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B.Sc. Honours

in

Mathematics

2015

Contents

1 Practical Information

1.1 Stellenbosch University and Department of Mathematical Sciences

Stellenbosch University is situated in a picturesque wine growing region nestled in the mountains, approximately 50km from Cape Town. The University has a proud history dating to 1874, and a strong research tradition.

Mathematics forms one division of the Department of Mathematical Sciences. This is the oldest department in the University and has 49 full-time academic staff with active research interests in a range of mathematical disciplines. (The other divisions are Applied Mathematics and Computer Science.) These research interests in Mathematics are reflected in the optional modules that form part of the B.Sc. Honours degree curriculum.

On successful completion of the Honours degree further study towards Masters and Ph.D. degrees in Mathematics is possible.

1.2 Degree structure

The B.Sc. Honours degree in Mathematics is a one year degree, studying full time on the Stellenbosch campus of the University.

Students must complete at least 9 modules totaling 128 credits towards the degree. (Details of the modules are given in Sections 3 and 4 below.) One of the modules takes the form of a research project of the student's choice. Adequate performance in all modules and a weighted average of at least 50% is required to obtain the degree.

In the first semester the programme consists of four modules (each 16 credits), each with a lecture load of three hours per week. In the second semester the programme consists of four modules (each 8 credits), each with a lecture load of two hours per week, and the research project (32 credits).

The programme for each student will be arranged to accommodate the student's background and interests. Subject to the Department's approval, a maximum of half of the degree credits may be taken in other divisions of the Department or in other University departments. *The guiding principle is the formation of a coherent, well-focused curriculum.*

Due to departmental expertise and the career and research opportunities they provide, the following possible focuses are suggested:

- *Mathematics; and*
- *Biomathematics.*

This list serves as a guideline, and suggested curricula for these focus areas are given below. The focus you choose is not reflected on your final degree certificate; it merely serves to give direction to your programme.

1.3 Requirements for admission

A B.Sc. degree with Mathematics as major subject or an equivalent qualification is needed to gain entrance to the Honours programme. A mark of at least 60% for Mathematics 3 is required.

Stellenbosch University is a multilingual university. At the undergraduate level classes are taught primarily in Afrikaans. At graduate level the language of instruction (Afrikaans and/or English) is in general determined by the orientation of the students. Proficiency in Afrikaans is not a prerequisite for admission to the honours degree, but academic competence in English is necessary.

1.4 Facilities

All students have access to the excellent facilities of Stellenbosch University. There are shared computers with e-mail and internet access and students have access to the well equipped University library as well as the departmental library where current mathematical journals are housed.

1.5 Financial Support

All eligible graduate students are encouraged to apply for bursaries through the University as well as the National Research Foundation. Additional income can be earned by being employed on a part-time basis as a tutor for undergraduate mathematics modules. Details about application procedures can be obtained from the head of the Department or the secretary of Mathematics Division.

1.6 Contact Information

The head of the Department of Mathematical Sciences is Prof. I.M. Rewitzky (rewitzky@sun.ac.za), and the head of the Mathematics Division is Prof. F. Breuer (fbreuer@sun.ac.za). The Mathematics Honours Coordinator is

Prof. L. van Wyk (LvW@sun.ac.za), and the secretary of the Mathematics Division is Mrs. O. Marais (omarais@sun.ac.za). The departmental address is:

Department of Mathematical Sciences Mathematics Division University of Stellenbosch Private Bag X1 Matieland 7602 South Africa	Tel: (021) 808-3282 Fax: (021) 808-3828

2 Suggested Focus Areas for the Degree

The Honours programme is flexible, and exact module choices for the second semester will be decided upon in consultation with individual students. The choice of modules should give a coherent focus to the programme, leading to opportunities for further study and employment. Suggested curricula with corresponding focus are outlined below.

2.1 Mathematics

This focus is for students wanting a rigorous mathematics education. It consists principally of modules taught in the Mathematics Division and is usually followed by students who have a love for "pure mathematics", in particular those who intend to follow a career in research and/or teaching. (The number of credits of each module is given in brackets.)

First Semester	Second Semester
Algebra (16) Functional Analysis and Measure Theory (16) Real and Complex Analysis (16) Set Theory and Topology (16)	four 8-credit Choice Modules subject to departmental approval Honours Project (32)

2.2 Biomathematics

The Biomathematics focus aims to train students to formulate and analyse precise models for experimental data arising from real-life research problems within the fields of biology and medicine, from predicting the influence of HIV, Aids, malaria and tuberculosis to the effects of climate change on South Africa. (The number of credits of each module is given in brackets.)

First Semester	Second Semester
Computational And Discrete Methods in Biomathematics (16)	Honours project (32)
Non-linear Dynamical Systems in Biomathematics (16)	Advanced Topics in Biomathematics III (16)
Advanced Topics in Biomathematics I (8)	Advanced Topics in Biomathematics IV (8)
Advanced Topics in Biomathematics II (8)	Choice module (8)
Selected Topics from Biological Sciences (8)	
Selected Topics from Biomedical Sciences (8)	

Students registering for this focus will spend the first part of the year (January–June) at AIMS (African Institute for Mathematical Sciences), where they will attend a number of special modules presented by local and international specialists in modelling of biological and biomedical systems, population dynamics, biomathematics, and bio-informatics. For the second part of the year the students will be based at Stellenbosch University and involved in project work supervised by a researcher in mathematics and a researcher in biological or biomedical sciences.

3 First Semester Modules for Focus: Mathematics

The modules offered in the first semester are the core modules for the honours programme. Each module is worth 16 credits and is taught in three lectures per week over the semester.

3.1 Algebra (711)

The first and second quarters are dedicated to group theory and Galois theory, respectively.

In the group theory course we will introduce basic notions such as conjugates, normalisers and normal subgroups, after which we will treat various examples, such as the circle group, dihedral groups and the quaternions. (The additive group of integers modulo n and other cyclic groups are already known from 3rd year courses). We will also treat the conjugate class equation of a group, p-groups, Cauchy's Theorem and the Sylow Theorems. The Galois theory course builds up on the field theory from the 3rd year algebra course. This theory arose from investigating solutions to polynomial equations, and combines central themes from classical and modern algebra. It is closely linked with the theory of solvable groups, and some of the greatest mathematicians of the last 200 years have contributed to this subject.

Requirements: A 3rd year course in basic algebra (Mathematics 314).

Textbook: A. Clark: *Elements of Abstract Algebra,* Dover Publications, New York, 1984.

Lecturers: Prof. L. van Wyk (Group Theory) and Prof. F. Breuer (Galois Theory).

3.2 Functional Analysis and Measure Theory (712)

Functional Analysis: Metric and Banach spaces, bounded linear operators, functionals and dual spaces. Introduction to Hilbert spaces. The Hahn Banach theorem and its consequences, the Baire category theorem, the uniform boundedness theorem.

Measure Theory: Lebesgue outer measure, measurable set and measure, measurable functions, Littlewood's Principles. Shortcomings of the Riemann integral, the Lebesgue integral and convergence theorems. The L^p spaces.

Requirements: A 3rd year course in metric spaces or real analysis (Mathematics 365).

Textbooks:
Functional Analysis: E. Kreyszig: *Introductory Functional Analysis with Applications,* John Wiley & Sons Inc., New York, 1978.
Measure Theory: H. L. Royden: *Real Analysis*, Macmillan Publishing Co., Inc., New York, 1968.

Lecturer: Prof. S. Mouton.

3.3 Real and Complex Analysis (713)

This course is a continuation of the third year course in complex analysis. In the process, topics from real analysis are also encountered. Covers conformal mappings, harmonic functions, Dirac sequences, the Riemann mapping theorem, analytic continuation, Weierstrass products, Mittag-Leffler theorem, the gamma and zeta functions.

Requirements: A 3rd year course in complex analysis (Mathematics 324).

Lecturer: Dr. G. Boxall and Dr. D. Ralaivaosoana.

3.4 Set Theory and Topology (714)

This course is mainly based on assignments from the following areas of mathematics: axiomatic set theory (Zermelo-Frankel axioms, Zorns lemma and the well ordering principle, cardinal and ordinal arithmetic), general topology (topology via neighborhoods, closure and interior, compactness, separation axioms, continuous functions and homeomorphisms), duality theory (lattices and Boolean algebras, Stone, Birkhoff and Priestly dualities), algebraic topology (homotopy of paths, definition and computation of fundamental group/groupoid of a topological space), and categorical topology (basic topological constructions viewed as limits and colimits in the category of topological spaces, topological functors). Students with a background in some of these areas from their undergraduate studies will be required to complete assignments that complement their background.

Lecturer: Prof. I.M. Rewitzky.

4 Second Semester Choice Modules for Focus: Mathematics

Modules in the second semester take the form of *capita selecta.* Students choose four of the available 8-credit modules. These modules are taught in two lectures per week during the semester.

The list of modules available is given below. Not all modules are taught every year. The exact modules taught in a specific year depend on the availability of the lecturers and on the students' choices, and once choices have been made, course codes are confirmed.

4.1 Functional Analysis II (751)

In this course further functional analytic topics, including spectral theory, will be covered. Spectral theory is one of the main branches of modern functional analysis and its applications. Roughly speaking, it is concerned with certain inverse operators, which arise quite naturally in connection with the problem of solving equations (eg. differential and integral equations). Spectral theory can also be considered a generalization of matrix eigenvalue theory.

Contents: Adjoint operators, reflexivity, uniform, strong and weak convergence of sequences of operators, the open mapping theorem, the closed graph theorem. Important classes of operators: bounded linear operators on Banach spaces, finite rank and compact operators, and the spectral theory of these operators. Theory of Banach algebras, spectral theory in Banach algebras.

Prerequisite honours modules: Functional Analysis and Measure Theory, Set Theory and Topology, Real and Complex Analysis.

Textbooks:
E. Kreyszig: *Introductory Functional Analysis with Applications,* John Wiley & Sons Inc., New York, 1978.
B. Aupetit: *A Primer on Spectral Theory*, Springer-Verlag, New York, 1991.

Lecturer: Prof. S. Mouton.

4.2 Category Theory (753)

Category theory provides a conceptual organization to mathematics, by developing and applying techniques for abstraction, unification, and deeper understanding of parallel phenomena across different areas of mathematics. This course will focus on basic concepts of category theory and their use in algebra, topology, lattice theory and logic.

Text: Notes.

Lecturer: Dr. J. Gray.

4.3 Logic (754)

This module aims to give a modern well-rounded introduction to first-order logic and its decidable fragments. A central theme of the module is that of 'logic engineering' which involves considering various interpretations (useful in mathematics, applied mathematics and computer science) of the basic logical formulae, and deciding what axioms and rules should be engineered.

Propositional logic and first-order logic provide the starting point and backbone for the module. Two perspectives, syntactic and semantic, are studied and shown, via soundness and completeness theorems, to be 'similar' in the sense that the set of provable formulae is the same as the set of formulae that are always true.

While first-order logic has enough expressivity to define, among other things, all of mathematics, and any digital computer, it lacks some desirable computational properties such as decidability. However, various decidable fragments of first-order logic have been engineered with a wide variety of applications. Such logics based on modal logic are studied in the second part of this module. Notions of modal expressivity and key ideas on completeness are covered, as well as some popular extensions of the basic modal logic (including temporal logic, hybrid logic, multi-dimensional modal logic).

Material for the module includes printed lecture notes, based on several research papers, and the following books:

- Blackburn, P., M. de Rijke and Y. Venema. [2001]. Modal Logic. Cambridge Tracts in Theoretical Computer Science 53. Cambridge University Press.
- Hedman, S. [2004]. A First Course in Logic: An Introduction to Model Theory, Proof Theory, Computability and Complexity. Oxford Texts in Logic 1. Oxford: Oxford University Press.
- Mendelson, E. 1987. Introduction to Mathematical Logic. Van Nostrand. Second edition.
- Priest, G. [2001]. An Introduction to Non-Classical Logic. Cambridge University Press.

Lecturer: Prof. J.W. Sanders.

4.4 An Introduction to Valuation Theory

When studying the arithmetic of the integers of in the field of rational numbers, or in number fields in general, the primes play a key role. Similarly when studying algebraic varieties, for example curves and surfaces, local properties are determined by the points, or sub-varieties of lower dimension, such as points and curves on surfaces. In each of these situations this is done by studying a class of rings with special properties called local rings. In the typical and best possible situation (of smoothness) these rings are valuation rings. These rings, introduced by Krull in the form we will be studying, can be studied with purely algebraic tools, but with astonishing and interesting links to number theory, analysis and geometry. The aim of this course is to give a short introduction to this theory, emphasizing examples in different situations and discussing selected key results in the field.

Familiarity with the typical third year algebra course and a willingness to use or become aquainted with elementary facts from Galois Theory is prerequisite to this course.

Lecturer: Prof. B.W. Green.

4.5 Categorical Algebra

This course is mainly intended for those students who wish to continue their studies in any field of pure mathematics where algebra plays an important role. Students will be exposed to a category theoretic insight to some of the selected results and constructions in abstract algebra, which will deepen and broaden their understanding of algebra, and its connections with other fields of pure mathematics. It is advisable to take this module in conjunction with Category Theory and Universal Algebra modules.

Much of this module is based on the very recent developments in the field, however, it also includes some of the classical material, where the best reference is: S. Mac Lane, *Categories for the Working Mathematician* (2nd edition), Graduate Texts in Mathematics 5, 1998, Springer.

Lecturer: Prof. Z. Janelidze.

4.6 Algebraic Number Theory

I hope the following brief remarks will whet your appetite for some algebraic number theory, where algebra is applied to solving problems in number theory.

An algebraic number field is an extension field K of the rational numbers $\mathbb{Q}$ of finite degree. In the simplest case, $K = \mathbb{Q}$ is the field of fractions of the ring of integers $\mathbb{Z}$, which is a unique factorisation domain. Similarly, in an algebraic number field K there is a ring of integers $\mathcal{O}_K$ which plays as important a role in the arithmetic of K as $\mathbb{Z}$ does in $\mathbb{Q}$.

For example, if $K = \mathbb{Q}(i)$ then $\mathcal{O}_K = \mathbb{Z}[i]$. Fermat proved that the odd primes which are a sum of two squares are those $\equiv 1 \pmod 4$. To prove this, it is natural to factorise $a^2 + b^2 = (a + bi)(a - bi)$ and to work in $\mathbb{Z}[i]$.

An early attempt at proving Fermat's Last Theorem, that for an integer $n \geq 3$ there are no integer solutions of $a^n + b^n = c^n$, $abc \neq 0$, proceeds as follows. Fix a primitive n-th root of unity, say $\zeta = e^{2\pi i/n}$, and factorise

$$a^n + b^n = \prod_{i=0}^{n-1} (a + \zeta^i b).$$

Then assume that the ring $\mathbb{Z}[\zeta]$ has unique factorisation to get a contradiction. Here $\mathbb{Z}[\zeta]$ is the ring of integers of the cyclotomic field $\mathbb{Q}(\zeta)$. But it was soon realised that, while $\mathbb{Z}$ and $\mathbb{Z}[i]$ are unique factorisation domains, in general $\mathbb{Z}[\zeta]$ is not!

In the mid nineteenth century, Kummer saw that, although $\mathbb{Z}[\zeta]$ is not always a unique factorisation domain, it has certain good properties which allow substantial progress on Fermat's Last Theorem. Dedekind generalised Kummer's results to show that although the ring of integers $\mathcal{O}_K$ of a number field K may not always have unique factorisation of **elements**, it does have unique factorisation of every **ideal** of $\mathcal{O}_K$ as a product of **prime ideals**. This is the first important result in algebraic number theory.

Another classical theorem describes, for an extension K/F of number fields, how prime ideals of $\mathcal{O}_F$ factorise as a product of prime ideals of $\mathcal{O}_K$. For example, let $F = \mathbb{Q}$ and suppose that $\mathcal{O}_K = \mathbb{Z}[\alpha]$, where α has minimal polynomial $f(X)$ over $\mathbb{Z}$. Let p be a prime number, and write $\bar{f}(X) \in (\mathbb{Z}/p\mathbb{Z})[X]$ for the polynomial obtained from $f(X)$ by reducing all its coefficients modulo p. Then the factorisation of $p\mathbb{Z}$ as a product of prime ideals in $\mathcal{O}_K$ has the same shape as the factorisation of $\bar{f}(X)$ as a product of irreducibles in $(\mathbb{Z}/p\mathbb{Z})[X]$. Applying this result for $K = \mathbb{Q}(i)$ gives another solution of the problem of which prime numbers are sums of squares.

When an extension K/F of number fields is Galois, there is a beautiful and very useful theory of how, with finitely many exceptions, the primes of $\mathcal{O}_K$ induce so-called **Frobenius** elements of the Galois group.

Lecturer: Dr. A. Keet.

4.7 A Course in Universal Algebra

This module aims to give a basic introduction to the ideas of universal algebra, and may be of interest to algebraists and theoretical computer scientists. The basic text, *A Course in Universal Algebra*, S. Burris and H.P Sankapannavar, Springer 1981, is now available free online at http://www.math.uwaterloo.ca/~snburris/htdocs/ualg.html (or email me: peter_ouwehand@sun.ac.za).

Syllabus

- Introductory lattice theory.
- Algebras, subalgebras and associated algebraic lattices.
- Congruences and homomorphic images.
- Products of algebras and direct decompositions.
- Subdirect products and subdirectly irreducible algebras.
- Varieties of algebras.

- Free algebras.
- Equational classes of algebras and the Birkhoff theorem.
- Mal'cev conditions.
- Boolean algebras, Boolean rings, Boolean spaces and Stone duality.
- Boolean powers
- Basic model theory and first-order logic.
- Ultraproducts and reduced products.
- Congruence distributive varieties.

Requirements: An algebra course at the third-year level.

Lecturer: Dr. P. Ouwehand.

4.8 Analytic Number Theory

In this module, we will discuss how real and complex analysis, in particular the study of the analytic properties of the Riemann Zeta function and similar functions, can be used to solve number-theoretic problems. Typical questions in this area ask for the "average" or "typical" order of number-theoretic functions (e.g. the number of divisors). The highlight of the course will certainly be the celebrated prime number theorem and its generalisation (Dirichlet's theorem on primes in arithmetic progressions).

Textbook: The course follows the book *Introduction to Analytic Number Theory* by T. Apostol (but it is not necessary to buy the book).

Prerequisites: Basic knowledge of real and complex analysis.

Lecturer: Dr. D. Ralaivaosoana.

4.9 Commutative Algebra

The purpose of this course is to attain competence in the introductory aspects of Commutative Algebra.

Content: Commutative rings and subrings; ideals; prime and maximal ideals; primary decomposition; quotient rings; modules; chain conditions on modules and Noetherian rings.

Prerequisites: Basic knowledge of Abstract Algebra, as was done in the first semester of the honours course.

Textbook: R.Y. Sharp, *Steps in Commutative Algebra*, London Mathematical Society Student Texts 19, Cambridge University Press.

Lecturer: Dr. C. Naude.

4.10 Independence Proofs in Set Theory

This is a short course in set theory, with focus on the independence of the *Axiom of Choice* and the *Generalized Continuum Hypothesis* (with respect to the Zermelo-Fraenkel axioms). We will cover a selection of chapters of the books *Set Theory*, by Thomas Jech, and *Set Theory — An Introduction to Independence Proofs*, by Kenneth Kunen. The prerequisites are a basic knowledge of elementary logic and set theory, including ordinals and transfinite induction, as well as cardinals and cardinal arithmetic.

Syllabus

I. **Zermelo–Fraenkel Set Theory**: A quick review of the axioms, including a discussion of the Axiom of Foundation

II. **Models of Set Theory**: We rapidly introduce the first–order logic and model theory required for independence proofs: Elementary submodels and skolemization; Löwenheim–Skolem theorems; Compactness Theorem; Ultraproducts. After this we consider transitive models of set theory, and prepare the way for the definition of the constructible universe L by formalizing the notion of definability.

III. **The Constructible Universe**: Here we introduce Gödel's universe L of constructible sets, show that $L \models AC + GCH$, and then prove that AC +GCH is consistent with ZF.

IV. **Forcing**: We introduce the notion of forcing and generic extensions via Boolean-valued models and prove the independence of AC and GCH.

Lecturer: Dr. P. Ouwehand.

4.11 Elliptic Curves

Elliptic curves are structurally extremely rich objects which are found at the intersection of Number Theory, Algebraic Geometry and Complex Analysis. They are algebraic curves of genus one, and the points on an elliptic curve form an abelian group. The subgroups of points defined over various fields have very interesting properties. The complex points form a torus, and are related to elliptic functions (hence the name). Over a number field, the group of points is finitely generated (by the Mordell-Weil Theorem), and the rank of this group is very mysterious and the subject of the Birch and Swinnerton-Dyer Conjecture, one of the six remaining Millennium Problems. Over a finite field, the group is finite and of immense practical importance for cryptography.

This will be a reading course based on the book "The Arithmetic of Elliptic Curves by J.H. Silverman. We will start with a background on Algebraic Geometry and specifically the geometry of projective algebraic curves, before focusing on the geometry of elliptic curves, their group structure and morphisms between elliptic curves. After that we will treat a selection of chapters, depending on the preferences of the students: Elliptic curves over the complex numbers, over finite fields, over local fields or over number fields.

Lecturer: Prof. F. Breuer.

5 Honours Project (746)

In the second semester all honours students have to complete a research project on a topic of their choice. This will be evaluated through a written report and an oral presentation at the end of the year. The project is worth 32 credits.

The projects below have been proposed. You are welcome to discuss them with the lecturers concerned. (These projects are for Mathematics focus Honours students; not in the Biomathematics focus. For the latter the students should consult the Biomathematics Honours convenor.)

5.1 Circulant Matrices

A circulant matrix is an $n \times n$ matrix with the property that each row is a cyclic shift of the row above it. For example, the 3×3 matrix with entries $[[1, 2, 3], [2, 3, 1], [3, 1, 2]]$ is a circulant matrix, as is the identity matrix. These matrices, with their deceptively simple definition, exhibit a variety of very interesting properties.

For example, all $n \times n$ circulant matrices with complex entries have the same set of orthonormal eigenvectors, whose entries are n-th roots of unity. Their characteristic polynomials and eigenvalues, however, are more subtle and interesting.

The set of all $n \times n$ complex circulant matrices can be represented in a number of interesting ways. Most trivially, this set forms a complex vector space of dimension n. This set can also be given the structure of a commutative C-algebra, which is isomorphic to $C[X]/ < x^n - 1 >$. Thirdly, since all $n \times n$ complex circulant matrices are diagonalizable by the same matrix of eigenvectors, this algebra is again isomorphic to the algebra of $n \times n$ diagonal matrices.

Circulant matrices have links to a variety of other topics in Algebra, Number Theory, Algebraic Geometry and Analysis. They occur wherever roots of unity play a role, they are related to solutions of polynomial equations and Galois Theory, they are related to the mythical "field of one element" and are even linked to Toeplitz operators.

A very nice introduction to this topic can be found in the March 2012 edition of the Notices of the American Mathematical Society, which also contains many further references. Other aspects that a student taking on this project can pursue is circulant matrices over finite fields and links with Ducci sequences, Moore determinants and Carlitz modules. This can also lead directly to an M.Sc. project.

Prerequisite: 80% or more for Honours Algebra course

Project Supervisor: Prof. F. Breuer.

5.2 Mal'tsev Categories

The notion of a Mal'tsev category occupies one of the important places in modern categorical abstract algebra. This notion has a long history, and its roots lie in universal algebra. Mal'tsev categories provide a convenient context for unifying many properties of groups, rings, topological groups and other group-like structures. The aim of the project will be to learn some of the recently obtained results on Mal'tsev categories, with a possibility to arriving to some new results. This should be an excellent project for a student who wishes to continue her/his Master studies in category theory, universal algebra or any other related field.

Recommended modules: Categorical Abstract Algebra, Category Theory, Universal Algebra, Set Theory.

Project Supervisor: Prof. Z. Janelidze.

5.3 Universal Arrows

Universal arrows play an illuminating role in the understanding of many fundamental concepts and constructions of modern mathematics. Cartesian products of sets, of groups and all other mathematical structures, free groups, monoids and other free structures, quotient structures, etc. — all can be defined via universal arrows. This leads to the notions of a limit in a category, and adjoint functors between categories, and to study these two deep concepts together with their examples will be the main theme of the project. This project is particularly suitable for those students who seek to obtain a general picture of how concepts and constructions in different branches of abstract mathematics are interrelated with each other.

Recommended modules: Category Theory, Set Theory, Algebra, Topology, Logic, Universal Algebra, Categorical Abstract Algebra.

Project Supervisor: Prof. Z. Janelidze.

5.4 Tropical Algebra

During July 2004, Bernd Sturmfeld gave a lecture in Park City, Utah, with a title: "The tropical approach in Mathematics." This approach was in its infancy at this stage, but has since matured and is now an integral part of geometric combinatorics and algebraic geometry. It also has expanded into mathematical physics, number theory, computational biology and beyond.

The basic object of the study is the tropical semi-ring $(\mathbb{R} \cup \infty, \oplus, \Box)$. As a set, this is just the real numbers together with an extra element ∞. The basic operations of addition and multiplication of real numbers are redefined as follows:

$$x \oplus y = \min(x; y); \qquad x \Box y = x + y$$

In this project the student will give an elementary introduction to this subject, touching upon arithmetic, polynomials, curves, phylogenetics and linear spaces.

Project Supervisor: Dr. C. Naude.

5.5 Linking structures using duality theory

An important undertaking in many branches of mathematics is the translation of concepts, theorems or mathematical structures in one context into other concepts, theorems or structures in a different context, in a one-one fashion. Duality theory is a mathematical tool employed for this purpose.

In this project you will be acquainted with the basic duality theory principles and study dualities for compact Hausdorff spaces including Stone duality and Gelfand duality.

Recommended modules: Set Theory and Topology, Universal Algebra.

Project Supervisor: Prof. I.M. Rewitzky.

5.6 Van der Waerden's Theorem

Suppose that all integers are coloured with n different colours. Then van der Waerden's theorem holds:

For any positive integer k, there is an arithmetic progression of length k all of whose members have the same colour.

The aim of this project is to discuss the proof of van der Waerden's surprising result, and to treat related problems and generalisations.

Project Supervisor: Prof. S. Wagner.

5.7 Partition identities and congruences

A partition of an integer n is a way to represent it as an sum of positive integers.

For example, we can write 5 as

$$5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1.$$

Partitions have many interesting properties: for instance, there are exactly as many partitions of n whose summands are all odd as there are partitions whose summands are all distinct. A famous result due to Ramanujan states that the number of partitions of $5m + 4$ is always divisible by 5. These are just two examples of a number of properties (there are enough for five honours projects), and there are also interesting relations between partitions and the theory of elliptic functions and modular forms.

Project Supervisor: Prof. S. Wagner.

5.8 Integer compositions

A composition of an integer n is a way of writing it as an ordered sum of integers, e.g.:

$$4 = 3 + 1 = 1 + 3 = 2 + 2 = 2 + 1 + 1 = 1 + 2 + 1 = 1 + 1 + 2 = 1 + 1 + 1 + 1.$$

It is a classical combinatorial result that there are precisely 2^{n-1} compositions of n. The purpose of this project would be to survey some statistical properties of compositions, such as: average length, distribution of terms (how many 1s, how many 2s, etc.), maximum, to name a few.

Project Supervisor: Prof. S. Wagner.

5.9 The "dimer model" in statistical physics

Consider an $m \times n$-grid (n even). In how many ways can one group the mn points (think of them as atoms) into $mn/2$ pairs such that the points that form a pair are either horizontally or vertically connected (bonds between atoms)? In graph-theoretical terms: how many perfect matchings of the $m \times n$-grid are there? Yet another equivalent formulation is: in how many ways can a $m \times n$-rectangle be covered with non-overlapping 1×2-pieces? The suprising answer, found by the physicist Kasteleyn, reads as follows:

$$\prod_{k=1}^{\lfloor m/2 \rfloor} \prod_{l=1}^{n/2} \left(4\cos^2 \frac{\pi k}{m+1} + 4\cos^2 \frac{\pi l}{n+1} \right).$$

The proof makes use of a number of clever arguments that combine combinatorics, graph theory and linear algebra (e.g., the Pfaffian of a matrix).

Project Supervisor: Prof. S. Wagner.

5.10 Jacobi's Triple Product

Jacobi's celebrated triple product expresses a series as a product of 3 terms. It is relevant in various branches of mathematics. There exist many proofs for it. Some of them have a combinatorial (bijective) flavour. The task is to collect these proofs and represent them in an attractive way.

Project Supervisor: Prof. H. Prodinger.

5.11 Approximate Counting

This is a procedure to count a population in an approximate fashion. The pioneering paper about it was written by Philippe Flajolet. The analysis leads to many fascinating things, involving generating functions, q-analysis, asymptotic techniques. The task is to digest the literature and present it in a coherent way.

Project Supervisor: Prof. H. Prodinger.

5.12 The Sum-of-Digits Function

If one writes an integer n using the base b representation with digits $0, 1, \ldots, b-1$, one forms the sum of its digits. This is popular in base 10 since it leads to some divisibility tests. The question is "what is the average of the sum-of-digits function when one considers the first n non-negative integers." Delange provided a very simple and sweet analysis. A more powerful and more sophisticated approach was developed later: the Mellin-Perron formula. The task is to understand these methods and present them in an attractive way. The student should have done a basic course in complex analysis.

Project Supervisor: Prof. H. Prodinger.

5.13 Horton-Strahler Numbers

Horton-Strahler numbers measure the complexity of a river network. They were independently discovered in a totally different area, as the register function in computer science. The task is to read the relevant literature and present it in an attractive way.

Project Supervisor: Prof. H. Prodinger.

5.14 The Gelfand theory for commutative Banach algebras

If a Banach algebra A is commutative, then there exist many multiplicative linear functionals on A. These functionals play a very important role in the representation of A, as was shown by I. M. Gelfand at around 1940. Among other things, it can be shown that if $x \in A$, then the spectrum of x consists of all complex numbers of the form $\chi(x)$, where χ is a multiplicative linear functional on A. This fact, and the related theory, has interesting applications regarding topics such as automatic continuity of homomorphisms, the existence and equivalence of Banach algebra norms, and Fourier series. In this project we shall investigate this theory, as well as some applications. Besides functional analysis, we will use topology, complex analysis and a little algebra in our study.

To be able to manage this project, it is essential that the student had taken/will take the following honours modules:

Functional Analysis and Measure Theory (first semester)
Real and Complex Analysis (first semester)
Set theory and Topology (first semester)
Functional Analysis II (second semester)

Project Supervisor: Prof. S. Mouton.

5.15 Categorical Mathematics

Depending on the interest of the student, this project will study a topic in classical mathematics from the category-theoretic perspective.

Recommended modules: Category Theory.

Project Supervisor: Dr. J. Gray.