The André-Oort conjecture for Drinfeld modular varieties

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Abstract
We state an analogue of the André-Oort Conjecture for subvarieties of Drinfeld modular varieties, and prove it in two special cases. To cite this article: F. Breuer, The André-Oort conjecture for Drinfeld modular varieties, C. R. Acad. Sci. Paris, Ser. I ??? (2007).

Résumé

1. Introduction

The André-Oort conjecture for complex Shimura varieties states that any closed subvariety \( X \) of a Shimura variety \( S/\mathbb{C} \) containing a Zariski-dense set of special points must be of Hodge type. For an overview, see [3] or [5]. Recently, Bruno Klingler, Emmanuel Ullmo and Andrei Yafaev have announced a proof of this conjecture assuming the Generalised Riemann Hypothesis.

In the current note we will study an analogue of this conjecture for Drinfeld modular varieties, and outline a proof for two special cases.

Let \( C/\mathbb{F}_q \) be a smooth projective geometrically connected algebraic curve with function field \( K \), choose a closed point \( \infty \) on \( C \), and define \( A = H^0(C \smallsetminus \infty, \mathcal{O}_C) \). We denote by \( K_\infty \) the completion of \( K \) at \( \infty \), and by \( \mathcal{O}_\infty = \hat{K}_\infty \) the completion of an algebraic closure of \( K_\infty \). We denote by \( A_f = A \otimes K \) the ring of finite adèles of \( K \).

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For an open subgroup $\mathcal{K} \subset \text{GL}_r(\hat{A})$, we denote by $M^r_{\mathcal{A}}(\mathcal{K})$ the coarse moduli scheme for rank $r$ Drinfeld $A$-modules with $\mathcal{K}$-level structure. We write $M^r_{\mathcal{A}}(1)$ for $M^r_{\mathcal{A}}(\text{GL}_r(\hat{A}))$, the coarse moduli scheme for Drinfeld modules without level structure.

**Definition 1.1** A closed irreducible subvariety $X \subset M^r_{\mathcal{A}}(\mathcal{K})_{\mathbb{C}_\infty}$ is called special if $X$ is an irreducible component of the locus of Drinfeld $A$-modules with endomorphism ring containing a given ring. A point $x \in M^r_{\mathcal{A}}(\mathcal{K})(\mathbb{C}_\infty)$ is called a CM point if it corresponds to a Drinfeld module $\varphi$ with complex multiplication.

We see that CM points are precisely the special subvarieties of dimension zero. Our analogue of the Andr´e-Oort Conjecture is the following.

**Conjecture 1.2** A closed irreducible subvariety $X \subset M^r_{\mathcal{A}}(\mathcal{K})_{\mathbb{C}_\infty}$ contains a Zariski-dense set of CM points if and only if $X$ is special.

We will sketch the following theorems, the first being an (unconditional) analogue of a result of Bas Edixhoven and Andrei Yafaev [3, Theorem 1.2], and the second an analogue of a result of Ben Moonen [4, §5]. Our approach is an adaptation of that of Edixhoven and Yafaev.

**Theorem 1.3** Let $X \subset M^r_{\mathcal{A}}(\mathcal{K})_{\mathbb{C}_\infty}$ be an irreducible algebraic subcurve. Then $X$ contains infinitely many CM points if and only if $X$ is special. In particular, Conjecture 1.2 is true if $r = 3$.

**Theorem 1.4** Let $p \subset A$ be a non-zero prime, and $n \in \mathbb{N}$ a positive integer. Let $X \subset M^r_{\mathcal{A}}(\mathcal{K})_{\mathbb{C}_\infty}$ be a closed irreducible subvariety containing a Zariski-dense set of CM points $x$ with the following property: Denote by $\mathcal{O}_{x}$ the endomorphism ring of a Drinfeld module representing $x$. Then there is an unramified prime $\mathfrak{p} | p$ of $\mathcal{O}_{x} \otimes_{\mathcal{A}} K$ such that the residue degree $f(\mathfrak{p} | p) = 1$, and $p^n$ does not divide the conductor of $\mathcal{O}_{x}$ in its integral closure. Then $X$ is a special subvariety.

2. Analytic theory

Denote by $\Omega^r := \mathbb{P}^{r-1}(\mathbb{C}_\infty) \setminus \{\text{linear subspaces defined over } K_{\infty}\}$ Drinfeld’s upper half-space, on which $\text{GL}_r(\mathbb{K}_\infty)$ acts. Then we have, as rigid analytic varieties, $M^r_{\mathcal{A}}(\mathcal{K})_{\mathbb{C}_\infty} \cong \text{GL}_r(\mathcal{K}) \setminus \Omega^r / \text{GL}_r(\mathcal{A}_f)/\mathcal{K} \cong \prod_{s \in S} \Gamma_s \backslash \Omega^r,$

where $S$ denotes a finite set of representatives for $\text{GL}_r(\mathcal{K})/\text{GL}_r(\mathcal{A}_f)/\mathcal{K}$, and $\Gamma_s = sKs^{-1} \cap \text{GL}_r(\mathcal{K}).$

The group $\text{GL}_r(\mathcal{A}_f)$ acts from the left on $\Omega \times \text{GL}_r(\mathcal{A}_f)$ via $g \cdot (\omega, h) = (\omega, hg^{-1})$, and this induces the Hecke correspondence $T_g$ on $M^r_{\mathcal{A}}(\mathcal{K})$, which factors through $M^r_{\mathcal{A}}(\mathcal{K}_g)$, where $K_g = K \cap g^{-1}Kg.$ For any non-zero ideal $n \subset A$ we denote by $T_n$ and $T_\mathbb{A}$ the Hecke correspondences associated to $\text{diag}(n, 1, \ldots, 1)$ and $\text{diag}(1, n, \ldots, n)$, respectively, viewed as elements in $\text{GL}_r(\mathcal{A}_f).$ In the moduli interpretation of $M^r_{\mathcal{A}}(\mathcal{K})_{\mathbb{C}_\infty}$, the correspondence $T_n$ encodes all isogenies of kernel isomorphic to $A/n$, and $T_\mathbb{A}$ encodes their dual isogenies, with kernels isomorphic to $(A/n)^{r-1}$.

We can now give an equivalent definition of special subvarieties. Let $r' | r$, and let $K' / K$ be an imaginary extension (which means that only one place of $K'$ lies above $\infty$) of degree $[K' : K] = r/r'$, and denote by $A'$ the integral closure of $A$ in $K'$. Then any rank $r'$ Drinfeld $A'$-module is also a rank $r$ Drinfeld $A$-module, which gives an embedding of moduli spaces $M^{r'}_{\mathcal{A}}(1)_{\mathbb{C}_\infty} \hookrightarrow M^r_{\mathcal{A}}(1)_{\mathbb{C}_\infty}.$ A closed subvariety $X \subset M^r_{\mathcal{A}}(1)_{\mathbb{C}_\infty}$ is special if and only if $X$ is an irreducible component of $T_g(M^{r'}_{\mathcal{A}}(1)_{\mathbb{C}_\infty})$ for some $g \in \text{GL}_r(\mathcal{A}_f)$ and some $A'$ and $r'$ as above. A closed irreducible subvariety $X \subset M^r_{\mathcal{A}}(\mathcal{K})_{\mathbb{C}_\infty}$ is special if its image under the canonical projection $M^r_{\mathcal{A}}(\mathcal{K})_{\mathbb{C}_\infty} \rightarrow M^r_{\mathcal{A}}(1)_{\mathbb{C}_\infty}$ is special.

Our first step is to show that suitable Hecke orbits are Zariski-dense.
Proposition 2.1 Let \( n \subset A \) be a non-trivial principal ideal. Then for any \( x \in M^*_A(1)(\mathbb{C}_\infty) \), the Hecke orbit \((T_n + T_\kappa)^\infty(x)\) is Zariski-dense in the irreducible component of \( M^*_A(1)(\mathbb{C}_\infty) \) containing \( x \).

It follows that \( M^*_A(1)(\mathbb{C}_\infty) \), and hence any special subvariety, contains a Zariski-dense set of CM points. The idea of the proof is the following. Let \( Z \subset \Omega' \) denote an irreducible component of the preimage of the Zariski-closure of the Hecke orbit \((T_n + T_\kappa)^\infty(x)\). Then one explicitly constructs a smooth point \( \omega \in Z \) which is approximated by sequences of points of \( Z \) lying on lines in sufficiently many directions (in \( \Omega \) viewed as a subspace of \( A^{r-1}(\mathbb{C}_\infty) \)) to conclude that the tangent space of \( Z \) at \( \omega \) must have dimension \( r - 1 \). The result follows. \( \square \)

A subvariety \( X \subset M^*_A(K)_{\mathbb{C}_\infty} \) is called Hodge generic if it is not contained in any proper special subvariety. The next step is to show that suitable Hecke images of irreducible Hodge generic subvarieties are again irreducible:

Proposition 2.2 Let \( X \subset M^*_A(1)_{\mathbb{C}_\infty} \) be an irreducible Hodge generic subvariety, and \( \dim(X) \geq 1 \). Then

(i) There exists a non-zero ideal \( m_X \subset A \), such that \( T_n(X) \) is irreducible for any non-zero ideal \( n \subset A \) prime to \( m_X \).

(ii) There exists an open subgroup \( K \subset \text{GL}_r(A) \) and an irreducible component \( X' \subset M^*_A(K)_{\mathbb{C}_\infty} \) of the preimage of \( X \) such that \( T_n(X') \) is irreducible for all non-zero ideals \( n \subset A \).

Proof sketch. We first replace \( X \) by its non-singular locus. We assume for simplicity that \( X^{an} \subset \text{GL}_r(A) \setminus \Omega' \). Let \( \Xi \subset \Omega' \) be an irreducible component of the preimage of \( X^{an} \) and \( \Delta = \text{Stab}_{\text{GL}_r(A)}(\Xi) \), so that \( X^{an} \cong \Delta \setminus \Xi \). Denote by \( \tilde{\Delta} \) the closure of \( \Delta \) in \( \text{GL}_r(A) \), then one can show using [1, Theorem 1.1] that \( \tilde{\Delta} \) is open in \( \text{GL}_r(A) \). It follows that there is a non-zero ideal \( m_X \subset A \) such that \( \Delta \to \text{GL}_r(A/n) \) is surjective for all non-zero ideals \( n \subset A \) prime to \( m_X \).

To prove (i), let \( n \subset A \) be a non-zero ideal prime to \( m_X \). Let \( \mathcal{K} = \text{GL}_r(\hat{A}) \), and define \( \Gamma = \mathcal{K} \cap \text{GL}_r(K) \) and \( \mathcal{K}_0(n) = \mathcal{K} \cap n^{-1} \mathcal{K} n \). Denote by \( \pi : M^*_A(K_0(n)) \to M^*_A(\mathcal{K}) \) the canonical projection. Since \( \Delta \) and \( \Gamma \) have the same image in \( \text{GL}_r(A/n) \), it follows that \( \Delta \) acts transitively on the fibres of \( \pi \), hence \( \pi^{-1}(X) \), and thus also \( T_n(X) = \pi(\text{diag}(n,1,\ldots,1)\pi^{-1}(X)) \), is irreducible.

To prove (ii), we make similar definitions as above, but with \( K = \tilde{\Delta} \). We choose \( X' \subset M^*_A(K)_{\mathbb{C}_\infty} \) such that \( X^{an} \cong \Delta \setminus \Xi \). Then we see that for each non-zero ideal \( n \subset A \), \( \Delta \) and \( \Gamma \) have the same image in \( \text{GL}_r(A/n) \), and thus again \( T_n(X') \) is irreducible. \( \square \)

3. Arithmetic theory

Let \( x \in M^*_A(K)_{\mathbb{C}_\infty} \) be a CM point represented by a CM Drinfeld module \( \varphi \) with endomorphism ring \( \text{End}(\varphi) = \mathcal{O} \). Then \( \mathcal{O} \) is an order in an imaginary extension \( K'/K \) of degree \( [K' : K] = r \). Denote by \( A' \) the integral closure of \( A \) in \( K' \). By the theory of complex multiplication, the field \( K'(x) \) of definition of \( x \) over \( K' \) is the ring class field associated to the order \( \mathcal{O} \), in particular \( \text{Gal}(K'(x)/K') \cong \text{Pic}(\mathcal{O}) \).

Now let \( p \subset A \) be an unramified prime of residue degree 1 in \( K'/K \), and which does not divide the conductor \( \epsilon \) of \( \mathcal{O} \) in \( A' \). Then \( \sigma_p(x) \in T_p(x) \), where \( \sigma_p \in \text{Gal}(K'(x)/K') \) is the Frobenius element associated to \( p \).

Denote by \( g(K') \) the genus of \( K' \), then we define the CM height of \( x \) (and of \( \varphi \)) to be

\[
H_{\text{CM}}(x) = H_{\text{CM}}(\varphi) := q^{g(K')} \cdot \#(A'/\mathbb{Q})^{1/r}.
\]

One can show
Proposition 3.1  

(i) There are only finitely many CM points in \( M_f^{\text{is}}(K)(\mathbb{C}_\infty) \) with CM-height bounded by a given constant.

(ii) For every \( \varepsilon > 0 \) there is a computable constant \( C_{\varepsilon} > 0 \) such that the following holds. Let \( \varphi \) be a Drinfeld module with complex multiplication by an order \( \mathcal{O} \), as above. Then \( \# \text{Pic}(\mathcal{O}) > C_{\varepsilon} H_{\text{CM}}(\varphi)^{1-\varepsilon} \).

Proof sketch of Theorem 1.3. Let \( X \subset M_f^{\text{is}}(K)_{\mathbb{C}_\infty} \) be an irreducible subcurve. We may ignore the level structure, and after replacing \( X \) by a Hecke translate and \( A \) by some \( A' \) if necessary, we may assume that \( X \subset M_f^{\text{is}}(1)_{\mathbb{C}_\infty} \) is Hodge generic.

Let \( F \) be a field of definition of \( X \) containing the Hilbert class field of \( K \), we may assume \( [F: K] \) is finite. Using explicit polynomial equations for the Hecke correspondence \( \text{Hecke}(X, p) \), one can show that the intersection degree of \( X \cap \text{Hecke}(X, p) \) is bounded by \( c|p|^n \) for constants \( c, n > 0 \) depending only on \( X \), where \( |p| := \#(A/p) \). Let \( m_X \subset A \) denote the ideal given by Proposition 2.2 (i).

Let \( x \in X(\mathbb{C}_\infty) \) be a CM point with endomorphism ring \( \mathcal{O} \subset K' \). Then if \( H_{\text{CM}}(x) \) is sufficiently large, it follows from the Cebotarev Theorem for function fields, and Proposition 3.1, that there exists a non-zero prime \( p \subset A \) with the following properties:

(i) \( p \) divides neither \( m_X \) nor the conductor of \( \mathcal{O} \), and has residue degree one in \( F K'/K \).

(ii) \( \# \text{Pic}(\mathcal{O}) > c(F : K)|p|^n \).

Denote by \( E \) the separable closure of \( K \) in \( F \), and by \( L \) the Galois closure of \( F_p K'(x) \) over \( K' \).

Let \( \sigma \in \text{Aut}(FL/FK') \) be an extension of the Frobenius element associated to a prime of \( L \) above \( p \). Then \( \sigma \) fixes \( F \) (a field of definition of \( X \) and of \( T_p(X) \)) and \( \sigma(x) \in T_p(x) \) complex multiplication. Thus \( X \cap T_p(X) \) contains the entire \( \text{Gal}(FK'(x)/FK') \)-orbit of \( x \), which by (ii) above is larger than the intersection degree. It follows that \( X \subset T_p(X) \). Since \( T_p(X) \) is irreducible by Proposition 2.2, we get \( X = T_p(X) = T_{p^n}(X) = T_{p^n}(X) \), where we choose \( m \in \mathbb{N} \) such that \( p^m \) is principal. Thus \( X \) contains the entire Hecke orbit \( (T_{p^n} + T_{p^n})^{\infty}(x) \), which is Zariski-dense in a component of \( M_f^{\text{is}}(1)_{\mathbb{C}_\infty} \), by Proposition 2.1. The result follows. \( \Box \)

Proof sketch of Theorem 1.4. By a sequence of simplifications one may reduce to the case where \( X \subset M_f^{\text{is}}(1)_{\mathbb{C}_\infty} \) is Hodge generic, and contains a Zariski-dense set \( \Sigma \) of CM points \( x \) with endomorphism rings \( \mathcal{O}_x \) in which \( p \) has residue degree one and does not divide the conductor. Next, let \( K \subset \text{GL}_2(\mathbb{A}) \) and \( X' \subset M_f^{\text{is}}(K)_{\mathbb{C}_\infty} \) be given by Proposition 2.2 (ii). We lift \( \Sigma \) to a Zariski-dense set \( \Sigma' \subset X'(\mathbb{C}_\infty) \). As in the proof of Theorem 1.3, we now see that \( \sigma(x) \in X' \cap T_p(X') \) for all \( x \in \Sigma' \), where \( \sigma \) is again a Frobenius element associated to \( p \). It follows that \( X' \subset T_p(X') \), and since \( T_p(X') \) is irreducible, it follows as above that \( X' \), and hence also \( X \), is special. \( \Box \)

Remark. The main obstacle to proving Conjecture 1.2 for subvarieties of higher dimension is the lack of control over the ideal \( m_X \) and the level structure \( K \) given by Proposition 2.2.

References


