, $c - b \leq n - 1$.

**Example 3.** Each line of points corresponds to an element of $\mathcal{C}_2$ or $\mathcal{D}_2$. This way it is easy to check that each $C_i$ in $\mathcal{C}_2$ happens to be a minimal transversal of $\mathcal{D}_2$, and each $D_j$ in $\mathcal{D}_2$ a minimal transversal of $\mathcal{C}_2$.

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**Theorem 3.** The clusters $\mathcal{C}_n$ and $\mathcal{D}_n$ are coupled.
Proof. By Theorem 1(b) it suffices to show that

(a) all the sets of \( \mathcal{D}_n \) are minimal transversals of \( \mathcal{C}_n \);
(b) each transversal of \( \mathcal{C}_n \) contains at least one of the sets of \( \mathcal{D}_n \).

As to (a), each set \( \{a, a+n\} \) of type (A) cuts the type (i) sets of \( \mathcal{C} := \mathcal{C}_n \) because \( a \) and \( a+n \) are too close to squeeze a set \( [k-n, k] \) between them. On the other hand, since \( a \) and \( a+n \) are too far apart to fit between \( [j-n, n] \) and \( [i, i+n] \), the set \( \{a, a+n\} \) cuts all the type (ii) sets of \( \mathcal{C} \). Thus \( \{a, a+n\} \) is a transversal of \( \mathcal{C} \), and obviously a minimal one.

In a similar way, the reader may check that all type (B), (C1), (C2), (C3) sets are minimal transversals of \( \mathcal{C} \).

As to (b), let \( T \subseteq [-2n, 2n] \) be any transversal of \( \mathcal{C} \). Obviously \( T \cap [-n, n] \) is nonempty, and if \( d \) and \( e \) are the smallest and biggest elements, respectively, of this intersection, then \( -n \leq d \leq e \leq n \) since e.g. \( d \leq e < 0 \) would imply \( T \cap [0, n] = \emptyset \). Let \( d' \in T \) be maximal with \( d' < -n \), and let \( e' \in T \) be minimal with \( e' > n \). Note that neither \( d' \) nor \( e' \) need exist.

Case 1: \( e - d - n - 1 \).

Case 1.1. Either \( d - d' \leq n + 1 \) (implying \( d' \) exists) or \( e' - e \leq n + 1 \) (implying \( e' \) exists).

First subcase. \( d \leq 0 < e \) or \( d < 0 \leq e \). By the symmetry of \( \mathcal{C} \) (why?) it suffices to discuss \( d \leq 0 < e \). Suppose first \( d - d' \leq n + 1 \) takes place. If \( d < 0 \) then \( e > 0 \) implies \( e - d' \geq n + 2 \). So \( \{d', d, e\} \subseteq T \) is a (C2) transversal. If \( d = 0 \) then \( d' < -n \) and \( d - d' \leq n + 1 \) force \( \{-d, d\} = \{-(n+1), 0\} \) to be a type (B) subset of \( T \). Suppose now that \( e' - e \leq n + 1 \) takes place. If \( e' - d' \geq n + 2 \) then \( \{d, e, e'\} \subseteq T \) is of type (C1). If \( e' - d' \leq n + 1 \) then \( d \leq 0 \) and \( e' > n \) force \( \{d, e'\} = \{0, n + 1\} \) to be a type (B) subset of \( T \).

Second subcase. \( d = e = 0 \). If \( e' - e \leq n + 1 \) then \( e' > n \) implies \( \{e, e'\} = \{0, n + 1\} \subseteq T \).

The case \( d - d' \leq n + 1 \) is analogous.

Case 1.2. \( d - d' \geq n + 2 \) and \( e' - e \geq n + 2 \). Hereby neither \( d' \) nor \( e' \) need exist, but in any case one gets the contradiction \( T \cap ([d - 1 - n, d - 1] \cup [e + 1, e + 1 + n]) = \emptyset \) (set \( i := e + 1 \) and \( j := d - 1 \) in (ii)). Hence this case is impossible.

Case 2: \( e - d \geq n \). We may make the general assumption that

\( \text{(GA)} \) there are no \( a, c, \in T \) with \( -n \leq a < c \leq n \) and \( c - a \in \{n, n + 1\} \).

Since otherwise \( \{a, c\} \subseteq T \) is type (A) or (B). In particular \( \text{(GA)} \) implies \( e - d \geq n + 2 \). Let \( d' \in T \) be maximal with \( d \leq d' \leq 0 \) and \( e' \in T \) minimal with \( 0 \leq e' \leq e \).

First subcase. \( e - d' \leq n - 1 \) and \( e' - d \leq n - 1 \). From \( e - d \geq n + 2 \) it follows that \( d < d' \leq 0 \leq e' < e \). Whence \( \{d, d', e\} \subseteq T \) is type (C2). In view of \( \text{(GA)} \) the remaining subcase is \( e - d' \geq n + 2 \) or \( e' - d \geq n + 2 \). Say \( e - d' \geq n + 2 \) (the other case is similar).

If \( 0 \in T \) then \( \text{(GA)} \) implies that \( \{d', 0, e\} \subseteq T \) is type (C3). If \( 0 \notin T \) then \( d' < 0 < e' \) and \( T \cap \{d' + 1, \ldots, e' - 1\} = \emptyset \). Since \( T \) cuts the type (i) sets we must have \( e' - d' \leq n + 1 \), whence \( e' - d' \leq n - 1 \) by \( \text{(GA)} \). Thus \( \{d', e', e\} \subseteq T \) is type (C1). \( \square \)

A brief word on complexity is appropriate here. Let \( \mathcal{C} \) and \( \mathcal{D} \) be clusters of subsets of a finite set \( A \). Putting \( n := |\mathcal{C}| + |\mathcal{D}| \) one can check in quasipolynomial time \( n^{O(\log n)} \) whether or not \( \mathcal{C} \) and \( \mathcal{D} \) are coupled [2]. Phrased equivalently, given two positive Boolean functions, one can decide in quasi-polynomial time whether they are mutually dual. Thus, as opposed
to arbitrary Boolean functions, duality testing for positive Boolean functions is unlikely to be NP-hard.

3. Normal forms of some stack filters

We begin by pointing out some obvious properties of a frequently used nonlinear operator.

Example 4. Fix natural numbers \( n \geq 1 \) and \( k \leq 2n + 1 \). Given a series \( f = \{ f_i | i \in \mathbb{Z} \} \) the \( k \)th order statistic \( \Phi = R_{n,k} \) looks at each “window” \( \{ f_{s-n}, \ldots, f_s, \ldots, f_{s+n} \} \) and selects the \( k \)th smallest element of it. More formally \( \Phi : \mathbb{R}^\mathbb{Z} \to \mathbb{R}^\mathbb{Z} \) is defined by \( [\Phi f]_s := \bar{f}_k \), where

\[
\{ \bar{f}_1, \bar{f}_2, \ldots, \bar{f}_{2n+1} \} := \{ f_{s-n}, \ldots, f_s, \ldots, f_{s+n} \} \quad \text{and} \quad \bar{f}_1 \leq \bar{f}_2 \leq \cdots \leq \bar{f}_{2n+1} \tag{2}
\]

Obviously \( \Phi \) is local in the sense that \( [\Phi f]_s \) is determined by finitely many components of \( f \). Furthermore, translating \( f \) say 10 units to the right and then applying \( \Phi \) clearly yields the same result as first applying \( \Phi \) and then translating 10 units to the right. In this sense \( \Phi \) is translation invariant.\(^2\) It is equally clear that \( \Phi \) commutes with contrast changes, meaning for all monotone maps \( g : \mathbb{R} \to \mathbb{R} \) one has \( \Phi(g \circ f) = g \circ \Phi(f) \).

This leads us to the definition of a stack filter\(^3\) as a local, translation invariant operator \( \Phi : \mathbb{R}^\mathbb{Z} \to \mathbb{R}^\mathbb{Z} \) that commutes with contrast changes. There is a neat equivalent definition: Instead of \( x_i \in \{ 0, 1 \} \), put any \( f_i \in \mathbb{R} \) into (1), where \( \bigwedge_{i \in C} f_i \) and \( \bigvee_{j \in D} f_j \) are taken to be \( \min \{ f_i | i \in C \} \) respectively \( \max \{ f_j | j \in D \} \). Thus define a min–max operator as an operator \( \Phi : \mathbb{R}^\mathbb{Z} \to \mathbb{R}^\mathbb{Z} \) which for some fixed finite cluster \( C \) of finite subsets of \( \mathbb{Z} \) maps series \( f = \{ f_i | i \in \mathbb{Z} \} \) to series \( \Phi f \) defined by

\[
[\Phi f]_s = \bigvee_{C \in \mathcal{C}} \left( \bigwedge_{i \in C} f_{s+i} \right) \quad (s \in \mathbb{Z}). \tag{3D}
\]

The cluster \( \mathcal{C} \) can be chosen as an antichain and is then uniquely determined. We then say that (3D) is the DNF of \( \Phi \). Dually there is an antichain \( \mathcal{D} \), as characterized in Theorem 1, which yields the CNF of \( \Phi \):

\[
[\Phi f]_s = \bigwedge_{D \in \mathcal{D}} \left( \bigvee_{j \in D} f_{s+j} \right) \quad (s \in \mathbb{Z}). \tag{3C}
\]

\(^2\) More formally, let \( \bar{f} \) be the series \( f \) pushed \( h \) units to the right (so \( \bar{f}_i := f_{i+h} \) for all \( i \in \mathbb{Z} \)). If \( \Psi(\bar{f}) = \Psi(f) \) for all series \( f \) and all \( h \in \mathbb{Z} \) then \( \Psi \) is called translation invariant. In particular, translation invariant operators \( \Psi : \mathbb{R}^\mathbb{Z} \to \mathbb{R}^\mathbb{Z} \) are completely defined by \( [\Psi f]_0 \).

\(^3\) The name is adapted from the signal processing literature, see e.g. [1].
It is easy to see that a min–max operator \( \Phi \) is monotone in the sense that \( f \leq g \) implies \( \Phi f \leq \Phi g \).

**Theorem 4.** A map \( \Phi : \mathbb{R}^Z \to \mathbb{R}^Z \) is a stack filter if and only if it is a min–max operator.

This result is well known. In fact, more general min–max operators are dealt with in [10]: The lattice \((\mathbb{R}, \leq)\) can be replaced by any distributive lattice \((\mathcal{L}, \leq)\), instead of \(\mathbb{Z}\) any index set \(S\) will do, and the restriction of translation invariance is done away with. But here we stick to translation invariant stack filters \( \Phi : \mathbb{R}^Z \to \mathbb{R}^Z \) as defined by (3D) or (3C), in order to avoid unnecessary distractions.

The composition \( \Phi \circ \Psi \) of two stack filters is again a stack filter. This is clear from either the three defining properties of a stack filter, or from Theorem 4. The DNF (or CNF) of \( \Phi \circ \Psi \) is obtained from the DNFs of \( \Phi \) and \( \Psi \) by applying the distributive laws to

\[
[(\Phi \circ \Psi)(f)]_0 = \bigvee_{C \in \mathcal{C}^1} \bigwedge_{i \in C} [\psi f]_i = \bigvee_{C \in \mathcal{C}^1} \bigwedge_{i \in C} \left( \bigvee_{C' \in \mathcal{C}^2} \bigwedge_{j \in C'} f_{i+j} \right).
\]

A stack filter property may or may not be inherited under composition. On the negative side, it is generally difficult to relate the DNF of \( \Phi \circ \Psi \) to the DNFs of \( \Phi \) and \( \Psi \) (a case in point is Theorem 6), or to carry over idempotency (see Section 4.1). On the positive side, e.g. the property of being neighbourly trend preserving is inherited (see Section 4.4). It also happens that good properties only arise in \( \Phi \circ \Psi \) (e.g. \( \bigwedge^n \) and \( \bigvee^n \) are not idempotent but \( \bigwedge^n \circ \bigvee^n \) is; see the remark after (9)).

One referee raised the interesting question of a decomposition theory of stack filters: What stack filters \( \Phi \) can be written as \( \Phi = \Phi_1 \circ \Phi_2 \circ \cdots \circ \Phi_k \) with “irreducible” (not necessarily unique) stack filters \( \Phi_i \) that preferably possess useful properties? The author doubts the existence of such a theory; in fact he would not know how to check, given \( \Phi \) in DNF, whether or not \( \Phi = \Phi_1 \circ \Phi_2 \) for some \( \Phi_1, \Phi_2 \neq \text{id!} \).

Given a stack filter \( \Phi \) let us show by example how one gets the normal forms of \( \Phi \) viewed as a min–max operator.

**Example 5.** Referring to Example 4, consider the stack filter \( \Phi := R_{6,4} \) which replaces each component \( f_i \) of a series \( f = \{ f_i \} \in \mathbb{Z} \) by the fourth smallest element \( \Phi f \) of \( \{ f_{i-6}, \ldots, f_s, \ldots, f_{s+6} \} \). By Theorem 4 it must be a min–max operator. What kind? Its cluster \( \mathcal{C} \) in (3D) has leaves which are subsets of \( A := \{ -6, 6 \} \). Let \( b : \{ 0, 1 \}^A \to \{ 0, 1 \} \) be the positive Boolean function defined by \( \mathcal{C} \). If \( x = (x_{-6}, \ldots, x_0, \ldots, x_6) \) in \( \{ 0, 1 \}^A \) has at most three components 0, then the fourth smallest component is 1. Therefore \( b(x) = 1 \). If \( x \) has at least four components 0, then the fourth smallest component of \( x \) is \( b(x) = 0 \). Hence the minimal 1-sets of \( b \) are exactly the subsets \( C \subseteq A \) of cardinality \( 13 - 3 = 10 \). Thus the DNF(1) of \( b \), and whence the DNF (3D) of \( \Phi \), is given by the cluster

\[
\mathcal{C} := \{ C \subseteq [-6, 6] : |C| = 10 \}.
\]
In view of the above and since \((2n + 1) - (k - 1) = 2n + 2 - k\) it is clear that the disjunctive normal form of \(\Phi := R_{n,k}\) is given by the cluster
\[
\mathcal{C}_{n,k} := \{C \subseteq [-n, n] : |C| = 2n + 2 - k\}. \tag{4}
\]

Other important stack filters can be obtained by composing order statistics in various ways.

**Example 6.** The *winsorizer* \(W_n\) is defined as the unique translation invariant operator \(\mathbb{R}^Z \to \mathbb{R}^Z\) with
\[
[W_n f]_0 := \begin{cases} f_n & \text{if } f_0 > f_n, \\ f_0 & \text{if } f_n \leq f_0 \leq f_{-n}, \\ f_{-n} & \text{if } f_0 < f_{-n}. \end{cases} \tag{5}
\]

Similar to (2) the numbers \(\mathcal{T}_i\) are defined by
\[
\mathcal{T}_{-2n},\ldots,\mathcal{T}_0,\ldots,\mathcal{T}_{2n} := \{f_{-2n},\ldots,f_0,\ldots,f_{2n}\} \quad \text{and} \quad \mathcal{T}_{-2n} \leq \mathcal{T}_{-2n+1} \leq \cdots \leq \mathcal{T}_{2n-1} \leq \mathcal{T}_{2n}.
\]

From definition (5) it is not immediately clear that \(W_n\) is monotone. This is implied by Theorem 5 but it also follows from (6) below which expresses \(W_n\) as a strange max-plus combination\(^4\) of monotone operators that is interesting in its own right.

To simplify notation we write \(Q_u\) for the order statistic \(R_{2n, 3n}\) (upper quartile) and \(Q_\ell\) for \(R_{2n, n}\) (lower quartile).

We claim that
\[
W_n = I + ((Q_u - I) \land (I - Q_\ell) \land 0) + (0 \lor (2Q_\ell - 2I)) \tag{6}
\]

In order to see (6) set \(\Phi := (Q_u - I) \land (I - Q_\ell) \land 0\). Then
\[
[\Phi f]_0 = (\mathcal{T}_n - f_0) \land (f_0 - \mathcal{T}_{-n}) \land 0 = \begin{cases} \mathcal{T}_n - f_0 & \text{if } f_0 > \mathcal{T}_n, \\ 0 & \text{if } \mathcal{T}_{-n} \leq f_0 \leq \mathcal{T}_n, \\ f_0 - \mathcal{T}_{-n} & \text{if } f_0 < \mathcal{T}_{-n}. \end{cases}
\]

Setting \(\Psi := 0 \lor (2Q_\ell - 2I)\) one has
\[
[\Psi f]_0 = 0 \lor (2\mathcal{T}_{-n} - 2f_0) = \begin{cases} 0 & \text{if } f_0 > \mathcal{T}_n, \\ 0 & \text{if } \mathcal{T}_{-n} \leq f_0 \leq \mathcal{T}_n, \\ 2\mathcal{T}_{-n} - 2f_0 & \text{if } f_0 < \mathcal{T}_{-n}. \end{cases}
\]
and therefore
\[
[(I + \Phi + \Psi) f]_0 = \begin{cases} f_0 & \text{if } f_0 > \mathcal{T}_n, \\ f_0 & \text{if } \mathcal{T}_{-n} \leq f_0 \leq \mathcal{T}_n, \\ \mathcal{T}_n & \text{if } f_0 < \mathcal{T}_{-n}. \end{cases}
\]

This proves (6).

\(^4\) When speaking of max-plus combinations we also allow minimum and minus to occur.
We mention that each max-plus combination of stack filters yields a function \( \Phi \) which is nonexpansive in the \( \ell^\infty \) norm (so \( ||\Phi(f) - \Phi(g)|| \leq ||f - g|| \)) and homogeneous in the sense that \( \Phi(f + c) = \Phi(f) + c \) for all constant series \( c \). Nonexpansive and homogeneous operators \( \Phi \) on \( \mathbb{R}^n \) (or \( \mathbb{R}^2 \)) are called \textit{topical} in [3]. As outlined in [3] topical functions arise in a remarkable variety of mathematical disciplines. We might add that nonlinear image analysis is another discipline raising interesting questions. Such as: When is a topical function a min–max operator? For instance, it is a priori not clear that the right-hand side of (6) is a min–max operator. How can one decide whether a topical function is idempotent? A polynomial algorithm only exists for min–max operators (Section 4.1). When \( \Phi \) is a min–max operator then the topical operator \( I - \Phi \) is never a min–max operator, but again there is a polynomial algorithm to decide its idempotency (Section 4.2).

**Theorem 5.** The winsorizer \( W_n : \mathbb{R}^Z \to \mathbb{R}^Z \) is a min–max operator. The cluster \( \mathcal{C} \) in its disjunctive normal form (3D) is given by all \( C \subseteq [-2n, 2n] \) such that

\[
\text{Either } (0 \in C \text{ and } |C| = n + 1) \text{ or } (0 \notin C \text{ and } |C| = 3n + 1). \tag{7}
\]

**Proof.** By definition (Example 6) \( W := W_n \) is translation invariant and local. It is easily seen to commute with contrast changes, so \( W \) is a min–max operator by Theorem 4. Thus, proceeding as in Example 5, put \( A := [-2n, 2n] \). Suppose \( W \) applied to the 0, 1-series \( x \in \{0, 1\}^A \) assumes a value \( b(x) := [Wx]_0 \) equal to 1. Let us distinguish two cases.

\textbf{First case:} \( x_0 = 1 \). Then the only way for \( b(x) \) to be 1 is when at least \( n \) other \( x_i \) are 1 within \( (x_{-2n}, \ldots, x_{2n}) \). Indeed, then \( \overline{x}_n = 1 \), so \( \overline{x}_n \leq x_0 \leq x_n \) and so \( b(x) = x_0 = 1 \) by definition (5); otherwise, if less than \( n \) other \( x_i \) are 1, then \( x_n = 0 \), so \( x_0 > x_n \) and so \( b(x) = x_n = 0 \). Hence the minimal 1-sets of \( b \) are the \( C \)'s in (7) with \( 0 \in C \) and \( |C| = n + 1 \).

\textbf{Second case:} \( x_0 = 0 \). Then the only way for \( b(x) \) to be 1 is when at most \( n - 1 \) other \( x_i \) are 0. Indeed, then \( \overline{x}_n = 1 \), so \( x_0 < \overline{x}_n \) and so \( b(x) = \overline{x}_n = 1 \); otherwise, if at least \( n \) other \( x_i \) are 0, then \( \overline{x}_n = 0 \), so \( \overline{x}_n \leq x_0 \leq \overline{x}_n \) and so \( b(x) = x_0 = 0 \). Hence the minimal 1-sets of \( b \) are the \( C \)'s in (7) with \( 0 \notin C \) and \( |C| = 3n + 1 \). This proves (7). \( \square \)

More prominent than winsorizers \( W_n \) are the \textit{medians} \( M_n := \text{R}_{n,n+1} \). Although in practice the median \( [M_n f]_0 \) of the set \( \{f_{-n}, \ldots, f_0, \ldots, f_n\} \) is not computed (as suggested by putting \( k := n + 1 \) in (4)) by taking the maximum of \( \binom{2n+1}{n+1} \) minima, its computation is still quite time consuming.

This leads us to the stack filters \( L_n \) and \( U_n \) defined by their DNF respectively CNF as follows (here it is convenient to have a general index \( k \) rather than 0):

\[
[L_n f]_k := (f_{-n} \land f_{-n+1} \land \cdots \land f_k) \lor (f_{k-n+1} \land \cdots \land f_{k+1})
\]
\[
\lor \cdots \lor (f_k \land \cdots \land f_{k+n}), \tag{8}
\]
\[
[U_n f]_k := (f_{-n} \lor f_{-n+1} \lor \cdots \lor f_k) \land (f_{k-n+1} \lor \cdots \lor f_{k+1})
\]
\[
\land \cdots \land (f_k \lor \cdots \lor f_{k+n}). \tag{9}
\]
This definition is more appealing upon noticing that \( L_n = \lor^n \circ \land^n \) and \( U_n = \land^n \circ \lor^n \) where

\[
\begin{bmatrix}
\lor^n f
\end{bmatrix}_0 := f_0 \lor f_1 \lor \cdots \lor f_n
\quad \text{and} \quad
\begin{bmatrix}
\land^n f
\end{bmatrix}_0 := f_{-n} \land f_{-n+1} \land \cdots \land f_0
\]

(extensive use of \( \land^n \) and \( \lor^n \) is made in [6] but is not necessary here). Each conjunction in (8) contains \( f_k \) which implies that \( [L_n f]_k \) is \((\leq f_k) \lor (\leq f_k) \lor \cdots \lor (\leq f_k)\) which is \( \leq f_k \). Thus \( L_n f \leq f \) for all \( f \in \mathbb{R}^Z \), i.e. \( L \) is antiextensive. The stack filters \( L_n \) and \( U_n \) are duals of each other in the sense that the DNF of one is obtained from the CNF of the other by switching \( \land \) and \( \lor \). Dual stack filters have dual properties. For instance \( U_n f \geq f \) for all \( f \in \mathbb{R}^Z \), i.e. \( U_n \) is extensive.

The behaviour of \( L_n \) and \( U_n \) on 0, 1-sequences is easily determined (we do not bother to introduce \( b(x) \) here). Say \( x \) is such that \( x_0 = x_1 = \cdots = x_m = 1 \) but \( x_{-1} = x_{m+1} = 0 \). Then \( Lx \leq x \) forces \([L_n x]_{-1} = [L_n x]_{m+1} = 0\). What about \([L_n x]_0\) to \([L_n x]_m\)?

**Case 1**: \( m < n \). Fix \( k \) in \([0, m]\). Since \([0, m]\) has cardinality \( m + 1 \leq n \) each of the conjunctions \((x_{k-n} \land \cdots \land x_k)\) to \((x_k \land \cdots \land x_{k+n})\) in (8) either contains \( x_{-1} = 0 \) or \( x_{m+1} = 0 \). Hence each conjunction is 0. Hence \([L_n x]_k\) is 0.

**Case 2**: \( m \geq n \). Fix \( k \) in \([0, m]\). Since \([0, m]\) has cardinality \( \geq n + 1 \) there is an \( i \) with \( k \in [i, i + n] \subseteq [0, m]\). From \( x_i \land x_{i+1} \land \cdots \land x_{i+n} = 1 \land 1 \land \cdots \land 1 = 1 \), we have

\[
[L_n x]_k = (x_{k-n} \land \cdots \land x_k) \lor \cdots \lor (x_i \land \cdots \land x_{i+n}) \lor \cdots \lor (x_k \land \cdots \land x_{k+n}) = 1
\]

Since \( L_n \) is translation invariant we can summarize as follows: \( L_n \) flattens short 1-pulses \((\cdots, 0, 1, 1, \cdots, 0, \cdots)\) with \( \leq n \) ones, and preserves long 1-pulses with \( > n \) ones. Dually \( U_n \) lifts short 0-pulses \((\cdots, 1, 0, 0, \cdots, 0, 0, 1, \cdots)\) (i.e. makes the 0’s to 1) and preserves long 0-pulses.

**Example 7.**

\[x = (\ldots, 0, 0, 1, 1, 1, 0, 1, 1, 1, 0, 0, 0, 1, 0, 1, 0, 0, 1, 1, \ldots)\]

\[L_3 x = (\ldots, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 0, 0, 0, 1, 1, 1, \ldots)\]

\[U_3 x = (\ldots, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 1, 1, \ldots)\].

In [6] it is argued that the composition \( U_n \circ L_n \), and dually \( L_n \circ U_n \), outperforms the median filter \( M_{2n} \) in several ways. Foremost \( U_n \circ L_n \) is easier to compute (see [6, p. 157] for details), idempotent, co-idempotent, and neighbourly trend preserving. The latter three concepts will be discussed in Sections 4.1, 4.2, 4.4. For now we are content to exhibit the DNF and CNF.

**Theorem 6.** The DNF and CNF of \( U_n \circ L_n \) are given by the coupled clusters \( \mathcal{C}_n \) and \( \mathcal{D}_n \) in Theorem 3.
A strong filter is clearly idempotent but has additional nice properties. All openings and closings are obviously strong filters. Also the winsorizer $W_n$ for $n \geq 2$ is not even idempotent [10, Examples 9, 12]. Generally, according to [9, Corollary 5], a stack filter $\Phi$ with cluster $\mathcal{C}$ in (3D) is a $\wedge$-overfilter iff

$$\forall C \in \mathcal{C} \ (\forall i \in C) (\exists C' \in \mathcal{C}) \ C' \subseteq C - i.$$ (10)
Here $C - i$ is defined as $\{ j - i \mid j \in C \}$. Dually, $\Phi$ is an $\vee$-underfilter iff condition (10) holds for the cluster $\mathcal{D}$ in (3C).

**Theorem 7.** The stack filters $U_n \circ L_n$ and $L_n \circ U_n$ are strong filters.

**Proof.** By Theorem 6 we have to show that (10) is satisfied by the cluster $\mathcal{D}_n$, respectively $\mathcal{D}_n$. Consider for instance a set $D := \{ a, b, c \}$ of type (C1). The set $D - a = \{ b - a, c - a \}$ is again (C1) since $(b - a) - 0 \leq n - 1$ and $(c - a) - (b - a) = c - b \leq n + 1$. Thus $D' := D - a$ holds for the cluster $h$ for the co-idempotency of any stack filter $D$. But $(c - b) - 0 = c - b \leq n + 1$ and $0 - (a - b) = b - a \leq n - 1$. If $c - b$ is $n + 1$ or $n$ then $D' := [0, c - b]$ is a type (B) or (A) subset of $D - b$. Otherwise $c - b \leq n - 1$ and $b - a \leq n - 1$ imply that $D' := D - b$ is type (C3). Similarly $D - c = \{ a - c, b - c, 0 \}$ either contains at type (A) or (B) subset or is itself a (C2) set. We leave it to the reader to check the remaining sets $D$ in $\mathcal{D}_n$ and $C$ in $\mathcal{E}_n$. □

**Problem.** Find a proof for the strongness of $U_n \circ L_n$ which is shorter than the combinations of Theorems 7, 6, and 3.

4.2. Co-idempotency and noise

Let $\Phi : \mathbb{R}^Z \rightarrow \mathbb{R}^Z$ be any stack filter. Call $h \in \mathbb{R}^Z$ a noise series if $h = f - \Phi f$ for some $f \in \mathbb{R}^Z$. This definition is most suggestive when $\Phi$ is idempotent. Then, as seen in 4.1, the series $g := \Phi f$ can be viewed as the original series underlying the corrupted message $f$; whence the difference $h = f - g$ is the noise added during transmission (or whenever). If Noise ($\Phi$) is the set (to be described) of all noise series, and Zero($\Phi$) is the set of all $h \in \mathbb{R}^Z$ with $\Phi h = 0$, then always Zero($\Phi$) $\subseteq$ Noise($\Phi$), for $\Phi h = 0$ implies $h = h - \Phi h$. It is desirable to have Noise ($\Phi$) = Zero($\Phi$) since we shall see in a moment that Zero($\Phi$) can be computed in a systematic way provided the DNF and CNF of $\Phi$ are known. This and similar considerations lead one [4,7] to define $\Phi$ as co-idempotent if $I - \Phi$ is idempotent. As opposed to linear maps $\mathbb{R}^Z \rightarrow \mathbb{R}^Z$, for stack filters $\Phi : \mathbb{R}^Z \rightarrow \mathbb{R}^Z$ idempotency is not equivalent to co-idempotency! However, since $(I - \Phi) \circ (I - \Phi) = (I - \Phi) - \Phi \circ (I - \Phi)$, the co-idempotency of $\Phi$ boils down to $\Phi \circ (I - \Phi) = 0$. But $\Phi (f - \Phi f) = 0$ ($f \in \mathbb{R}^Z$) is equivalent to the desired equality Noise ($\Phi$) = Zero($\Phi$). A sufficient and necessary condition for the co-idempotency of any stack filter $\Phi$ is derived in [10, Theorem 14]. It can be checked in polynomial time, provided the DNF and CNF of $\Phi$ are known.

For instance $L_n$ and $U_n$ are co-idempotent [4, Corollary 1], so Noise($L_n$) = Zero($L_n$) and Noise($U_n$) = Zero($U_n$). In view of the discussion preceding Example 7 one can easily determine:

**Corollary 8.** The set Noise($L_n$) consists of all series $p \geq 0$ such that each string of positive adjacent components has cardinality $\leq n$. The set Noise($U_n$) consists of all series $q \leq 0$ such that each string of negative adjacent components has cardinality $\leq n$.

For any series $h \in \mathbb{R}^Z$ define its positive shadow $h^+ \in \mathbb{R}^Z$ by $h^+_i = (h^+_i)_i = \max\{h_i, 0\}$, and its negative shadow $h^- \in \mathbb{R}^Z$ by $h^-_i = (h^-)_i = \min\{h_i, 0\}$.
Lemma 9. Let $\Phi : \mathbb{R}^Z \to \mathbb{R}^Z$ be a stack filter and $h \in \mathbb{R}^Z$ a series. Then

$$h \in \text{Zero}(\Phi) \iff (h^+ \in \text{Zero}(\Phi) \text{ and } h^- \in \text{Zero}(\Phi))$$

Proof. As to $\Leftarrow$, from $h^- \leq h \leq h^+$ and $\Phi(h^-) = \Phi(h^+) = 0$ follows $\Phi h = 0$.

As to $\Rightarrow$, from $h^- \leq h \leq h^+$ and $\Phi h = 0$ follows $\Phi(h^-) \leq 0 \leq \Phi(h^+)$. Suppose one had $0 < \Phi(h^+)$, say

$$\left[ \Phi(h^+) \right]_s = \bigvee_{C \in \mathcal{C}} \left( \bigwedge_{i \in C} h^+_{s+i} \right) > 0$$

(using the DNF). Then $\bigwedge_{i \in C} h^+_{s+i} > 0$ for some $C \in \mathcal{C}$, whence $h^+_{s+i} > 0$ ($i \in C$), whence $h_{s+i} > 0$ ($i \in C$), whence $\bigwedge_{i \in C} h_{s+i} > 0$, whence the contradiction $[\Phi h]_s > 0$. Therefore $\Phi(h^+) = 0$. Dually (using the CNF) it follows that $\Phi(h^-) = 0$. □

Thus it suffices to determine all the nonnegative kernel series $p \geq 0$ and all the nonpositive kernel series $q \leq 0$. Any $h$ with $q \leq h \leq p$ is in $\text{Zero}(\Phi)$, and each kernel series $h$ is sandwiched in that way (by Lemma 9 we may choose $p := h^+$ and $q := h^-$). This result is useful because non-negative (dually non-positive) kernel series have the following neat properties which are easily verified:

If $p \geq 0$ and $p' \geq 0$ and $(\forall i) p_i = 0 \iff p'_i = 0,$

then $\Phi p = 0$ implies $\Phi p' = 0.$ (11)

Let

$$[\Phi p]_s = \bigwedge_{D \in \mathcal{D}} \left( \bigvee_{j \in D} p_{j+s} \right) \text{ (CNF)}, \text{ then } p \in \text{Zero}(\Phi) \text{ if and only if}$$

for each $s \in \mathbb{Z}$ there is some $D \in \mathcal{D}$ with $p_{s+j} = 0$ ($j \in D$). (12)

Summarizing: the CNF of a stack filter $\Phi$ “yields” all series $p \geq 0$ in $\text{Zero}(\Phi)$, and the DNF “yields” all series $q \leq 0$ in $\text{Zero}(\Phi)$. By Lemma 9 this describes the whole of $\text{Zero}(\Phi)$. But remember that $\text{Zero}(\Phi) = \text{Noise}(\Phi)$ only when $\Phi$ is co-idempotent. We put “yields” in quotes because further insights beyond (12) and its dual might be required for an elegant description of the nonnegative, respectively nonpositive kernel series. This point is illustrated by the Theorem below. If, say, we speak of a nonpositive string of length $> n$ we mean a maximal sequence of $> n$ nonpositive adjacent components of our series $h$ at hand.

Theorem 10. The set $\text{Noise}(L_n \circ U_n)$ consists of all series $h$ with the following properties:

(i) each negative string has length $\leq n$;

(ii) each string between subsequent nonpositive strings, both of length $> n$, has length $\leq n$. 
**Proof.** First of all \( \text{Noise}(L_n \circ U_n) = \text{Zero}(L_n \circ U_n) \) since \( L_n \circ U_n \) is co-idempotent. In view of the discussion before Example 7 one verifies at once that the nonpositive kernel series of \( L_n \circ U_n \) are precisely the series

\[
q \leq 0 \quad \text{such that each negative string has length } \leq n; \tag{13}
\]

Almost as evident, the nonnegative kernel series of \( L_n \circ U_n \) are precisely the series

\[
p \geq 0 \quad \text{such that each string between subsequent zero strings, both of length } > n, \text{ has length } \leq n. \tag{14}
\]

Now let \( h \) be any series satisfying (i) and (ii). Then clearly \( h^+ \geq 0 \) satisfies (14), and \( h^- \) satisfies (13). Thus \( h \in \text{Zero}(L_n \circ U_n) \) by Lemma 9. If \( h \) violates (i) or (ii), then either \( h^- \) violates (13) or \( h^+ \) violates (14). Therefore, again by Lemma 9, \( h \notin \text{Zero}(L_n \circ U_n) \). \( \square \)

By duality \( \text{Noise}(U_n \circ L_n) \) is similarly characterized.

### 4.3. Order relations among stack filters

For operators \( \Phi, \Psi : \mathbb{R}^\mathbb{Z} \to \mathbb{R}^\mathbb{Z} \) put \( \Phi \leq \Psi \) if \( \Phi f \leq \Psi f \) for all series \( f \in \mathbb{R}^\mathbb{Z} \). With respect to this relation the set of all operators \( \mathbb{R}^\mathbb{Z} \to \mathbb{R}^\mathbb{Z} \) becomes a distributive lattice. Let us focus on the sublattice \( \mathcal{M} \) of all stack filters. As explained in [8] or [6] it is important to know whether or not \( \Phi \leq \Psi \) for given \( \Phi, \Psi \in \mathcal{M} \).

**Theorem 11.** Let \( \Phi \) and \( \Psi \) have disjunctive normal forms

\[
[\Phi f]_s = \bigvee_{C \in \mathcal{C}} \left( \bigwedge_{i \in C} f_{s+i} \right), \quad \text{respectively,} \quad [\Psi f]_s = \bigvee_{D \in \mathcal{D}} \left( \bigwedge_{i \in D} f_{s+i} \right)
\]

Then \( \Phi \leq \Psi \) if and only if \( (\forall C \in \mathcal{C}) (\exists D \in \mathcal{D}) D \subseteq C \).

The easy proof is given in [6, Theorem 9]. Observe that Theorem 11 only settles the question “\( \Phi \leq \Psi \)” for specific \( \Phi \) and \( \Psi \) but might not yield the overall picture of order relations among the members of a given family of stack filters. For instance, a complete description of the order relations among products of stack filters \( L_m \) and \( U_n \) is given in [6, Theorem 16].

**Example 8.** By [6, Theorem 13] the semigroup \( S_{M,N} \) generated by \( L_1, L_2, \ldots, L_M \) and \( U_1, U_2, \ldots, U_N \) has cardinality \( \left( \binom{M+N+2}{N+1} - 2 \right) \). Because \( S_{M,N} \subseteq \mathcal{M} \) is not a sublattice it is not clear whether the partially ordered set \( (S_{M,N}, \leq) \) is a lattice for all \( M, N \)!

---

\(^5\)This was first proven in [4, Theorem 4]; in fact each strong filter is co-idempotent [9, Corollary 7].
But \((S_{2,2}, \leq)\) is a lattice and this is its diagram [6, Example 16]:

![Lattice Diagram](image)

### 4.4. Neighbourly trend preserving stack filters

Recall that each min–max operator \(\Phi : \mathbb{R}^Z \to \mathbb{R}^Z\) is monotone, so \(f \leq g\) implies \(\Phi f \leq \Phi g\). Restating it componentwise, one has

\[
(\forall s \in \mathbb{Z}) \quad f_s \leq g_s \Rightarrow (\forall s \in \mathbb{Z}) \quad [\Phi f]_s \leq [\Phi g]_s.
\]

(15)

Compare this with the following definition of Rohwer. An operator \(\Phi : \mathbb{R}^Z \to \mathbb{R}^Z\) is **neighbourly trend preserving** (n.t.p.) if for all series \(f\) and all indices \(s\) one has

\[
f_s \geq f_{s+1} \Rightarrow [\Phi f]_s \geq [\Phi f]_{s+1}\quad \text{and} \quad f_s \leq f_{s+1} \Rightarrow [\Phi f]_s \leq [\Phi f]_{s+1}.
\]

(16)

**Example 9.** If \(f \in \mathbb{R}^Z\) is defined by \(f_s := (-1)^s\) then \([M_1 f]_s = (-1)^{s+1}\). Hence the median \(M_1\) is **not** n.t.p., neither is \(M_n\) for \(n > 1\). On the other hand, from \(f_s \leq f_{s+1}\) (so \(f_s \land f_{s+1} = f_s\)) and (8) it follows that

\[
[L_1 f]_s \leq f_s \leq f_s \lor (f_{s+1} \lor f_{s+2}) = (f_s \land f_{s+1}) \lor (f_{s+1} \lor f_{s+2}) = [L_1 f]_{s+1}.
\]

Thus \(L_1\) satisfies half of (16). Now let \(f_s \geq f_{s+1}\). Then

\[
[L_1 f]_s = (f_{s-1} \land f_s) \lor (f_s \land f_{s+1}) \geq f_{s+1} = (f_s \land f_{s+1}) \lor (f_{s+1} \land f_{s+2}) = [L_1 f]_{s+1}
\]

so \(L_1\) satisfies the other half as well, whence is n.t.p.
Obviously n.t.p. is preserved under composition of stack filters and under duality. See [5] for more on the relevance of neighbourly trend preservation. Here we content ourselves with characterizing n.t.p. stack filters by their DNF (an analogous characterization in terms of the CNF holds).

**Theorem 12.** Let \( \Phi : \mathbb{R}^Z \rightarrow \mathbb{R}^Z \) have the disjunctive normal form \( [\Phi f]_0 = \bigvee_{C \in \mathcal{C}} (\bigwedge_{i \in C} f_i) \). Then \( \Phi \) is n.t.p. if and only if for all \( C \in \mathcal{C} \) the following properties hold:

\[
\begin{align*}
(N1) \quad 0 \notin C & \Rightarrow (\exists D \in \mathcal{C}) \quad C \supseteq D + 1 \\
0 \in C & \Rightarrow (\exists D \in \mathcal{C}) \quad C \supseteq D + 1 - \{1\}
\end{align*}
\]

\[
\begin{align*}
(N(-1)) \quad 0 \notin C & \Rightarrow (\exists D \in \mathcal{C}) \quad C \supseteq D + 1 \\
0 \in C & \Rightarrow (\exists D \in \mathcal{C}) \quad C \supseteq D + 1 - \{1\}
\end{align*}
\]

**Proof.** Let \( f_s \leq f_{s+1} \). We show that \((N1)\) implies \([\Phi f]_s \leq [\Phi f]_{s+1}\). First observe that

\[
[\Phi f]_{s+1} = \bigvee_{D \in \mathcal{C}} \left( \bigwedge_{i \in D} f_{s+1+i} \right) = \bigvee_{D \in \mathcal{C}} \left( \bigwedge_{j \in D+1} f_{s+j} \right).
\]

In order to have \([\Phi f]_s \leq [\Phi f]_{s+1}\) it thus suffices that each \( \bigwedge_{i \in C} f_{s+i} \) is \( \leq \) some \( \bigwedge_{j \in D+1} f_{s+j} \).

First case: \( 0 \notin C \). By \((N1)\) there is a \( D \in \mathcal{C} \) with \( C \supseteq D + 1 \). Thus \( \bigwedge_{i \in C} f_{s+i} \leq \bigwedge_{j \in D+1} f_{s+j} \).

Second case: \( 0 \in C \). By \((N1)\) there is a \( D \in \mathcal{C} \) with \( C \supseteq D + 1 - \{1\} \).

Thus \( z := \bigwedge_{j \in D+1-\{1\}} f_{s+j} \geq \bigwedge_{i \in C} f_{s+i} \). If \( 1 \notin D + 1 \) then \( \bigwedge_{j \in D+1} f_{s+j} = z \) and we are done. If \( 1 \in D + 1 \) then

\[
\bigwedge_{j \in D+1} f_{s+j} = z \wedge f_{s+1} \geq \bigwedge_{i \in C} f_{s+i} \wedge f_{s+1} = \bigwedge_{i \in C} f_{s+i}.
\]

The last equality holds because of \( 0 \notin C \) and \( f_s \wedge f_{s+1} = f_s \).

Conversely, suppose \((N1)\) is violated. Then there is a \( \overline{C} \in \mathcal{C} \) such that either (a) or (b) takes place:

(a) \( 0 \in \overline{C} \) and \( (\forall D \in \mathcal{C}) \quad \overline{C} \not\supseteq D + 1 - \{1\} \),
(b) \( 0 \notin \overline{C} \) and \( (\forall D \in \mathcal{C}) \quad \overline{C} \not\supseteq D + 1 \).

As to (a), define \( f \in \mathbb{R}^Z \) by \( f_i := 1 \) \((i \in \overline{C} \cup \{1\})\) and \( f_i := 0 \) otherwise. Then \( f_0 \leq f_1 \) (both are 1) but we claim that \([\Phi f]_0 \not\leq [\Phi f]_1\). Indeed, for all \( D \in \mathcal{C} \) there is \( j_D \in D + 1 - \{1\} \) with \( j_D \notin \overline{C} \). Because all \( f_{j_D} \) are 0 one has \( \bigwedge_{j \in D+1} f_j = 0 \) for all \( D \in \mathcal{C} \). Therefore

\[
[\Phi f]_0 = \bigvee_{C \in \mathcal{C}} \left( \bigwedge_{i \in C} f_i \right) \geq \left( \bigwedge_{i \in \overline{C}} f_i \right) = 1 \quad \text{but} \quad [\Phi f]_1 = \bigvee_{D \in \mathcal{C}} \left( \bigwedge_{j \in D+1} f_j \right) = 0.
\]
As to (b), define \( f \in \mathbb{R}^Z \) by \( f_i := 1 \) \((i \in \overline{C})\) and \( f_i := 0 \) otherwise. Then \( f_0 = 0 \leq f_1 \) but \( [\Phi f]_0 \geq [\Phi f]_1 \) by an argument analogous to the above.

We see that \((N1)\) is sufficient and necessary for the implication \((f_s \leq f_{s+1} \Rightarrow [\Phi f]_s \leq [\Phi f]_{s+1})\) to hold for all \( s \in \mathbb{Z} \). Similarly the reader may verify that \((N(-1))\) is sufficient and necessary for the implication \((f_s \geq f_{s+1} \Rightarrow [\Phi f]_s \geq [\Phi f]_{s+1})\) to hold for all \( s \in \mathbb{Z} \). \( \square \)

Observe that the truth of \( f_s \geq f_{s+1} \Rightarrow [\Phi f]_s \geq [\Phi f]_{s+1} \) for all \( s \in \mathbb{Z} \) is equivalent to the truth of \( f_s \leq f_{s-1} \Rightarrow [\Phi f]_s \leq [\Phi f]_{s-1} \) for all \( s \in \mathbb{Z} \). This explains the notation \((N1)\) and \((N(-1))\) in Theorem 12. In fact, a general min–max operator \( \Phi : \mathbb{R}^S \rightarrow \mathbb{R}^S \) may be said to be “trend preserving with respect to a set of directions \( E \)” if for all \( e \in E \) and \( s \in S \) and \( f \in \mathbb{R}^S \) one has

\[
f_s \leq f_{s+e} \Rightarrow [\Phi f]_s \leq [\Phi f]_{s+e}.
\]

Assuming translation invariance the conditions \((Ne)\) \((e \in E)\) analogously to the ones in Theorem 12 are necessary and sufficient for this property. For instance, when \( S = \mathbb{Z}^2 \) then \( E := \{(1, 0), (-1, 0), (0, 1), (0, -1)\} \) may be a natural choice.

**Corollary 13.** All stack filters \( L_n \) and \( U_n \) and arbitrary products thereof are n.t.p.

**Proof.** Because n.t.p. is inherited under composition and duality it suffices to show that \( L_n \) is n.t.p. By (8) the \( n + 1 \) conjunctions in the DNF of \([L_n f]_0\) are given by the sets \( C := [j - n, j] \) \((0 \leq j \leq n)\) all of which contain 0. As to \((N1)\), if \( j \neq 0 \) then \( D := C - 1 \) is one of these sets and \( C \supseteq C - 1 = D + 1 - \{1\} \). For \( j = 0 \) we set \( D := C \) and also get that \( C = [-n, 0] \supseteq [-n + 1, 0] = D + 1 - \{1\} \). Similarly, distinguishing \( j \neq n \) and \( j = n \), one verifies \((N(-1))\). \( \square \)

We mention that a fifth benefit of normal forms, namely the swift computation of probability distributions of stack filters, is dealt with in a forthcoming article.

**References**


