The Integer Partition Function

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Abstract

The function $p(n)$ counts the number of integer partitions of a positive integer $n$. The purpose of this text is to deduce a remarkable formula for $p(n)$: the convergent infinite series obtained by Hans Rademacher. Initially the reader is familiarised with the basic concepts of generating functions, Möbius transformations, Farey fractions and Ford circles, before these are called upon to aid in the explanation of the proof of Rademacher’s formula. Thereafter a few auxillary results are provided to conclude the text, among them the asymptotic formula first presented by Hardy and Ramanujan.
1 Introduction

First things first: the partition function \( p(n) \) counts the ways in which \( n \) can be represented as sums of positive integers; such a representation is known as a partition. These partitions are unordered, and \( p(0) \) is defined as 1. As an example, the partitions of 5 are
\[
\{(5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1, 1)\},
\]
so that \( p(5) = 7 \). The partition function is strictly positive, as \( p(n) \geq n \). It is also an increasing function, as every partition of \( n \) can be made into a partition of \( n + 1 \) by adding 1 to the set, and \( n + 1 \) has at least one more partition — namely \( (n + 1) \).

The theory of partitions owes much to Leonhard Euler — he penned numerous important theorems on the topic, which would garner attention from many of the most prolific mathematicians of the last three centuries (Andrews 1984, p. xv). The most illustrious result in this area, however, is due to Hardy & Ramanujan (1918). The pair obtained an asymptotic formula for \( p(n) \); a finite sum with error \( O(n^{-\frac{1}{4}}) \), and in doing so, they also created and used what would later come to be known as the Hardy-Littlewood circle method. Two decades later, H. Rademacher refined the approach of Hardy and Ramanujan (1974, pp. 100-103) and was rewarded with an exact formula for \( p(n) \), which shall be the focus of this text.

The circle method has become an important tool in additive number theory, but its application to the partition function remains its most endearing achievement. The intuitive idea of the method can be seen in the case where a function is defined inside the unit disk but has singularities along the unit circle. The contour integral of this function is sought, and a circle with radius slightly less than 1 is used as the curve of integration. This circle is dissected into arcs related to each singularity; two sets of integrals are formed: the ‘major’ and ‘minor’ arcs, whose singularities are more and less ‘important’ respectively. An integrable function which approximates the integrand is then required. Replacing the integrand of the major arcs with this function and integrating, and showing that the contribution of the minor arcs is of an order less than the major arcs, will hopefully yield an expression with a small enough error term to render it an asymptotic equivalent of the original integral. An introduction to the circle method is given in Chapter 13 of Miller & Takloo-Bighash (2006), and a popular reference for its applications is Vaughan (1997).

The motivation behind the circle method will hopefully become clear during the investigation of its application to \( p(n) \), as the handiwork of three mathematicians make this application rather striking. The proof of the exact formula for \( p(n) \) follows the same course as that outlined in Chapter 5 of Apostol (1990), and as far as possible the same or similar function labels have been used.

1.1 Proof outline

The coefficients of the function
\[
F(x) = \prod_{k=1}^{\infty} \frac{1}{1 - x^k} = \sum_{k=0}^{\infty} p(k)x^k
\]
contain the sequence \( (p(n)) \). We seek a formula for \( p(n) \). With a little bit of manipulation,
\[
\frac{F(x)}{x^{n+1}} = \sum_{k=-n-1}^{\infty} p(k + n + 1)x^k,
\]
so that the coefficient of $x^{-1}$ is $p(n)$, and Cauchy’s residue theorem gives us

$$p(n) = \frac{1}{2\pi i} \int_C \frac{F(x)}{x^{n+1}} \, dx.$$ 

We are thus interested in the behaviour of $\frac{F(x)}{x^{n+1}}$ within the unit disk, as this is where it is defined, and we must integrate it. The cornerstone of the proof is the fact that when $F(x)$ is calculated near a complex root of unity, $F(x)$ can be approximated by a function which can be integrated. Thus we choose a path $C$ near the roots of unity on the unit circle. We split our path up so that we consider it as the sum of integrals related to each root of unity, and hope that the difference between our approximation of $F$ near each root and the actual value of $F(x)$ is very small. It turns out that Rademacher’s method for integrating near the roots of unity is so effective that this difference vanishes as we consider the infinite sum of these integrals.
2 Generating Functions

To begin, we must obtain an infinite series which contains information about the function $p$ within its terms. For this cause, generating functions (Chapters 5, 6 of Wagner (2009)) provide an elegant and intuitive process. To a sequence of complex numbers $a_0, a_1, a_2, \ldots$, we associate the power series

$$A(x) = \sum_{n=0}^{\infty} a_n x^n,$$

where the coefficient of $x^n$ is $a_n$. Throughout this text, unless otherwise stated, $x$ and $z$ will be regarded as complex. If we consider the geometric progression $1, r, r^2, r^3, \ldots$, we obtain the generating function

$$R(x) = \sum_{r=0}^{\infty} r^n x^n = \frac{1}{1 - rx},$$

which gives the important representation of $1, 1, 1, \ldots$:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}.$$

In combinatorics, one is often concerned with structures of different sizes, and specifically, counting the number of structures of a given size. Approaching generating functions from this perspective, we can view $x^n$ as a structure of size $n$ and the coefficient $a_n$ as the number of such structures present.

For an example, consider the family of positive integers $I = \{1, 2, 3, \ldots\}$. If we say integer $k$ has size $k$, then there is exactly one integer of each size, providing us with the generating function

$$I(x) = \sum_{n=1}^{\infty} x^n = x + x^2 + x^3 + \cdots = x(1 + x + x^2 + \cdots) = x \frac{1}{1 - x}.$$

This view of generating functions lets us handle families of structures simply. If $A$ and $B$ are two disjoint families, then it is not difficult to see that the generating function of $A \cup B$ will be $A(x) + B(x)$.

Pairs of structures $(a, b)$, with $a \in A$ and $b \in B$, can also be considered. A pair $(a, b)$ has size equal to the sum of the sizes of $a$ and $b$. Thus the number of pairs of size $n$ is

$$\sum_{k=0}^{n} a_k b_{n-k},$$

and the generating function for pairs of structures from $A$ and $B$ is

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} x^n = A(x) B(x).$$
A similar result holds for other tuples (triples, quadruples, etc).

Sequences of structures from a family $A$ build upon this principle. A sequence can be of any length, and differentiating them as such provides a simple way of understanding their generating function. A sequence of length 1 would be any $a \in A$. Sequences of length 2 are all the pairs $(a_1, a_2)$; of length 3 are all the triples; and so forth. This implies that

$$\text{Seq}(A) = \{ \epsilon \} \cup A \cup (A \times A) \cup (A \times A \times A) \cup \cdots$$

is the set of all sequences, and thus that the generating function $S(x)$ for sequences of structures on $A$ is

$$S(x) = \sum_{n=0}^{\infty} A(x)^n = \frac{1}{1-A(x)}.$$

### 2.1 Integer compositions

Rather remarkably, these tools are already sufficient to solve a problem only slightly different from the focus of this text. Namely, how many partitions of an integer $n$ are there when these partitions are ordered? Ordered partitions are known as compositions, and we shall denote the number of compositions of a positive integer as $c(n)$. Then

- $c(1) = 1; \quad \{(1)\}$
- $c(2) = 2; \quad \{(2), (1, 1)\}$
- $c(3) = 4; \quad \{(3), (2, 1), (1, 2), (1, 1, 1)\}$
- $c(4) = 8; \quad \{(4), (3, 1), (1, 3), (2, 2), (2, 1, 1), (1, 1, 2), (1, 1, 1), (1, 1, 1, 1)\}$

Compositions can easily be deconstructed in terms of generating functions, as a composition of $n$ is a sequence of any length with size $n$. Thus the set of compositions of $n$ is the set of sequences of size $n$. Recall that the generating function for the positive integers $I$ is

$$I(x) = \frac{x}{1-x},$$

so that the generating function for sequences of positive integers $\text{Seq}(I) = C$ is

$$C(x) = \frac{1}{1-\frac{x}{1-x}} = \frac{1-x}{1-2x},$$

and with a small amount of manipulation we find that

$$C(x) = \frac{\frac{1}{2}}{1-2x} + \frac{\frac{1}{2} - x}{1-2x}$$

$$= \frac{1}{2} \left( \frac{1}{1-2x} + 1 \right)$$

$$= \frac{1}{2} \left( 1 + \sum_{n=0}^{\infty} 2^n x^n \right)$$

$$= \frac{1}{2} + \sum_{n=0}^{\infty} 2^{n-1} x^n.$$

The $\frac{1}{2}$ only affects the constant term in the sum, so that $c(0) = \frac{1}{2} + 2^{-1} = 1$, giving us 1 composition of 0. For any $n \geq 1$, $c(n) = 2^{n-1}$. 
2.2 Integer partitions

We can expand on the idea of sequences of structures with powersets and multisets. The powerset of a family $A$ is the set of all subsets of $A$, and is represented by

$$\text{PSet}(A) = \bigotimes_{a \in A} \{a, \varepsilon\},$$

where the set $\{a, \varepsilon\}$ shows that structure $a$ can either be present or absent in the subset. If $a$ is of size $n$, then $\{a, \varepsilon\}$ has generating function $(1 + z^n)$, and if there are $a_n$ such structures, we find the generating function of the powerset to be:

$$B_1(x) = \prod_{n=1}^{\infty} (1 + z^n)^{a_n}.$$

A multiset is similar to a powerset except that each structure can be included multiple times — essentially, a sequence (which may be of length 0) of each structure occurs. Hence,

$$\text{MSet}(A) = \bigotimes_{a \in A} \text{Seq}(\{a\}),$$

and since the generating function of Seq($\{a\}$) is $(1 - z^n)^{-1}$,

$$B_2(x) = \prod_{n=1}^{\infty} (1 - z^n)^{-a_n}.$$

Now that we possess a foundation of descriptive generating functions, it should be clear that a partition of an integer $n$ is technically just a multiset of size $n$ over the family of integers $I$. Recalling that $a_n = 1$ as there is only one integer of size $n$, this yields a generating function for $p(n)$ quite easily, namely

$$P(x) = \prod_{n=1}^{\infty} (1 - x^n)^{-1} = \sum_{n=0}^{\infty} p(n)x^n.$$

The problem of finding an exact formula for $p(n)$ is hardly as simple as it was for $c(n)$, but is far more interesting to investigate. In time we shall come upon this formula, but before we do we must establish an elementary knowledge of a few related topics.
3 The Modular Group

The theory of partitions is linked to that of elliptic modular functions, and the approach of Hardy and Ramanujan in obtaining their asymptotic formula for $p(n)$ makes use of the Dedekind eta function $\eta(\tau)$, an important modular function in number theory. As such, a very simple overview of the necessary properties of Möbius transformations and the modular group, as presented in Chapter 2 of Apostol (1990), will be helpful.

Firstly, considering a transformation of the form

$$f(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d$ are complex numbers, we can see that $f$ is defined on the extended complex number system for all values of $z$ except $z = -\frac{d}{c}$ and $z = \infty$. Then $f$ can be extended, according to the intuition that $\frac{-d}{c} = \infty$ for non-zero $z$, by letting

$$f\left(\frac{-d}{c}\right) = \infty \quad \text{and} \quad f(\infty) = \frac{a}{c}.$$

If one then considers another $w$ in the extended complex system, then

$$f(w) - f(z) = \frac{aw + b}{cw + d} - \frac{az + b}{cz + d} = \frac{acwz + bcz + adw + bd}{(cw + d)(cz + d)} - \frac{acwz + bcw + adz + bd}{(cw + d)(cz + d)} = \frac{(ad - bc)(w - z)}{(cw + d)(cz + d)}$$

so that if $ad - bc = 0$, then $f(w) - f(z) = 0$ and $f$ is constant. We thus consider only transformations which satisfy $ad - bc \neq 0$, and call these functions on the extended complex system Möbius transformations.

Furthermore, if $ad - bc = k \neq 0$, we can instead consider the analogous transformation

$$f(z) = \frac{\frac{a}{\sqrt{k}}z + \frac{b}{\sqrt{k}}}{\frac{c}{\sqrt{k}}z + \frac{d}{\sqrt{k}}} = \frac{az + b}{cz + d}$$

which satisfies $ad - bc = 1$. We can thus replace the requirement that $ad - bc \neq 0$ by that of $ad - bc = 1$.

If we restrict our coefficients $a, b, c, d$ to the integers, and denote $f(\tau)$ as $\tau'$, then the set of Möbius transformations of the form

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

is called the modular group and is depicted by $\Gamma$. Each transformation can also be written as a $2 \times 2$ integer matrix with determinant 1:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with $A$ and $-A$ both representing the same transformation. Then, if two transformations $\tau'$ and $\mu'$ have relevant matrices $A$ and $B$, the matrix product $BA$ represents the composition transformation $\mu' \circ \tau'$. Hence, because the set of $2 \times 2$ matrices with integer coefficients and determinant 1 form a group with respect to multiplication, the modular group is, in fact, a group.
4 Farey Fractions

It was Rademacher who noted that one can make use of the theory of Farey fractions to obtain a more accurate estimate of $p(n)$, and as such we shall require a few simple properties of these fractions.

A reduced fraction is one in which the numerator and denominator have no common factors. The set of Farey fractions of order $n$ is the set of reduced fractions lying in $[0, 1]$ whose denominator is at most $n$, and shall be denoted by $F_n$. So we have

\[ F_1 = \left\{ \frac{0}{1}, \frac{1}{1} \right\} \]
\[ F_2 = \left\{ \frac{0}{1}, \frac{1}{2}, \frac{1}{1} \right\} \]
\[ F_3 = \left\{ \frac{0}{1}, \frac{1}{3}, \frac{2}{3}, \frac{1}{1} \right\} \]
\[ F_4 = \left\{ \frac{0}{1}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{1}{1} \right\} \]

and so forth. We can concisely define $F_n$ as

\[ \left\{ \frac{h}{k} : 0 \leq h \leq k \leq n; (h, k) = 1 \right\} \]

It can also be easily seen in the examples above that $F_n \subset F_{n+1}$.

If we consider two fractions in $F_n$, say $\frac{a}{b} < \frac{c}{d}$, then an important new fraction called the mediant is defined as

\[ \frac{h}{k} = \frac{a + c}{b + d} \]

The mediant lies between these two fractions, as $\frac{a}{b} < \frac{c}{d}$ implies that $bc - ad > 0$, so

\[ \frac{a + c}{b + d} - \frac{a}{b} = \frac{bc - ad}{b(b + d)} > 0 \]

and

\[ \frac{c}{d} - \frac{a + c}{b + d} = \frac{bc - ad}{d(b + d)} > 0. \]

Also, in the case that $bc - ad = 1$, then

\[ k = (bc - ad)k = b(ck) - d(ak) + (d(bh) - b(dh)) \]
\[ = b(ck - dh) + d(bh - ak), \quad (4.1) \]

so that $ck - dh = bh - ak = 1$. This means that if the relation $bc - ad = 1$ holds for two Farey fractions, it holds between each fraction and the mediant as well.

Equation (4.1) also gives some information revealing when two Farey fractions $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive for the case where $bc - ad = 1$: if $\frac{p}{q}$ is any reduced fraction with $\frac{a}{b} < \frac{p}{q} < \frac{c}{d}$ then

\[ q = b(cq - dp) + d(bp - aq), \]

and because $cq - dp > 0$ and $bp - aq > 0$ must be integer values, we have $cq - dp \geq 1$ and $bp - aq \geq 1$, so the denominator of any fraction which lies between $\frac{a}{b}$ and $\frac{c}{d}$ must be at least $b + d$. Combining this
with the fact that \( \max(b, d) \leq n \) because \( \frac{a}{b} \) and \( \frac{c}{d} \) are in \( F_n \), we find that \( \frac{a}{b} \) and \( \frac{c}{d} \) are consecutive in \( F_n \) for any \( n \) satisfying
\[
\max(b, d) \leq n \leq b + d - 1.
\]
This provides us with enough information to fully describe the way in which the sequence of Farey fractions \( F_1, F_2, \ldots \) is built. Consider the construction of \( F_{n+1} \) from \( F_n \). If \( \frac{a}{b} \) and \( \frac{c}{d} \) are consecutive in \( F_n \) and \( bc - ad = 1 \) then they will remain consecutive until \( n \geq b + d \). Forming the mediant
\[
\frac{h}{k} = \frac{a + c}{b + d},
\]
we know that \( (h, k) = 1 \) because
\[
bh - ak = 1 \quad \text{and} \quad ck - dh = 1,
\]
and any \( t \) which divides both \( h \) and \( k \) must divide 1 as well. Thus, \( \frac{h}{k} \) is in \( F_k \), and these two equations also imply that both the pairs \( \frac{a}{b} < \frac{h}{k} \) and \( \frac{h}{k} < \frac{c}{d} \) are consecutive in \( F_k \), as \( \max(b, k) = k = \max(k, d) \). Noting that the consecutive fractions in \( F_1 \) satisfy \( 1 \cdot 1 - 0 \cdot 1 = 1 \), we have inductively shown that consecutive Farey fractions satisfy a relation of the form \( bc - ad = 1 \), as when passing from \( F_n \) to \( F_{n+1} \), each new fraction is the mediant of two consecutive fractions satisfying such a relation and the new consecutive pairs satisfy a similar relation.
5 Ford Circles

Originally, Rademacher used only Farey fractions to construct his exact formula for \( p(n) \). However, he later refined this approach by making use of Ford circles — geometric constructs which relate elegantly to sets of Farey fractions.

The Ford circle of a reduced fraction \( \frac{h}{k} \) is defined as the circle in the complex plane centered at \( \frac{h}{k} + i \frac{1}{2k^2} \) with radius \( \frac{1}{2k^2} \), and is denoted by \( C(h, k) \). These circles shall be of high importance when addressing the proof of Rademacher’s formula later in this text, and as such some valuable properties shall be developed here.

![Ford circle](image)

\[ \text{Figure 5.1: The Ford circle } C(h, k) \]

Firstly, consider \( C(a, b) \) and \( C(c, d) \), the Ford circles of two different reduced fractions \( \frac{a}{b} \) and \( \frac{c}{d} \). These circles have centers

\[
\frac{a}{b} + i \frac{1}{2b^2} \quad \text{and} \quad \frac{c}{d} + i \frac{1}{2d^2}
\]

respectively, and if \( D \) is the distance between these centers, then

\[
D^2 = \left( \frac{a}{b} - \frac{c}{d} \right)^2 + \left( \frac{1}{2b^2} - \frac{1}{2d^2} \right)^2.
\]

The square of the sum of their radii is

\[
(r + R)^2 = \left( \frac{1}{2b^2} + \frac{1}{2d^2} \right)^2.
\]

The difference between these two lengths is

\[
D^2 - (r + R)^2 = \left( \frac{ad - bc}{bd} \right)^2 + \left( \frac{1}{2b^2} - \frac{1}{2d^2} \right)^2 - \left( \frac{1}{2b^2} + \frac{1}{2d^2} \right)^2
= \left( \frac{ad - bc}{bd} \right)^2 - 4 \frac{1}{4b^2d^2}
= \frac{(ad - bc)^2}{b^2d^2} - 1.
\]

Because \( \frac{a}{b} \) and \( \frac{c}{d} \) are different, \( ad - bc \neq 0 \), and \( D^2 - (r + R)^2 \geq 0 \). Equality holds when \( (ad - bc)^2 = 1 \).

Geometrically, this means that two different Ford circles never intersect, and that they touch if, and only if, \( bc - ad = \pm 1 \). More importantly, this tells us that the Ford circles of consecutive Farey fractions are tangent to each other.
Let us investigate this further, and determine these points of tangency. Let $\frac{h_1}{k_1} < \frac{h}{k} < \frac{h_2}{k_2}$ be three consecutive Farey fractions. To find $\alpha_1(h, k)$, the point of contact of $C(h_1, k_1)$ and $C(h, k)$ as in Figure 5.2, consider the large right-angled triangle above the line joining the centers of the two circles, with this line as hypotenuse. This triangle is similar to the triangle which has as hypotenuse the radius of $C(h, k)$, and horizontal and vertical lengths $a$ and $b$ respectively. Then,

$$\alpha_1(h, k) = \left(\frac{h}{k} - a\right) + i\left(\frac{1}{2k^2} + b\right). \quad (5.1)$$

The large triangle has a hypotenuse of length $\frac{1}{2k^2} + \frac{1}{2k^2}$, vertical length $\frac{1}{2k^2} - \frac{1}{2k^2}$ and horizontal length $\frac{h}{k} - \frac{h_1}{k_1}$, whereas the only known side of the smaller triangle is the hypotenuse — with length $\frac{1}{2k^2}$. However, because the two triangles are similar, we have

$$\frac{a}{\frac{h}{k} - \frac{h_1}{k_1}} = \frac{\frac{1}{2k^2}}{\frac{1}{2k^2} + \frac{1}{2k^2}} = \frac{1}{2k^2} \frac{2k_1^2k_2^2}{k^2 + k_1^2} = \frac{k_1^2}{k^2 + k_1^2}$$

and, also using the hypotenuse of the large triangle,

$$\frac{b}{\frac{1}{2k^2} - \frac{1}{2k^2}} = \frac{k_1^2}{k^2 + k_1^2},$$

so that, remembering that $hk_1 - h_1k = 1$,

$$a = \frac{k_1^2}{k^2 + k_1^2} \frac{1}{kk_1} = \frac{k_1}{k(k^2 + k_1^2)}$$

and

$$b = \frac{k_1^2}{k^2 + k_1^2} \frac{k^2 - k_1^2}{2k^2k_1^2} = \frac{k^2 - k_1^2}{2k^2(k^2 + k_1^2)}.$$
Combining this with equation (5.1), we find that

\[ \alpha_1(h, k) = \left( \frac{h}{k} - \frac{k_1}{k(k^2 + k_1^2)} \right) + i \left( \frac{1}{2k^2} + \frac{k^2 - k_1^2}{2k^2(k^2 + k_1^2)} \right) \]

\[ = \frac{h}{k} - \frac{k_1}{k(k^2 + k_1^2)} + i \frac{2k^2}{2k^2(k^2 + k_1^2)} \]

\[ = \frac{h}{k} - \frac{k_1}{k(k^2 + k_1^2)} + \frac{i}{k^2 + k_1^2}. \]

In this investigation we have subtly assumed that \( k_1 > k \) (see Figure 5.2). If the converse holds, we would have

\[ \alpha_1(h, k) = \left( \frac{h}{k} - a \right) + i \left( \frac{1}{2k^2} - b \right). \]

But in this case we would then divide \( b \) by \( \frac{1}{2k^2} - \frac{1}{2k_1^2} \) to compare the ratio with the larger triangle, and this would yield the term \( k_1^2 - k^2 \) in place of \( k^2 - k_1^2 \) in the formula for \( b \), and leave the formula for \( \alpha_1 \) unchanged.

Following an analogous course, it is shown that

\[ \alpha_2(h, k) = \frac{h}{k} + \frac{k_2}{k(k^2 + k_2^2)} + \frac{i}{k^2 + k_2^2}. \] (5.2)

This provides us with sufficient knowledge of the structure of the Ford circles formed by sets of Farey fractions. Let us draw nearer to \( p(n) \), and with it, the proof of an exact formula.
6 Rademacher’s Series

So far we have a generating function for $p$ in the infinite product

$$F(x) = \prod_{m=1}^{\infty} \frac{1}{1-x^m} = \sum_{n=0}^{\infty} p(n)x^n \quad (6.1)$$

and a basic knowledge of the properties of Farey fractions and Ford circles. Our first order of business will be to investigate further the function $F$, and ultimately its behaviour near the complex roots of unity.

6.1 The function $F$

Let us consider $F(x)$ as in (6.1): to see where the function converges we need only consider the infinite product, as our results will also apply to the series. An infinite product is said to converge when its limit exists and is not zero. Thus, for an infinite product to converge, it is necessary that its terms approach 1.

If $|x| \geq 1$, $|x^m|$ cannot have limit 0, so $\lim_{1-x^m} \neq 1$, and the infinite product diverges. However, consider the case where $|x| < 1$. If we can show that the reciprocal product $\prod (1-x^m)$ converges to $l$, it implies that $F(x)$ has limit $\frac{1}{l}$, and thus that $F(x)$ converges. The reciprocal product does converge throughout the unit disk as an infinite product of the form $\prod (1-x^m)$ converges if and only if the geometric series $\sum x^m$ converges.

So we shall restrict our focus to within the unit disk. If we divide $F(x)$ through by $x^{n+1}$ we obtain

$$\frac{F(x)}{x^{n+1}} = \sum_{k=0}^{\infty} \frac{p(k)x^k}{x^{n+1}}$$

for $n \geq 0$. This gives us the Laurent series representation of $\frac{F(x)}{x^{n+1}}$, as

$$\sum_{k=0}^{\infty} \frac{p(k)x^k}{x^{n+1}} = p(0)x^{-n-1} + p(1)x^{-n} + \cdots + p(n)x^{-1} + p(n+1) + \cdots$$

So $\frac{F(x)}{x^{n+1}}$ has a pole of order $n+1$ at $x = 0$ and residue $p(n)$. This allows us to apply Cauchy’s residue theorem and find that

$$p(n) = \frac{1}{2\pi i} \int_{C} \frac{F(x)}{x^{n+1}}dx. \quad (6.2)$$

6.2 The Rademacher path $P(N)$

Though we are yet to prove it, $F(x)$ can be approximated by an elementary function whenever $x$ lies near a complex root of unity – a result which follows from a functional equation satisfied by $F$. The reason for postponing this result is that the functional equation’s form is more understandable once a few transformations have been made. We shall encounter these transformations while elaborating on the path of integration used by Rademacher.

So, assuming that we have reliable information about $F$’s behaviour near the roots of unity, we would prefer $C$ to lie as near as possible to each root of unity. This is where Rademacher was inspired with a more
elegant approach than that of Hardy and Ramanujan. He originally divided the circular contour used by his fellows into arcs whose endpoints were defined by Farey fractions, as seen in Chapters 5 and 8 of Andrews (1984) and Hardy (1978) respectively, but later refined this even further by making use of Ford circles to integrate nearer to the singularities and thus simplify the estimations made (Apostol 1990). This latter approach has proved useful in other areas of modular function theory (Andrews 1984, p. 85), and shall be our elected course. The Ford circles of Section 5 reside in the upper half-plane \( \mathcal{H} = \{ \tau : \text{Im}(\tau) > 0 \} \), and we thus need to relate our domain, the unit disk, to \( \mathcal{H} \). An elegant mapping between the open unit disk and \( \mathcal{H} \) is found in the first transformation:

\[
x = e^{2\pi i \tau},
\]

which maps the open unit disk onto the vertical strip \( \mathcal{V} = \{ \tau : 0 \leq \text{Re}(\tau) < 1, \text{Im}(\tau) > 0 \} \) in \( \mathcal{H} \).

This can be seen by taking a point \( \tau = a + bi \) in \( \mathcal{H} \) from which we have

\[
x = e^{2\pi i (a + bi)} = e^{2\pi a i - 2b \pi} = e^{2\pi a i} e^{-2b \pi}.
\]

Because \( e^{2\pi a i} \) traverses the unit circle as \( a \) varies from 0 to 1, the integer part of \( a \) can be ignored. The denominator is a real constant \( > 1 \), as \( b > 0 \), so \( x \) lies within the punctured disk.

If we then let \( b = 1 \), so \( \tau \) varies from \( i \) to \( 1 + i \) along a horizontal line, we see that \( x \) traverses a circle of radius \( e^{-2\pi} \) with center 0.

Rademacher’s idea was to follow the curve in the \( \tau \)-plane created by the upper arcs of the Ford circles formed from \( F_n \) — the set of Farey fractions of order \( n \). These circles fit neatly into \( \mathcal{V} \), bounded by \( i \) on the imaginary axis. If \( \frac{h_1}{k_1}, \frac{h_2}{k_2} \) and \( \frac{h_3}{k_3} \) are consecutive fractions in \( F_n \), then there is an upper arc (which does not touch the real axis) from \( C(h, k) \)’s intersection with \( C(h_1, k_1) \) to its intersection with \( C(h_2, k_2) \). Note that for the fractions \( 0, \frac{1}{1} \) and \( \frac{1}{2} \) we consider only the parts of the circles contained within the vertical strip \( \mathcal{V} \). The path \( P(N) \), which begins at \( i \) and ends at \( 1 + i \), is the union of these arcs.

From Figure 6.2 it can be seen that \( P(N) \) arcs away from every \( k \)-th complex root of unity, where \( k < N \). Throughout the proof, \( n \) will remain fixed, as we seek to determine \( p(n) \), while \( N \) will be allowed to approach infinity to account for infinitely many complex roots of unity.
To make use of this path of integration, we note that the change of variable
\[ x = e^{2\pi i \tau} \]
yields
\[ dx = x \cdot 2\pi i d\tau, \]
so that (6.2) becomes
\[
\int_{P(N)} p(n) = \frac{2\pi i}{2\pi} \int_{P(N)} \frac{F(x)}{x^{n}} d\tau \\
= \int_{P(N)} F(e^{2\pi i \tau}) e^{-2\pi i n \tau} d\tau.
\]
If \( \gamma(h, k) \) denotes the upper arc of \( C(h, k) \), then we can split \( P(N) \) into separate arcs for each Ford circle so that we may treat each root of unity independently:
\[
\int_{P(N)} = \sum_{k=1}^{N} \sum_{\substack{0 \leq h < k \\ (h, k) = 1}} \int_{\gamma(h, k)} \gamma(h, k)
\]
\[
= \sum_{h, k} \int_{\gamma(h, k)}
\]
where \( \sum_{h, k} \) denotes the double sum over \( k \) and \( h \).
There remains one transformation to be made before we can deduce an alternate form of $F$:

$$z = -ik^2 \left( \tau - \frac{h}{k} \right),$$

so that

$$\tau = \frac{h}{k} + \frac{iz}{k^2}.$$

To see the effect on a Ford circle $C(h, k)$, the transformation can be broken down into elementary operations. The term $\tau - \frac{h}{k}$ shifts $C(h, k)$ a distance of $\frac{h}{k}$ to the left, centering the circle above the origin. The factor $k^2$ expands the radius to

$$k^2 \cdot \frac{1}{2k^2} = \frac{1}{2},$$

while multiplication by $-i = e^{-\pi i/2}$ rotates the figure through $-\frac{\pi}{2}$ radians. $C(h, k)$ is thus transformed from the $\tau$-plane onto a circle $K$ in the $z$-plane with center $\frac{1}{2}$ and a radius of $\frac{1}{2}$.

![Figure 6.3](image)

The point of contact $\alpha_1(h, k)$ with the preceding circle becomes

$$z_1(h, k) = -ik^2 \left( \alpha_1(h, k) - \frac{h}{k} \right)$$

$$= -ik^2 \left( \frac{i}{k^2 + k_1^2} - \frac{k_1}{k(k^2 + k_1^2)} \right)$$

$$= \frac{k^2}{k^2 + k_1^2} + i \frac{kk_1}{k^2 + k_1^2} \quad (6.3a)$$

and similarly the other point of contact $\alpha_2(h, k)$ is now

$$z_2(h, k) = \frac{k^2}{k^2 + k_2^2} - i \frac{kk_2}{k^2 + k_2^2} \quad (6.3b)$$

Because $\gamma(h, k)$ does not touch the real axis in the $\tau$-plane, the arc from $z_1(h, k)$ to $z_2(h, k)$ is the curve which does not touch the imaginary axis in the $z$-plane. Noting that

$$d\tau = \frac{i}{k^2} \, dz,$$
the effect of this change of variable on \( p(n) \) is:

\[
p(n) = \sum_{h,k} \int_{\gamma(h,k)} F(e^{2\pi i r}) e^{-2\pi i n r} d\tau
\]

\[
= \sum_{h,k} \int_{z_1(h,k)}^{z_2(h,k)} F \left( \exp \left( \frac{2\pi i h}{k} - \frac{2\pi z}{k^2} \right) \right) i e^{-2\pi i h/k} e^{2\pi n z/k^2} dz
\]

\[
= \sum_{h,k} \frac{i}{k^2} e^{-2\pi i h/k} \int_{z_1(h,k)}^{z_2(h,k)} e^{2\pi n z/k^2} F \left( \exp \left( \frac{2\pi i h}{k} - \frac{2\pi z}{k^2} \right) \right) dz.
\]

The challenge of finding a usable form of \( F \) can now be attempted.

\( F(x) \) is an elliptic modular function, and in general this class of functions satisfy functional equations which describe them near the unit circle. We proceed by obtaining such an equation for \( F \) from the known functional equation for \( \eta(\tau) \).

### 6.3 Dedekind’s functional equation

The eta function was introduced by Dedekind in 1877, and is defined on the upper half-plane \( \mathcal{H} = \{ \tau \in \mathbb{C} : \text{Im}(\tau) > 0 \} \) as

\[
\eta(\tau) = e^{\pi i \tau/12} \prod_{m=1}^{\infty} (1 - e^{2\pi i m \tau}).
\]  

(Dedekind’s eta function satisfies an important functional equation which we shall require (Apostol 1990, p. 52). Briefly, a functional equation is a characterisation of a function which relates its values at different points, e.g.,

\[
\Gamma(1 + x) = x\Gamma(x).
\]

To Dedekind’s functional equation: if \( \frac{a b}{c d} \in \Gamma, \ c > 0 \) and \( \tau \in \mathcal{H} \), then

\[
\eta \left( \frac{a\tau + b}{c\tau + d} \right) = \varepsilon(a, b, c, d) \{ -i(c\tau + d) \}^{\frac{1}{2}} \eta(\tau),
\]  

(6.5)

where

\[
\varepsilon(a, b, c, d) = \exp \left\{ \pi i \left( \frac{a + d}{12c} + s(-d, c) \right) \right\}
\]

and

\[
s(h, k) = \sum_{r=1}^{k-1} h \left( \frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2} \right).
\]
6.4 Application to $F$

One could hope that $F$ and $\eta$ are in some way linked when one considers the similarities in their basic equations (6.1) and (6.4). Thankfully, if $x = e^{2\pi i \tau}$,

$$\eta(\tau) = e^{\pi i \tau/12} \prod_{m=1}^{\infty} (1 - e^{2\pi i m \tau}) = \frac{e^{\pi i \tau/12}}{F(e^{2\pi i \tau})}$$

Dedekind’s functional equation for $\eta$ (6.5) can in fact be represented in terms of $F$. If we remember that $(a b\, c d) = (a + b/c + d)$, with $c > 0$, then the relation between $\eta$ and $F$ implies

$$\frac{e^{\pi i \tau'/12}}{F(e^{2\pi i \tau'})} = \frac{e^{\pi i \tau/12}}{F(e^{2\pi i \tau})} \left(-i(c\tau + d)\right)^{1/2} \exp\left\{\frac{\pi i}{12c} (a + d + s(-d, c))\right\},$$

so that

$$F(e^{2\pi i \tau}) = F(e^{2\pi i \tau'}) \exp\left(\frac{\pi i (\tau - \tau')}{12c}\right) \left(-i(c\tau + d)\right)^{1/2} \exp\left\{\frac{\pi i}{12c} (a + d + s(-d, c))\right\}.$$

With a careful choice of $(a \ b \ c \ d)$, it can be shown (see Section A.1) that

$$F\left(\exp\left(\frac{2\pi i h}{k} - \frac{2\pi z}{k^2}\right)\right) = \omega(h, k) \left(\frac{z}{k}\right)^{1/2} F\left(\exp\left(\frac{2\pi i H}{k} - \frac{2\pi i}{z}\right)\right) \exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right)$$

where

$$\omega(h, k) = e^{\pi i s(h,k)}, \quad hH \equiv -1 \pmod{k}, \quad (h, k) = 1.$$  

Note that $s(h,k) \in \mathbb{Q}$ so that $\omega(h, k)$ is a complex root of unity, with modulus 1.

After two transformations we have reached the point where

$$x = e^{2\pi i \tau} = \exp\left(\frac{2\pi i h}{k} - \frac{2\pi z}{k^2}\right).$$

Letting

$$x' = e^{2\pi i \tau'} = \exp\left(\frac{2\pi i H}{k} - \frac{2\pi i}{z}\right),$$

the functional equation can be rewritten as

$$F(x) = \omega(h, k) \left(\frac{z}{k}\right)^{1/2} \exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right) F(x'),$$

providing us, at length, with an alternate formula for $F(x)$.

This equation does indeed provide us with valuable information about $F$’s behaviour near the unit circle. In light of the above representations of $x$ and $x'$, note that $x$ is near to the root of unity $e^{2\pi i h/k}$ when $|z|$
is small. However, the smaller \( \frac{z}{z} \) becomes, the larger \( \frac{z}{z} \) becomes, and the nearer \( x' \) moves toward 0, the origin. So when \( x \) is near to a root of unity, \( F(x') \approx F(0) = 1 \) and \( F(x) \) can be approximated by

\[
\omega(h, k) \left( \frac{z}{k} \right)^{\frac{1}{2}} \exp \left( \frac{\pi i}{12z} - \frac{\pi z}{12k^2} \right),
\]

which is an elementary function. The error in approximating \( F \) in this way will be determined by just how close to 1 the value \( F(x') \) really is, i.e., by the difference \( (F(x') - 1) \).

Returning to our equation for \( p(n) \),

\[
p(n) = \sum_{h, k} ik^{-5/2}e^{-2\pi i nh/k} \int_{z_1(h, k)}^{z_2(h, k)} e^{2\pi nz/k^2} F \left( \exp \left( \frac{2\pi i h}{k} - \frac{2\pi z}{k^2} \right) \right) dz,
\]

we can apply Dedekind’s functional equation and obtain

\[
p(n) = \sum_{h, k} ik^{-5/2}\omega(h, k) e^{-2\pi i nh/k} \int_{z_1(h, k)}^{z_2(h, k)} e^{2\pi nz/k^2} \frac{1}{z^2} \exp \left( \frac{\pi i}{12z} - \frac{\pi z}{12k^2} \right) F(x') dz.
\]

We now perform the substitution \( F(x') = 1 + (F(x') - 1) \) to separate our elementary function from the error made in our approximation, and find

\[
p(n) = \sum_{h, k} ik^{-5/2}\omega(h, k) e^{-2\pi i nh/k} \int_{z_1(h, k)}^{z_2(h, k)} e^{2\pi nz/k^2} \Psi_k(z) \left\{ 1 + (F(x') - 1) \right\} dz,
\]

with \( \Psi_k(z) = \frac{1}{z^2} \exp \left( \frac{\pi i}{12z} - \frac{\pi z}{12k^2} \right) \).

If we define

\[
I_1(h, k) = \int_{z_1(h, k)}^{z_2(h, k)} \Psi_k(z) e^{2\pi nz/k^2} dz \quad (6.7a)
\]

and

\[
I_2(h, k) = \int_{z_1(h, k)}^{z_2(h, k)} \Psi_k(z) e^{2\pi nz/k^2} \left\{ F \left( \exp \left( \frac{2\pi i H}{k} - \frac{2\pi}{z} \right) \right) - 1 \right\} dz \quad (6.7b)
\]

then we are finally left with the more visually appealing

\[
p(n) = \sum_{h, k} ik^{-5/2}\omega(h, k) e^{-2\pi i nh/k} (I_1(h, k) + I_2(h, k)), \quad (6.8)
\]

in which \( I_1(h, k) \) is the elementary integral with which we wish to work and \( I_2(h, k) \) is the error made in approximating our integral in this way.

Let us take a moment to evaluate our position. We began with an integral (6.2) obtained from Cauchy’s residue theorem and proceeded to find an integration curve \( P(N) \) which we could split into a sum of integrals over the Farey fractions in \( F_n \). We have now applied Dedekind’s functional equation and split each integral into two integrals \( I_1 \) and \( I_2 \), each along the arc from \( z_1(h, k) \) to \( z_2(h, k) \) on the circle \( K \).
6.5 The contribution of \( I_2(h, k) \)

We now possess an almost reasonable-looking formula for \( p(n) \) — except for the term \( I_2(h, k) \), which is our difference term. Luckily, \( I_2(h, k) \) is \( O(N^{-1/2}) \), and thus becomes small for large values of \( N \). To see this, we show that the integrand is bounded and then integrate along the direct chord from \( z_1(h, k) \) to \( z_2(h, k) \) instead of the arc. Before we investigate the integrand, note that

\[
|e^z| = |e^{\Re(z)}| |e^{\Im(z)i}|
\]

as \( e^{\Im(z)i} \) is a point on the unit circle with modulus 1.

Now, considering the integrand of \( I_2(h, k) \) and making use of the infinite sum (with first term 1) in (6.1), we have

\[
|\Psi_k(z)| F \left( \exp \left( \frac{2\pi i H}{k} - \frac{2\pi}{z} \right) \right) - 1 \right| e^{2\pi n z/k^2}
\]

\[
= |z|^{\frac{1}{2}} \exp \left( \frac{\pi}{12z} - \frac{\pi z}{12k^2} \right) \exp \left( \frac{2\pi n z}{k^2} \right) \left\{ \sum_{m=1}^{\infty} p(m) \exp \left( \frac{2\pi i m H}{k} - \frac{2\pi m}{z} \right) \right\}
\]

\[
= |z|^{\frac{1}{2}} \exp \left( \frac{\pi \Re(1/z)}{12} - \frac{\pi \Re(z)}{12k^2} \right) \exp \left( \frac{2\pi n \Re(z)}{k^2} \right) \sum_{m=1}^{\infty} p(m) e^{2\pi i m/k} e^{-2\pi m \Re(1/z)}.
\]

Because \( z \) is a point on the chord from \( z_1 \) to \( z_2 \), \( 0 < \Re(z) < 1 \). To continue, we also need information about \( \Re(1/z) \). This is thus a good time to investigate the transformation \( 1/z \) for a point \( z = a + bi \) on or within the circle \( K \) (see Figure 6.3):

\[
\left( a - \frac{1}{2} \right)^2 + b^2 \leq \frac{1}{4}
\]

so

\[
a^2 - a + \frac{1}{4} + b^2 \leq \frac{1}{4}, \quad \text{and} \quad a^2 + b^2 \leq a.
\]

We can write

\[
\frac{1}{z} = \frac{a - bi}{a^2 + b^2},
\]

and then investigate each part separately:

\[
\Re(1/z) = \frac{a}{a^2 + b^2} \geq \frac{a}{a} = 1
\]

and

\[
\Im(1/z) = -\frac{b}{a^2 + b^2}.
\]

In the case where \( z \) lies on the circle \( K \), we have \( a^2 + b^2 = a \), so that \( \Re(1/z) = 1 \) and

\[
\Im(1/z) = -\frac{b}{a} = \pm \sqrt{\frac{a - a^2}{a^2}} = \pm \sqrt{\frac{1}{a} - 1},
\]
which can take on any real value as $0 \leq a \leq 1$.

Noting then that
\[
\exp\left(-\frac{\pi \Re(z)}{12k^2}\right) \leq 1,
\]
\[
\exp\left(\frac{2\pi n \Re(z)}{k^2}\right) < e^{2\pi n},
\]
\[
|e^{2\pi iH_m/k}| = 1,
\]
we can continue with our estimation, seeing that the modulus of the integrand is
\[
< |z|^{\frac{1}{2}} \exp\left(\frac{\pi \Re(1/z)}{12}\right) e^{2\pi n} \sum_{m=1}^{\infty} p(m)e^{-2\pi m \Re(1/z)}
\]
\[
= |z|^{\frac{1}{2}} e^{2\pi n} \sum_{m=1}^{\infty} p(m)e^{-2\pi \Re(1/z)(m-1/24)}
\]
\[
\leq |z|^{\frac{1}{2}} e^{2\pi n} \sum_{m=1}^{\infty} p(m)e^{-\pi(24m-1)/12}
\]
\[
< |z|^{\frac{1}{2}} e^{2\pi n} \sum_{m=1}^{\infty} p(24m-1)e^{-\pi(24m-1)/12}.
\]

Taking $y = e^{-\pi/12}$ and recognising that the infinite sum is $F(y)$ and thus convergent ($|y| < 1$), we are left with
\[
\frac{c}{|z|^2},
\]
where
\[
c = e^{2\pi n} \sum_{m=1}^{\infty} p(24m-1)y^{24m-1},
\]
so that $c$ is independent of $N$ and $z$, and thus remains unchanged throughout the double summation in our current formula for $p(n)$.

We have yet to consider the term $|z|^{\frac{1}{2}}$, and thereafter to investigate the length of the chord between $z_1(h, k)$ and $z_2(h, k)$. We will then be able to combine our results and obtain an estimate for $|I_2(h, k)|$.

Firstly, let us investigate $|z|$ along the chord.

Making use of (6.3),
\[
|z_1(h, k)|^2 = \left|\frac{k^2}{k^2 + k_1^2}\right|^2 + \left|\frac{kk_1}{k^2 + k_1^2}\right|^2
\]
\[
= \frac{k^4 + k^2k_1^2}{(k^2 + k_1^2)^2}
\]
\[
= \frac{k^2}{k^2 + k_1^2}.
\]
A similar equation holds for $z_2(h, k)$, so that

\[
|z_1(h, k)| = \frac{k}{\sqrt{k^2 + k_1^2}}
\]

\[
|z_2(h, k)| = \frac{k}{\sqrt{k^2 + k_2^2}}
\]

(6.9)

Now, the Cauchy-Schwarz inequality implies that

\[
(k \cdot 1 + k_1 \cdot 1)^2 \leq (k^2 + k_1^2)(1 + 1)
\]

so that

\[
\sqrt{k^2 + k_1^2} \geq \frac{k + k_1}{\sqrt{2}} \geq \frac{N + 1}{\sqrt{2}} \geq \frac{N}{\sqrt{2}}
\]

because we have consecutive Farey fractions, so $N \leq k + k - 1$. The same inequality holds when $k_1$ is replaced by $k_2$, so that

\[
|z_1| < \frac{\sqrt{2} k}{N} \quad \text{and} \quad |z_2| < \frac{\sqrt{2} k}{N}.
\]

Because $z$ is on the chord,

\[
|z| = |\alpha z_1 + (1 - \alpha) z_2| \leq \alpha |z_1| + (1 - \alpha) |z_2| \leq \max(|z_1|, |z_2|) \leq \sqrt{2} \frac{k}{N}.
\]

This implies that

\[
c |z|^{\frac{1}{2}} \leq 2^{\frac{1}{2}} \left( \frac{k}{N} \right)^{\frac{1}{2}}.
\]

Also, by way of the triangle inequality, the length of the chord is at most $|z_1| + |z_2| = 2\sqrt{2} k / N$. We can thus bound the integral by multiplying the length of the chord by an upper bound for the integrand on the chord, so that

\[
|I_2(h, k)| < C \left( \frac{k}{N} \right)^{\frac{3}{2}}
\]
where \( C \) is a constant. Hence, noting the roots of unity \( \omega(h, k) \) and \( e^{-2\pi i n h/k} \),

\[
\left| \sum_{h,k} ik^{-5/2} \omega(h, k) e^{-2\pi i n h/k} I_2(h, k) \right| \leq \sum_{h,k} k^{-5/2} |I_2(h, k)| \\
< \sum_{k=1}^{N} \sum_{0 \leq h < k} \frac{C}{k N^{3/2}} \\
= \frac{C}{N^{3/2}} \sum_{k=1}^{N} \sum_{0 \leq h < k} 1 \\
\leq \frac{C}{N^{3/2}} \sum_{k=1}^{N} 1 \\
= \frac{C}{\sqrt{N}}.
\]

Equation (6.8) then becomes

\[
p(n) = \sum_{h,k} ik^{-5/2} \omega(h, k) e^{-2\pi i n h/k} I_2(h, k) = O(N^{-1/2}). \tag{6.10}
\]

6.6 The contribution of \( I_1(h, k) \)

We now know that \( I_2(h, k) \) is of an encouragingly small order, and will even vanish when we allow \( N \to \infty \). But we are getting ahead of ourselves. Before we hastily get rid of our error term, we can show that integrating \( I_1(h, k) \) along the entire circle \( K \) instead of just a section of its circumference will provide us with an error also of \( O(N^{-1/2}) \).

Recall equation (6.7a):

\[
I_1(h, k) = \int_{z_1(h,k)}^{z_2(h,k)} \Psi_k(z) e^{2\pi i n z/k^2} dz.
\]

If \( K^- \) denotes the curve along the circumference of \( K \) in the negative direction then

\[
I_1(h, k) = \int_{K^-} - \int_{z_2(h,k)}^{z_1(h,k)}
\]

\[
= \int_{K^-} - \int_{z_2(h,k)}^{0} - \int_{0}^{z_1(h,k)}.
\]

We split the difference into two separate integrals (call them \( J_1 \) and \( J_2 \) respectively) because \( \frac{1}{z} \) does not exist when \( z = 0 \). It remains to show that both \( J_1 \) and \( J_2 \) are \( O(N^{-1/2}) \).
If one considers the straight line from 0 to \( z_1 \) to be the diameter of a new circle (Figure 6.4), then the arc on \( K \) from 0 to \( z_1 \) lies within the upper half of this circle, so that the length of the arc is less than half of the new circle’s circumference, i.e.,

\[
< \frac{\pi |z_1|}{2}
\]

\[
< \frac{\sqrt{2}k}{N}.
\]

Similarly, this same inequality holds for the length of the arc from \( z_2 \) to 0. Clearly, any \( z \) on either of these arcs lies within or on the circle about the origin with radius \( \max(|z_1|, |z_2|) \), so \( |z| < \sqrt{2}k/N \).

We already know that when applied to the circle \( K \), \( 1/z \) maps \( z \) onto the vertical line through 1, so \( \text{Re}(1/z) = 1 \). We consider the integrand:

\[
\left| \psi_k(z)e^{2\pi nz/k^2} \right|
= |z|^{1/2} \exp \left( \frac{\pi}{12z} - \frac{\pi z}{12k^2} \right) e^{2\pi nz/k^2}
\leq |z|^{1/2} e^{\pi/12} e^{-\pi e^{2\pi n}}
< c_1 \left( \frac{k}{N} \right)^{3/2},
\]

using the inequality for \( |z| \) obtained above. Combining this with our knowledge of the length of the arcs, we find that both

\[
|J_1| < C_1 \left( \frac{k}{N} \right)^{3/2} \quad \text{and} \quad |J_2| < C_1 \left( \frac{k}{N} \right)^{3/2},
\]

where \( C_1 \) is constant. Remembering that these together form the error when using the curve \( K^- \) to
calculate $I_1(h, k)$, we find

$$\left| \sum_{h,k} i k^{-5/2} \omega(h, k) e^{-2\pi i n h/k} (J_1 + J_2) \right| \leq \sum_{h,k} k^{-5/2} |J_1 + J_2|$$

$$< \sum_{k=1}^{N} \sum_{0 \leq h < k \atop (h,k)=1} \frac{2C_1}{kN^{-3/2}}$$

$$= \frac{2C_1}{N^{-3/2}} \sum_{k=1}^{N} \sum_{0 \leq h < k \atop (h,k)=1} 1$$

$$\leq \frac{2C_1}{\sqrt{N}}.$$

This, when applied to (6.10), gives us

$$p(n) = \sum_{h,k} i k^{-5/2} \omega(h, k) e^{-2\pi i n h/k} \int_{K^-} \Psi_k(z) e^{2\pi n z/k^2} dz + O(N^{-1/2}).$$

6.7 Letting go

At present we are considering only a finite number of Ford circles — namely those based on $F_N$. Letting $N \to \infty$, our equation becomes

$$p(n) = \sum_{k=1}^{\infty} \sum_{0 \leq h < k \atop (h,k)=1} i k^{-5/2} \omega(h, k) e^{-2\pi i n h/k} \int_{K^-} \Psi_k(z) e^{2\pi n z/k^2} dz$$

$$= \sum_{k=1}^{\infty} i k^{-5/2} A_k(n) \int_{K^-} z^{1/2} \exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right) e^{2\pi n z/k^2} dz,$$

$$= \sum_{k=1}^{\infty} i k^{-5/2} A_k(n) \int_{K^-} z^{1/2} \exp\left\{\frac{\pi}{12z} - \frac{2\pi z}{k^2} \left(n - \frac{1}{24}\right)\right\} dz,$$

where we let

$$A_k(n) = \sum_{0 \leq h < k \atop (h,k)=1} \omega(h, k) e^{-2\pi i n h/k}$$

as it shall remain unchanged for the remainder of the proof.

We then apply the change of variable

$$z = \frac{\pi}{12t} \quad \text{with} \quad dz = -\frac{\pi}{12t^2} dt,$$

noting that this is just a constant multiple of the transform $\frac{1}{z}$ which was investigated earlier, and thus maps the circle $K$ (and with it our path of integration) onto the line Re$(z) = \frac{\pi}{12}$. 

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Then
\[
p(n) = \sum_{k=1}^{\infty} -ik^{-5/2} A_k(n) \int_{\pi/12 - \infty i}^{\pi/12 + \infty i} \left( \frac{\pi}{12t} \right)^{1/2} \exp \left\{ t + \frac{\pi^2}{6k^2 t} \left( n - \frac{1}{24} \right) \right\} \frac{\pi}{12t} dt
\]
\[
= \sum_{k=1}^{\infty} -ik^{-5/2} A_k(n) \left( \frac{\pi}{12} \right)^{3/2} \int_{\pi/12 - \infty i}^{\pi/12 + \infty i} t^{-5/2} \exp \left\{ t + \frac{\pi^2}{6k^2 t} \left( n - \frac{1}{24} \right) \right\} dt
\]
\[
= 2\pi \left( \frac{\pi}{12} \right)^{3/2} \sum_{k=1}^{\infty} k^{-5/2} A_k(n) \frac{1}{2\pi i} \int_{\pi/12 - \infty i}^{\pi/12 + \infty i} t^{-5/2} \exp \left\{ t + \frac{\pi^2}{6k^2 t} \left( n - \frac{1}{24} \right) \right\} dt.
\]

We can evaluate the integral in terms of Bessel functions, but for the sake of continuity the technicalities of the evaluation are performed in Section A.2, with a narrative sketch provided here. Equation (B.1) found in Section B.1 states that
\[
\left( \frac{1}{2} \right)^{\nu} \int_{\pi/12 - \infty i}^{\pi/12 + \infty i} t^{-\nu-1} \exp \left( t + \frac{z^2}{4t} \right) dt = I_{\nu}(z),
\]
and if \( z \) and \( \nu \) in this integral are taken as
\[
z = \left\{ \frac{\pi^2}{6k^2} \left( n - \frac{1}{24} \right) \right\}^{1/2} \quad \text{and} \quad \nu = \frac{3}{2},
\]
we find that our integral can now be expressed as
\[
\frac{1}{2\pi i} \int_{\pi/12 - \infty i}^{\pi/12 + \infty i} t^{-5/2} \exp \left\{ t + \frac{\pi^2}{6k^2 t} \left( n - \frac{1}{24} \right) \right\} dt = \left\{ \frac{\pi^2}{6k^2} \left( n - \frac{1}{24} \right) \right\}^{-3/4} I_{3/2} \left( \frac{\pi}{k} \sqrt{2/3} \left( n - \frac{1}{24} \right) \right).
\]

Combining this equation with that of \( p(n) \) and simplifying gives us
\[
p(n) = \frac{2\pi (n - \frac{1}{24})^{-3/4}}{(24)^{3/4}} \sum_{k=1}^{\infty} k^{-1} A_k(n) I_{3/2} \left( \frac{\pi}{k} \sqrt{2/3} \left( n - \frac{1}{24} \right) \right).
\]

\( I_{3/2}(z) \) is a Bessel function of half odd order, and, specifically (B.2) gives
\[
I_{3/2}(z) = \sqrt{\frac{2z}{\pi}} \frac{d}{dz} \left( \sinh z \right),
\]
which implies that
\[
I_{3/2} \left( \frac{\pi}{k} \sqrt{2/3} \left( n - \frac{1}{24} \right) \right) = \frac{6^{3/4} k^{3/2}}{\pi^2} \left( n - \frac{1}{24} \right)^{\frac{3}{4}} \frac{d}{dn} \left( \sinh \left\{ \frac{\pi}{k} \sqrt{2/3} \left( n - \frac{1}{24} \right) \right\} \right).
\]

Once again simplifying our equation for \( p(n) \), we arrive at our infinite series representation:
\[
p(n) = \frac{1}{\pi \sqrt{2}} \sum_{k=1}^{\infty} k^{3/2} A_k(n) \frac{d}{dn} \left( \sinh \left\{ \frac{\pi}{k} \sqrt{2/3} \left( n - \frac{1}{24} \right) \right\} \right), \tag{6.11}
\]
\[
25
\]
where, as before
\[ A_k(n) = \sum_{0 \leq h < k \atop (h,k)=1} \omega(h,k) e^{-2\pi i h/k}. \]

We have now deduced the convergent series for \( p(n) \), our main objective for this text. A few auxiliary results will yet be presented.

### 6.8 Asymptotics

When Hardy and Ramanujan published their results on \( p(n) \), they included the asymptotic formula

\[ p(n) \sim \frac{e^{\pi \sqrt{\frac{2n}{3}}}}{4n\sqrt{3}} \quad \text{as } n \to \infty. \]

This formula can in fact be deduced from the first term of Rademacher's infinite series. To prove this, we need to show that the first term (denote the \( k \)-th term by \( R_k(n) \)) is of this form and that the weight of the rest of the terms combined is of an order less than \( R_1(n) \). Recall from (6.11) that

\[ R_k(n) = k^{\frac{1}{2}} A_k(n) \frac{d}{dn} \left( \frac{\sinh \left\{ \frac{\pi}{k} \sqrt{\frac{2}{3}} \left( n - \frac{1}{24} \right) \right\}}{\sqrt{n - \frac{1}{24}}} \right), \]

so that for the first term, where \( k = 1 \):

\[ A_1(n) = \sum_{0 < h < k \atop (h,k)=1} e^{\pi is(h,k)} e^{-2\pi i h/k} = e^{\pi i 0} e^{-2\pi n 0} = 1, \]

because \( s(h,k) \) is a sum from 1 to \( k - 1 = 0 \), and is thus 0. The remaining part of \( R_1(n) \) is the derivative,

\[ R_1(n) = \frac{d}{dn} \left( \frac{\sinh \left( \frac{\pi}{\sqrt{3}} \left( n - \frac{1}{24} \right) \right)}{\sqrt{n - \frac{1}{24}}} \right) \]

\[ = \frac{1}{4} \left( e^{\pi \sqrt{\frac{2}{3}} (n - \frac{1}{24})} + e^{-\pi \sqrt{\frac{2}{3}} (n - \frac{1}{24})} \right) \pi \sqrt{\frac{2}{3}} \left( n - \frac{1}{24} \right)^{-1} \]

\[ - \frac{1}{4} \left( e^{\pi \sqrt{\frac{2}{3}} (n - \frac{1}{24})} - e^{-\pi \sqrt{\frac{2}{3}} (n - \frac{1}{24})} \right) \left( n - \frac{1}{24} \right) ^{\frac{3}{2}}. \]

Also, we can make use of the binomial series and obtain:

\[ \sqrt{n - \frac{1}{24}} = \sqrt{n} \left( 1 - \frac{1}{24n} \right)^{\frac{1}{2}} = \sqrt{n} \sum_{m=0}^{\infty} \binom{\frac{1}{2}}{m} (-1)^k \left( \frac{1}{24n} \right)^m \]

\[ = \sqrt{n} \left( 1 - \frac{1}{2 \cdot 24n} - \frac{1}{8 \cdot (24n)^2} - \cdots \right) \]

\[ = \sqrt{n} \left( 1 + O \left( \frac{1}{n} \right) \right) = \sqrt{n} + O \left( \frac{1}{\sqrt{n}} \right), \]
so that, by the Taylor expansion of $e$,

$$e^{\pi \sqrt{\frac{2}{3}(n+\frac{1}{2})}} = e^{\pi \sqrt{\frac{2}{3}} \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)},$$

Following on from the above, and noting that the exponential terms with negative exponents diminish rapidly, we can further simplify our former result to:

$$R_1(n) = \frac{\pi}{4n} \sqrt{\frac{2}{3}} e^{\pi \sqrt{\frac{2}{3}} \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)},$$

We now have a representation of the first term which seems strikingly familiar to the asymptotic formula we wish to obtain. To further our knowledge of the weight of the subsequent terms, we must investigate the derivative term more closely. Letting

$$t = \frac{\pi}{k} \sqrt{\frac{2}{3}(n - \frac{1}{24})}, \quad \text{so that} \quad \left(n - \frac{1}{24}\right) = \left(\frac{kt}{\pi}\right)^2 \frac{3}{2},$$

we find that

$$\frac{d}{dn} \left(\frac{\sinh t}{\sqrt{n - \frac{1}{24}}}\right) = \frac{\pi}{2k} \sqrt{\frac{2}{3}(n - \frac{1}{24})} \frac{1}{2} \sinh t - \frac{\pi^2}{2k} \cdot \frac{\pi}{k} \left(\frac{2}{3}\right)^{\frac{3}{2}} \left(\frac{2}{3}\right)^{\frac{3}{2}} \sinh t$$

$$= \frac{2 \pi}{3 \sqrt{6} k^3} \cosh t - \frac{\sqrt{2 \pi} \sinh t}{3 \sqrt{3} k^3 t^3}$$

$$= \frac{2 \pi^3}{3 \sqrt{6} k^3} \left(\frac{\cosh t}{t^2} - \frac{\sinh t}{t^3}\right). \quad (6.12)$$

With $f(t) = \left(\frac{\cosh t}{t^2} - \frac{\sinh t}{t^3}\right)$, the Taylor series for cosh and sinh imply

$$f(t) = \left(\frac{\cosh t}{t^2} - \frac{\sinh t}{t^3}\right) = \left(\frac{1}{t^2} + \frac{t^2}{2!t^2} + \frac{t^4}{4!t^2} + \cdots\right) - \left(\frac{t}{t^3} + \frac{t^3}{3!t^3} + \frac{t^5}{5!t^3} + \cdots\right)$$

$$= \left(\frac{1}{2!} - \frac{1}{3!}\right) + t^2 \left(\frac{1}{4!} - \frac{1}{5!}\right) + t^4 \left(\frac{1}{6!} - \frac{1}{7!}\right) + \cdots,$$

so that $f$ is increasing and $\lim_{t \to 0} f(t) \neq \pm \infty$.

We can now apply this knowledge to the remaining terms. Because $A_k(n)$ is a sum of roots of unity with less than $k$ terms, it is clear that $|A_k(n)| \leq k$. With this in mind (and $C$ a constant), it follows that:

$$\left|\sum_{k \geq 2} k^2 A_k(n) \frac{2 \pi^3}{3 \sqrt{6} k^3} f \left(\frac{\pi}{k} \sqrt{\frac{2}{3}(n - \frac{1}{24})}\right)\right| \leq C \cdot \sum_{k \geq 2} k^{-\frac{3}{2}} f \left(\frac{\pi}{2} \sqrt{\frac{2}{3}(n - \frac{1}{24})}\right)$$

$$\leq C \cdot f \left(\frac{\pi}{2} \sqrt{\frac{2}{3}(n - \frac{1}{24})}\right) \sum_{k \geq 2} k^{-\frac{3}{2}},$$
because \( f(t) \) decreases as \( k \) increases. Furthermore, \( f(t) \) does not approach \(-\infty\) as \( k \) increases, and the infinite sum is a \( p \)-series with \( p > 1 \), and thus converges. So

\[
\sum_{k \geq 2} R_k(n) = O \left( f \left( \frac{\pi}{2} \sqrt{\frac{2}{3} \left( n - \frac{1}{24} \right)} \right) \right) = O(R_2(n)).
\]

It is clear from the above processes that each term is dominated by an exponential function, and although the exponent of \( e \) is similar for each term, the constant part of the exponent in \( R_2(n) \) (and subsequent terms) is smaller than that of \( R_1(n) \), so that the first term dominates the sum. From (6.11) then,

\[
p(n) \sim \frac{1}{\pi \sqrt{2}} R_1(n) = \frac{1}{\pi \sqrt{2}} \frac{\pi}{4n} \sqrt{\frac{2}{3}} \frac{\pi}{\sqrt{2}} \left( 1 + O \left( \frac{1}{\sqrt{n}} \right) \right) \\
\sim \frac{e^{\pi \sqrt{\frac{2}{3}}}}{4n \sqrt{3}} \quad \text{as } n \to \infty.
\]  

(6.13)

The actual weight of \( \frac{1}{\pi \sqrt{2}} R_1(n) \) (and the growth of \( p(n) \)), is demonstrated below:

<table>
<thead>
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<th>( n )</th>
<th>( p(n) )</th>
<th>( \frac{1}{\pi \sqrt{2}} R_1(n) )</th>
</tr>
</thead>
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<td>1</td>
</tr>
<tr>
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</tr>
<tr>
<td>5000</td>
<td>1.6982e+74</td>
<td>1.6982e+74</td>
</tr>
<tr>
<td>10000</td>
<td>3.61673e+106</td>
<td>3.61673e+106</td>
</tr>
</tbody>
</table>

Table 1: Comparison of \( p(n) \) and its first term approximation (rounded to the nearest integer).

The difference between the first term approximation and \( p(n) \) does grow infinitely large with \( n \), but at a far slower rate than the increase of the partition function. Because the contribution of the other terms becomes less important as \( n \) becomes large, the ratio of \( \frac{1}{\pi \sqrt{2}} R_1(n) \) to \( p(n) \) tends to 1.
6.9 Distinct summands

Another interesting result due to Goh & Schmutz (1995) is that the average number of distinct summands in a partition of \( n \) is

\[
D(n) = \frac{\sqrt{6n}}{\pi} \left( 1 + O \left( n^{-\frac{1}{6}} \right) \right).
\]  

(6.14)

Firstly, let us obtain a combinatoric formula for this quantity. Instead of considering each partition individually, regard the set of all partitions of \( n \). Summing the number of partitions containing a 1 with the number of partitions containing a 2, then a 3, and so forth, counts each distinct summand in every partition once, and so gives the total number of distinct summands over all partitions. Dividing this quantity by \( p(n) \) gives the average number of distinct summands in a typical partition. The number of partitions containing the integer \( r \) is given by \( p(n - r) \), as removing the \( r \) gives a partition of \( n - r \), and conversely attaching \( r \) to a partition of \( n - r \) results in a partition of \( n \). Hence,

\[
D(n) = \frac{p(n - 1) + p(n - 2) + \cdots + p(0)}{p(n)}.
\]

To understand the numerator, we recall equation (6.13), and see that

\[
p(n - r) = \frac{e^\sqrt{\frac{2(n - r)}{3}}}{4\sqrt{3}(n - r)} \left( 1 + O \left( \frac{1}{\sqrt{n - r}} \right) \right).
\]

Making use of the binomial series to expand and noting that \( \frac{r}{n} \leq 1 \) gives

\[
\sqrt{n - r} = \sqrt{n} \left( 1 - \frac{r}{n} \right)^{\frac{1}{2}} = \sqrt{n} \left( 1 - \frac{1}{2n} - \frac{r^2}{8n^2} - \frac{r^3}{16n^3} - \cdots \right) = \sqrt{n} \left( 1 - \frac{r}{2n} + O \left( \frac{r^2}{n^2} \right) \right),
\]

\[29\]
\[
\frac{1}{n-r} = \frac{1}{n} \left(1 - \frac{r}{n}\right)^{-1} = \frac{1}{n} \left(1 + \frac{r}{n} + \frac{r^2}{n^2} + \cdots \right) = \frac{1}{n} \left(1 + O\left(\frac{r}{n}\right)\right),
\]

so that
\[
p(n-r) = \exp\left\{\frac{\pi \sqrt{2} \sqrt{n}}{4 \sqrt{3} n} - \frac{\pi r}{\sqrt{n} \sqrt{n}} + O\left(\frac{r^2}{n^2}\right)\right\} \left(1 + O\left(\frac{r}{\sqrt{n-r}}\right)\right) \left(1 + O\left(\frac{1}{\sqrt{n-r}}\right)\right)
\]
\[
= p(n) \cdot \exp\left\{-\frac{\pi r}{\sqrt{6n}}\right\} \exp\left\{O\left(\frac{r^2}{n^2}\right)\right\} \left(1 + O\left(\frac{r}{n}\right) + O\left(\frac{1}{\sqrt{n-r}}\right)\right) \left(1 + O\left(\frac{1}{\sqrt{n-r}}\right)\right)
\]

Now, large values of \(r\) (greater than \(\sqrt{n}\)) will contract the value of \(p(n-r)\), as the effect of the first exponential function outweighs the second:
\[
\frac{r}{\sqrt{n}} = \frac{r^2}{\sqrt{r} \sqrt{n}} \geq \frac{r^2}{n^2}
\]

because \(r < n\). So also, if \(r > n^2\), then \(\frac{r}{\sqrt{n}} > n\), and \(p(n-r) \to 0\) exponentially as \(n \to \infty\). This shall prove to be an opportune value at which to separate the \(r\)'s. There are less than \(n\) terms with such an \(r\), so their contribution tends to 0. We thus consider terms \(p(n-r)\) with \(r \leq n^2\). The Taylor series for \(\exp(z)\) implies
\[
\exp\left(\frac{r^2}{n^2}\right) = 1 + \frac{r^2}{n^2} + \frac{1}{2!} \left(\frac{r^2}{n^2}\right)^2 + \cdots = 1 + O\left(\frac{r^2}{n^2}\right),
\]
as \(r^2 < n^2\). Applying this to our equation for \(p(n-r)\) yields
\[
p(n-r) = p(n) \cdot \exp\left\{-\frac{\pi r}{\sqrt{6n}}\right\} \left(1 + O\left(\frac{r^2}{n^2}\right) + O\left(\frac{r}{n}\right) + O\left(\frac{1}{\sqrt{n-r}}\right)\right).
\]

Because \(r \leq n^2\), the fractions can be ordered
\[
\frac{r^2}{n^2} \leq n^{-1/2} \quad \text{and} \quad \frac{r}{n} \leq n^{-1/2},
\]
and for sufficiently sized \(n\) (\(n \geq 8\), \(n^{3/2} \leq \frac{n}{2}\)), so that
\[
\sqrt{n} \geq \sqrt{n-r} \geq \sqrt{\frac{n}{2}},
\]
implying that \(\frac{1}{\sqrt{n-r}} = O\left(n^{-\frac{1}{2}}\right)\). This knowledge combined simplifies our equation to
\[
p(n-r) = p(n) \cdot \exp\left\{-\frac{\pi r}{\sqrt{6n}}\right\} \left(1 + O\left(n^{-1/2}\right)\right),
\]
and, when substituted into (6.14):

\[
D(n) = \frac{1}{p(n)} \sum_{r=1}^{\lfloor n^{2/3} \rfloor} p(n) \cdot \exp \left\{ -\frac{\pi r}{\sqrt{6n}} \right\} \left( 1 + O \left( n^{-\frac{1}{2}} \right) \right)
\]

\[
= \left( 1 + O \left( n^{-\frac{1}{2}} \right) \right) \sum_{r=1}^{\lfloor n^{2/3} \rfloor} \exp \left\{ -\frac{\pi r}{\sqrt{6n}} \right\}^r.
\]

The sum is recognisable as a finite geometric series, and can be simplified to

\[
D(n) = \frac{1 - \exp \left\{ -\frac{\pi \lfloor n^{2/3} \rfloor}{\sqrt{6n}} \left( \lfloor n^{2/3} \rfloor + 1 \right) \right\}}{1 - \exp \left\{ -\frac{\pi}{\sqrt{6n}} \right\}} \left( 1 + O \left( n^{-\frac{1}{2}} \right) \right).
\]

Once again, the exponential term in the numerator approaches 0 quickly, as \( \lfloor n^{2/3} \rfloor > \sqrt{n} \). Expanding the other exponential function yields

\[
D(n) = \frac{1}{1 - \left( 1 + \frac{\pi}{\sqrt{6n}} + O \left( \frac{\pi^2}{6n} \right) \right) \left( 1 + O \left( n^{-\frac{1}{2}} \right) \right)}
\]

\[
= \frac{1}{\frac{\pi}{\sqrt{6n}} + O \left( \frac{\pi^2}{6n} \right)} \left( 1 + O \left( n^{-\frac{1}{2}} \right) \right)
\]

\[
= \frac{\sqrt{6n}}{\pi} \left( 1 + O \left( \frac{\pi}{\sqrt{6n}} \right) \right)^{-1} \left( 1 + O \left( n^{-\frac{1}{2}} \right) \right).
\]

Lastly, the binomial series can once again be applied, this time to the reciprocal term:

\[
D(n) = \frac{\sqrt{6n}}{\pi} \left( 1 + O \left( \frac{\pi}{\sqrt{6n}} \right) \right) \left( 1 + O \left( n^{-\frac{1}{2}} \right) \right)
\]

\[
= \frac{\sqrt{6n}}{\pi} \left( 1 + O \left( n^{-\frac{1}{2}} \right) \right),
\]

and the result is obtained. With a similar but more extensive argument it can also be shown that the average number of summands (counting repeated summands) of a partition of \( n \) is

\[
\frac{\sqrt{6n}}{2\pi} \log n + O(\sqrt{n}).
\]
A Appendix of Calculations

A.1 Dedekind’s functional equation in terms of $F$

This deduction is outlined in Section 6.4.

The relationship between $F$ and $\eta$ implies that if $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma$, $c > 0$ and $\tau' = \frac{a \tau + b}{c \tau + d}$, then

$$F(e^{2\pi i \tau}) = F(e^{2\pi i \tau'}) \exp \left( \frac{\pi i (\tau - \tau'')}{12} \right) \left\{ -i(c \tau + d) \right\} \exp \left\{ \pi i \left( \frac{a + d}{12c} + s(d, c) \right) \right\}.$$ 

Now if $c = k$ and $d = -h$ so that $s(-d, c) = s(h, k)$ and if $a = H$, then $b = -\frac{hH + 1}{k}$ because $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma$. In our case, where $\tau = \frac{iz + h}{k^2}$,

$$\begin{align*}
a \tau + b &= \frac{Hiz + hHk}{k^2} - \frac{hHk + k}{k^2} \\
&= \frac{Hiz - k}{k^2}, \\
c \tau + d &= \frac{iz + hk}{k} - h \\
&= \frac{iz}{k}
\end{align*}$$

so

$$\tau' = \frac{Hiz - k}{k^2} = \frac{iz^{-1} k + H}{k}.$$ 

Setting this into our functional equation, we have

$$F \left( \exp \left( \frac{2\pi i k}{k^2} \frac{2\pi i z}{k^2} \right) \right) = F \left( \exp \left( \frac{2\pi i H}{k} - \frac{2\pi k}{z} \right) \right) \left\{ \frac{i}{k} \right\} \exp \left\{ \pi i \left( \frac{H - h}{12k} + s(h, k) \right) \right\}$$

$$\times \exp \left( \frac{\pi i}{12} \left( \frac{iz - iz^{-1} k^2 + hk - Hk}{k^2} \right) \right) \exp \left\{ \frac{-\pi i \left( \frac{H - h}{12k} + s(h, k) \right)}{12} \right\}$$

$$= F \left( \exp \left( \frac{2\pi i H}{k} - \frac{2\pi}{z} \right) \right) \left( \frac{z}{k} \right) \frac{1}{2}$$

$$\times \exp \left( \frac{\pi i}{12} \left( \frac{h - H}{k} + H - h + \frac{iz}{k^2} - \frac{i}{z} \right) \right) \exp \left( \pi i (s(h, k)) \right)$$

$$= F \left( \exp \left( \frac{2\pi i H}{k} - \frac{2\pi}{z} \right) \right) \left( \frac{z}{k} \right) \frac{1}{2} \exp \left( \frac{\pi}{12z} - \frac{\pi z}{12k^2} + \pi i (s(h, k)) \right),$$

as required.
A.2 Evaluation of the Bessel Integral

We have the function

\[ p(n) = 2\pi \left( \frac{\pi}{12} \right)^{\frac{3}{2}} \sum_{k=1}^{\infty} k^{-5/2} A_k(n) \frac{1}{2\pi i} \int_{\pi/12 - \infty i}^{\pi/12 + \infty i} t^{-5/2} \exp \left\{ t + \frac{\pi^2}{6k^2} \left( n - \frac{1}{24} \right) \right\} \, dt, \]

and wish to evaluate the integral. Equation (B.1) implies that

\[ \left( \frac{z}{2} \right)^{\nu} \int_{\pi/12 - \infty i}^{\pi/12 + \infty i} t^{-\nu-1} \exp \left( t + \frac{z^2}{4t} \right) \, dt = I_\nu(z), \]

and if \( z \) and \( \nu \) in this integral are taken as

\[ z = \left\{ \frac{\pi^2}{6k^2} \left( n - \frac{1}{24} \right) \right\}^{\frac{1}{2}} \quad \text{and} \quad \nu = \frac{3}{2}, \]

we have

\[ \frac{1}{2\pi i} \int_{\pi/12 - \infty i}^{\pi/12 + \infty i} t^{-5/2} \exp \left\{ t + \frac{\pi^2}{6k^2} \left( n - \frac{1}{24} \right) \right\} \, dt = \left\{ \frac{\pi^2}{6k^2} \left( n - \frac{1}{24} \right) \right\}^{-\frac{3}{2}} I_{\frac{3}{2}} \left( \frac{\pi}{k} \sqrt{\frac{2}{3} \left( n - \frac{1}{24} \right)} \right). \]

Hence,

\[ p(n) = 2\pi \left( \frac{\pi}{12} \right)^{\frac{3}{2}} \sum_{k=1}^{\infty} k^{-5/2} A_k(n) \frac{\pi^{-\frac{3}{2}} (n - \frac{1}{24})^{-\frac{3}{2}}}{6^{-\frac{3}{4}} k^{-\frac{3}{2}}} I_{\frac{3}{2}} \left( \frac{\pi}{k} \sqrt{\frac{2}{3} \left( n - \frac{1}{24} \right)} \right) \]

\[ = \frac{2\pi (n - \frac{1}{24})^{-\frac{1}{4}}}{24^{\frac{1}{4}}} \sum_{k=1}^{\infty} k^{-1} A_k(n) I_{\frac{3}{2}} \left( \frac{\pi}{k} \sqrt{\frac{2}{3} \left( n - \frac{1}{24} \right)} \right). \]

As obtained in (B.2), Bessel functions of half odd order can be reduced, in this case

\[ I_{\frac{3}{2}} (z) = \sqrt{\frac{2z}{\pi}} \frac{d}{dz} \left( \frac{\sinh z}{z} \right). \]

This yields

\[ I_{\frac{3}{2}} \left( \frac{\pi}{k} \sqrt{\frac{2}{3} \left( n - \frac{1}{24} \right)} \right) = \frac{2^{\frac{1}{2}} \pi^{\frac{3}{4}} 2^{\frac{1}{4}}}{\pi^{\frac{1}{2}} k^{\frac{3}{4}} 3^{\frac{1}{4}}} \left( n - \frac{1}{24} \right)^{\frac{1}{4}} \frac{d}{dz} \left( \frac{\sinh \left\{ \frac{\pi}{k} \sqrt{\frac{2}{3} \left( n - \frac{1}{24} \right)} \right\} }{\frac{\pi}{k} \sqrt{\frac{2}{3} \left( n - \frac{1}{24} \right)} \right). \]

Because \( k \) is constant we have

\[ dz = \frac{\pi}{2k} \left( \frac{2}{3} \left( n - \frac{1}{24} \right) \right)^{-\frac{1}{2}} \frac{2}{3} \, dn, \]
and thus

\[ I_{3/2} \left( \frac{\pi}{k} \sqrt{\frac{2}{3}} (n - \frac{1}{24}) \right) = \frac{2^{3/4}}{\sqrt{3}} \left( n - \frac{1}{24} \right)^{1/2} \frac{2k}{\pi^{3/4}} \left( n - \frac{1}{24} \right)^{1/4} \frac{3k}{2\pi^{3/4}} \frac{d}{dn} \left( \sinh \left\{ \frac{\pi}{k} \sqrt{\frac{2}{3}} (n - \frac{1}{24}) \right\} \right) \]

= \frac{6^{3/4} k^{3/2}}{\pi^2} \left( n - \frac{1}{24} \right)^{3/4} \frac{d}{dn} \left( \sinh \left\{ \frac{\pi}{k} \sqrt{\frac{2}{3}} (n - \frac{1}{24}) \right\} \right).

Inserting this into our equation for \( p(n) \) we obtain

\[ p(n) = \frac{2\pi}{2^{3/4}} \left( n - \frac{1}{24} \right)^{-3/4} 6^{3/4} \left( n - \frac{1}{24} \right)^{3/4} \sum_{k=1}^{\infty} k^{-1} k^{3/2} A_k(n) \frac{d}{dn} \left( \sinh \left\{ \frac{\pi}{k} \sqrt{\frac{2}{3}} (n - \frac{1}{24}) \right\} \right) \]

= \frac{1}{\pi \sqrt{2}} \sum_{k=1}^{\infty} k^{3/2} A_k(n) \frac{d}{dn} \left( \sinh \left\{ \frac{\pi}{k} \sqrt{\frac{2}{3}} (n - \frac{1}{24}) \right\} \right). \]
B Appendix of Required Results

B.1 Bessel functions

Bessel functions of the first kind, which are solutions to the Bessel differential equation, are defined as

\[ J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{\nu+2m}}{m! \Gamma(\nu + m + 1)} \].

The modified Bessel functions of the first kind are consequently defined as

\[ I_\nu(z) = e^{-\frac{1}{2} \nu \pi i} J_\nu(iz) \]
\[ = i^{-\nu} J_\nu(iz) \]
\[ = \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{\nu+2m}}{m! \Gamma(\nu + m + 1)} \].

These functions can be described in a myriad of forms, and for our purposes a specific integral representation is required. Making use of the Gamma integral

\[ \frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-\nu} e^t \, dt, \]

we find that

\[ I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{m=0}^{\infty} \int_{c-i\infty}^{c+i\infty} \frac{\left(\frac{z}{2}\right)^{2m}}{m!} t^{-\nu-m-1} e^t \, dt. \]

Replacing the sum of integrals by an integral of sums and continuing:

\[ I_\nu(z) = \left(\frac{z}{2}\right)^\nu \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \sum_{m=0}^{\infty} \left(\frac{z}{2}\right)^{2m} \frac{1}{m!} t^{-\nu-m-1} e^t \, dt \]
\[ = \left(\frac{z}{2}\right)^\nu \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp \left\{ \left(\frac{z}{2}\right)^2 \frac{1}{t} \right\} t^{-\nu-1} e^t \, dt. \]

This yields the necessary integral equation, in which \( c > 0 \) and \( \text{Re}(\nu) > 0 \):

\[ I_\nu(z) = \left(\frac{1}{2} \frac{z}{2}\right)^\nu \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-\nu-1} \exp \left( t + \frac{z^2}{4t} \right) \, dt. \]  

(B.1)

Bessel functions of half-odd order, i.e., those with \( \nu = n + \frac{1}{2} \), where \( n \in \mathbb{Z} \), can be expressed in finite terms by means of algebraic and trigonometrical functions of \( z \). In particular,

\[ I_{\frac{n}{2}}(z) = \sqrt{\frac{2z}{\pi}} \frac{d}{dz} \left( \frac{\sinh z}{z} \right). \]  

(B.2)
The proof of this result is rather technical. We rework the right and left hand sides respectively to a mutual ‘middle-point’.

\[
\sqrt{\frac{2z}{\pi}} \frac{d}{dz} \left( \frac{\sinh \, z}{z} \right) = \sqrt{\frac{2z}{\pi}} \left( \frac{\cosh \, z - \sinh \, z}{z^2} \right)
\]

\[
= \sqrt{\frac{2z}{\pi}} \left\{ \left( \frac{1}{2!} - \frac{1}{3!} \right) z + \left( \frac{1}{4!} - \frac{1}{5!} \right) z^3 + \left( \frac{1}{6!} - \frac{1}{7!} \right) z^5 + \cdots \right\}
\]

\[
= \sqrt{\frac{2z}{\pi}} \sum_{m=0}^{\infty} \left\{ \frac{1}{(2m+2)!} - \frac{1}{(2m+3)!} \right\} z^{2m+1}
\]

\[
= \sqrt{\frac{2z}{\pi}} \sum_{m=0}^{\infty} \frac{2m+2}{(2m+3)!} z^{2m+1}
\]

\[
= \sqrt{\frac{2z}{\pi}} \sum_{m=0}^{\infty} \frac{z^{2m+1}}{(2m)! (4m^2 + 8m + 3)}.
\]

Before we alter the left hand side, we require a few extra properties of the Gamma function, namely that

\[
\Gamma(z + 1) = z \Gamma(z) \quad \text{and} \quad \Gamma(n + \frac{1}{2}) = \frac{(2n - 1)!!}{2^n} \sqrt{\pi},
\]

and also a trick involving (double) factorials:

\[
2^n m! (2m - 1)!! = (2m)! (2m - 1)!! = (2m)!. \]

Now, we proceed as follows:

\[
I_\frac{1}{2}(z) = \left( \frac{z}{2} \right)^\frac{3}{2} \sum_{m=0}^{\infty} \frac{\left( \frac{z}{2} \right)^{2m}}{m! \Gamma(m + \frac{3}{2})}
\]

\[
= \left( \frac{z}{2} \right)^\frac{3}{2} \sum_{m=0}^{\infty} \frac{\left( \frac{z}{2} \right)^{2m}}{m! (m + \frac{1}{2}) (m + \frac{3}{2}) (m + \frac{5}{2})}
\]

\[
= \frac{1}{\sqrt{\pi}} \left( \frac{z}{2} \right)^\frac{3}{2} \sum_{m=0}^{\infty} \frac{\left( \frac{z}{2} \right)^{2m} 2^m}{m! (2m - 1)!! (m^2 + 2m + \frac{3}{4})}
\]

\[
= \frac{1}{4} \sqrt{\frac{2z}{\pi}} \sum_{m=0}^{\infty} \frac{z^{2m+1}}{m! (2m - 1)!! (m^2 + 2m + \frac{3}{4})}
\]

\[
= \sqrt{\frac{2z}{\pi}} \sum_{m=0}^{\infty} \frac{z^{2m+1}}{(2m)! (4m^2 + 8m + 3)}.
\]

and obtain the desired result, equation (B.2).

For a thorough resource on Bessel functions the reader is referred to Watson (1966), specifically pages 15, 52-54, 77 and 181.
C Supplementary Figures

C.1 The Rademacher path $P(N)$

Figure C.1: $P(5)$ and $P(20)$.  

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References


