Interpolatory refinement pairs with properties of symmetry and polynomial filling

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Subdivision schemes are important and efficient tools for generating smooth curves and surfaces used in computer aided geometric design (CAGD) and in wavelet decomposition.

- Suppose that we want to draw or design some curve or surface, using a computer graphic tool. We would like the curve or surface to be smooth, so as to be pleasing to the eye. A subdivision scheme provides a way of rendering the curve or surface: given a number of points, chosen by the user to determine the shape of the desired curve (we will call these points control points), this method determines a rule for iteratively generating new points from existing ones. With each iteration, the points become more dense, until it resembles the desired curve.

- We consider in this talk particularly interpolatory subdivision schemes (ISS), which have the property that the limit curve interpolates the original control points.
Subdivision schemes

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- We consider in this talk particularly interpolatory subdivision schemes (ISS), which have the property that the limit curve interpolates the original control points.
Outline

1. Introduction
2. A class of symmetric interpolatory subdivision scheme (ISS)
3. An explicit characterisation of the class $A_{m,n}$
4. Interpolatory refinable function existence
Outline

1. Introduction
   - Notation
   - Basic definitions

2. A class of symmetric interpolatory subdivision scheme (ISS)

3. An explicit characterisation of the class $A_{m,n}$

4. Interpolatory refinable function existence
Notation

- $\mathbb{Z}_+ :=$ the set of nonnegative integers
- $\sum := \sum_{j \in \mathbb{Z}}$
- $\mathcal{M}(\mathbb{Z}) :=$ the set of bi-infinite real-valued sequences, i.e., $c \in \mathcal{M}(\mathbb{Z}) \iff c = \{c_j : j \in \mathbb{Z}\} \subset \mathbb{R}$
- $\text{supp}(c) :=$ support of the sequence $c$, i.e., $\text{supp}(c) = \{j : c_j \neq 0\}$
- $\mathcal{M}_0(\mathbb{Z}) :=$ the set of sequences in $\mathcal{M}(\mathbb{Z})$ with finite support
Basic definitions

For a given sequence \( a = \{a_j\} \in M_0(\mathbb{Z}) \), we define the subdivision operator \( S_a : M(\mathbb{Z}) \rightarrow M(\mathbb{Z}) \) by

\[
(S_a c)_j = \sum_k a_{j-2k}c_k, \quad j \in \mathbb{Z}.
\]

For a given initial control point sequence \( c = \{c_j : j \in \mathbb{Z}\} \in M(\mathbb{Z}) \), the subdivision scheme \( (S_a, c) \) generates the sequence \( \{c^{(r)} : r \in \mathbb{Z}_+\} \subset M(\mathbb{Z}) \) by means of

\[
c^{(0)} = c, \quad c^{(r+1)} = S_a c^{(r)}, \quad r \in \mathbb{Z}_+
\]

\[\Leftrightarrow\]

\[
c^{(r)} = S_a^r c, \quad r \in \mathbb{Z}_+.
\]

\( a = \text{subdivision mask}. \) The corresponding mask symbol is the Laurent polynomial \( A \) defined by

\[
A(z) = \sum_j a_jz^j, \quad z \in \mathbb{C}\setminus\{0\}.
\]
Outline

1. Introduction

2. A class of symmetric interpolatory subdivision scheme (ISS)
   - The interpolatory condition
   - The symmetry condition
   - The polynomial filling condition
   - The class $A_{m,n}$

3. An explicit characterisation of the class $A_{m,n}$

4. Interpolatory refinable function existence
The interpolatory condition

**Definition**

For a given initial sequence \( c \in \mathcal{M}(\mathbb{Z}) \), we say that \((S_{a}, c)\) is an **interpolatory** subdivision scheme (ISS) if and only if it holds that

\[
c^{(r+1)}_{2j} = c^{(r)}_{j}, \quad j \in \mathbb{Z}, \quad r \in \mathbb{Z}^{+}
\]

\(\iff\)

\[
(S_{a} c)_{2j} = c_{j}, \quad j \in \mathbb{Z}, \quad c \in \mathcal{M}(\mathbb{Z})
\]

\(\iff\)

\[
a_{2j} = \delta_{j} = \begin{cases} 
1, & j = 0, \\
0, & j \neq 0,
\end{cases} \quad j \in \mathbb{Z}
\]

\(\iff\)

\[
A(z) + A(-z) = 2, \quad z \in \mathbb{C}\setminus\{0\}
\]

\(a \in \mathcal{M}_{0}(\mathbb{Z})\) is called an **interpolatory mask**
Example of an interpolatory subdivision scheme (ISS)

\[ \left\{ c_j^{(0)} : j \in \mathbb{Z} \right\} \in \mathbb{R}^2 \]

\[
\begin{align*}
  c_{2j}^{(1)} &= c_j^{(0)} \\
  c_{2j+1}^{(1)} &= \frac{1}{2} \left( c_j^{(0)} + c_{j+1}^{(0)} \right)
\end{align*}
\]
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Example of an interpolatory subdivision scheme (ISS)

Example

\[
\left\{ c_j^{(0)} : j \in \mathbb{Z} \right\} \in \mathbb{R}^2
\]

\[
\begin{align*}
   c^{(1)}_{2j} &= c_j^{(0)} \\
   c^{(1)}_{2j+1} &= \frac{1}{2} \left( c_j^{(0)} + c_{j+1}^{(0)} \right)
\end{align*}
\]
The symmetry condition

**Definition**

We say that the subdivision operator $S_a$ yields a symmetric subdivision scheme if and only if it holds for $a \in M_0(\mathbb{Z})$ that

$$a_j = a_{-j}, \quad j \in \mathbb{Z}$$

$$\Leftrightarrow$$

$$A(z) = A(z^{-1}), \quad z \in \mathbb{C}\{0\}$$

i.e., $A$ is a symmetric Laurent polynomial.

**Proposition**

Suppose $a \in M_0(\mathbb{Z})$ is the mask corresponding to a symmetric ISS. Then

(i) there is $n \in \mathbb{N}$ such that

$$a_j = 0, \quad j \notin \{-2n+1, \ldots, 2n-1\}, \text{ with } a_{-2n+1} \neq 0, \ a_{2n-1} \neq 0.$$  

(ii) $A(e^{ix}) \in \mathbb{R}, \quad x \in \mathbb{R}$.  


The polynomial filling condition

\[ \pi_n := \text{the linear space of polynomials of degree } \leq n, \text{ where } n \in \mathbb{Z}_+ \]

**Definition**

An ISS is said to possess the \( \pi_{2m-1} \) polynomial filling property if and only if the corresponding mask \( a \in M_0(\mathbb{Z}) \) satisfies the property

\[
\sum_k a_{2j+1-2k} p(k) = p\left(j + \frac{1}{2}\right), \quad j \in \mathbb{Z}, \quad p \in \pi_{2m-1} \tag{1}
\]

and where \( m \) is the largest integer for which (1) holds.

\[ \Leftrightarrow \]

\[ \exists \text{ a symmetric Laurent polynomial } C \text{ such that } \]

\[ A(z) = \frac{1}{2^{m-1}} \left(1 + \frac{z + z^{-1}}{2}\right)^m C(z), \quad z \in \mathbb{C} \setminus \{0\} \]

where

\[ C(1) = 1, \quad C(-1) \neq 0. \]
A class $A_{m,n}$ of Laurent polynomials

$[x] :=$ the smallest integer $\geq x$

**Definition (De Villiers & Hunter 2006 I)**

For $m, n \in \mathbb{N}$, with $n \geq \lceil \frac{m+1}{2} \rceil$, we define the class $A_{m,n}$ of L. polynomials

$$A \in A_{m,n} \iff A(z) = \sum_{j} a_{j} z^{j} = \sum_{j=-2n+1}^{2n-1} a_{j} z^{j}, \quad z \in \mathbb{C}\{0\}, \quad \text{with } a_{-2n+1} \neq 0, \quad a_{2n-1} \neq 0,$$

$$A(z) + A(-z) = 2, \quad z \in \mathbb{C}\{0\},$$

$$A(z^{-1}) = A(z), \quad z \in \mathbb{C}\{0\},$$

$\exists$ a symmetric Laurent polynomial $C$ such that

$$A(z) = \frac{1}{2^{m-1}} \left(1 + \frac{z + z^{-1}}{2}\right)^{m} C(z), \quad z \in \mathbb{C}\{0\}$$

where

$$C(1) = 1, \quad C(-1) \neq 0.$$
Outline

1. Introduction
2. A class of symmetric interpolatory subdivision scheme (ISS)
3. An explicit characterisation of the class $A_{m,n}$
   - Statement of the main result
4. Interpolatory refinable function existence
An explicit characterisation of the class $A_{m,n}$

**Theorem (De Villiers & Gavhi (2007))**

For $m, n \in \mathbb{N}$, the class $A_{m,n} \neq \emptyset \iff n \geq m$. Moreover, a Laurent polynomial $A \in A_{m,n} \iff$

$$A(z) = \frac{1}{2^{m-1}} (1+\zeta)^m \sum_{j=0}^{m-1} \binom{m+j-1}{j} \left[ \frac{1}{2} (1-\zeta) \right]^j + \zeta(1-\zeta)^m P(\zeta^2), \quad z \in \mathbb{C}\{0\}$$  \hspace{1cm} (2)

where

$$\zeta = \frac{z+z^{-1}}{2}, \quad z \in \mathbb{C}\{0\}$$

and where either $P = 0$, in which case $n = m$, or $P$ is an arbitrary polynomial, with $\deg(P) = n - m - 1$ if $n \geq m + 1$, satisfying the condition

$$P(1) \neq \frac{1}{2^m} \binom{2m-1}{m-1}.$$  

In addition,

$$A_{m,m} = \{A_m\}$$

where the Laurent polynomial $A_m$ is obtained by choosing $P = 0$ in (2), i.e.

$$A_m(z) = \frac{1}{2^{m-1}} (1+\zeta)^m \sum_{j=0}^{m-1} \binom{m+j-1}{j} \left[ \frac{1}{2} (1-\zeta) \right]^j, \quad z \in \mathbb{C}\{0\}.$$
Remark

Since $A_m \in \mathcal{A}_{m,m}$, $\exists$ a sequence $\{a_{m,j}, \ j \in \mathbb{Z}\} \in \mathcal{M}_0(z)$, with

$$a_{m,j} = 0, \ j \notin \{-2m+1, \ldots, 2m-1\}, \ a_{m,-2m+1} \neq 0, \ a_{m,2m-1} \neq 0,$$

$$a_{m,2j} = \delta_j, \ j \in \mathbb{Z}$$

and

$$a_{m,-j} = a_{m,j}, \ j \in \mathbb{Z}$$

such that

$$A_m(z) = \sum_{j=-2m+1}^{2m-1} a_{m,j}z^j, \ z \in \mathbb{C}\{0\}.$$

Observe that

$$A_m(z) = 1 + \sum_{j=-m}^{m-1} a_{m,2j+1}z^{2j+1}, \ z \in \mathbb{C}\{0\}. \quad (3)$$

We proceed to introduce an explicit closed formula for the mask coefficients $\{a_{m,2j+1} : j = -m, \ldots, m - 1\}$ in (3).
A explicit formula for \( \{a_{m,2j+1} : j = -m, \ldots, m - 1 \} \)

De Villiers, Hunter & Herbst 2003

A minimally supported mask \( a \in M_0(z) \) such that

\[
\sum_k a_{2j+1 - 2k} p(k) = p(j + 1/2), \quad j \in \mathbb{Z}, \quad p \in \pi_{2m-1}
\]

is the mask \( a = d_m = \{d_{m,j} : j \in \mathbb{Z}\} \in M_0(\mathbb{Z}) \) given by

\[
\begin{align*}
    d_{m,2j} &= \delta_j, \quad j \in \mathbb{Z}, \\
    d_{m,2j+1} &= \frac{m}{2^{4m-3}} \left( \frac{2m-1}{m} \right)^j \left( \frac{-1}{1+2j} \right) \left( \frac{2m-1}{m+j} \right), \quad j = -m, \ldots, m - 1, \\
    d_{m,2j+1} &= 0, \quad j \notin \{-m, \ldots, m - 1\}.
\end{align*}
\]

The subdivision scheme based on (4) was first introduced by Dubuc-Deslauriers (DD) (1986), and shall be referred to as DD subdivision. The corresponding DD mask symbol is defined by

\[
D_m(z) = \sum_j d_{m,j} z^j, \quad z \in \mathbb{C}\setminus\{0\}.
\]

From (4) we deduce that \( D_m \in A_{m,m} \). It follows that \( A_m = D_m \).
Alternative explicit formulation

**Corollary**

For $m, n \in \mathbb{N}$, the class $\mathcal{A}_{m,n} \neq \emptyset \iff n \geq m$. Moreover, $A \in \mathcal{A}_{m,n} \iff$

$$A(z) = D_m(z) + \zeta \left(1 - \zeta^2 \right)^m Q(\zeta^2), \quad z \in \mathbb{C}\{0\}$$

where $\zeta = \frac{z + z^{-1}}{2}$, $z \in \mathbb{C}\{0\}$ and where either $Q = 0$, if $n = m$, or $Q$ denotes an arbitrary polynomial, with $\deg(Q) = n - m - 1$ if $n \geq m + 1 \left( = \frac{1}{2^{m-1}} P \right)$ satisfying the condition $Q(1) \neq \frac{1}{2m} \binom{2m-1}{m-1}$. In addition, $\mathcal{A}_{m,m} = \{D_m\}$ where $D_m$ is the DD mask symbol as given by the explicit formula

$$D_m(z) = 1 + \frac{m}{2^{4m-3}} \binom{2m-1}{m} \sum_{j=-m}^{m-1} \frac{(-1)^j}{1 + 2j} \binom{2m-1}{m+j} z^{2j+1}, \quad z \in \mathbb{C}\{0\}.$$ 

**Theorem (JdV & Hunter 2006 I)**

For $m, n \in \mathbb{N}$, with $n \geq m$,

$$A \in \mathcal{A}_{m,n} \iff \left\{ \begin{array}{l}
A = \sum_{j=0}^{n-m} t_j D_{m+j} \\
\sum_{j=0}^{n-m} t_j = 1.
\end{array} \right.$$
Special cases of $A_{m,n}$

**Example (The case $n = m$)**

- $Q = 0 \Rightarrow A = A_m = D_m$

\[
\begin{align*}
D_1(z) &= \frac{1}{2} z^{-1} + 1 + \frac{1}{2} z, \\
D_2(z) &= -\frac{1}{16} z^{-3} + \frac{9}{16} z^{-1} + 1 + \frac{9}{16} z - \frac{1}{16} z^3, \\
& \quad z \in \mathbb{C} \setminus \{0\}.
\end{align*}
\]

**Example (The case $n = m + 1$)**

- $Q(z) = t, \ z \in \mathbb{C}$, with $t$ denoting an arbitrary non-zero constant (independent of $z$) such that $t \neq \frac{1}{2^{2m-1}} \binom{2m-1}{m-1}$

\[
\Rightarrow \quad A = A_m(t|\cdot)
\]

\[
A_1(t|z) = \frac{-t}{8} z^{-3} + \frac{4+t}{8} z^{-1} + 1 + \frac{4+t}{8} z - \frac{t}{8} z^3, \quad z \in \mathbb{C} \setminus \{0\}
\]
Outline

1. Introduction
2. A class of symmetric interpolatory subdivision scheme (ISS)
3. An explicit characterisation of the class $\mathcal{A}_{m,n}$
4. Interpolatory refinable function existence
   - Refinement pairs
   - A fundamental existence result
   - Positivity on the unit circle in $\mathbb{C}$ for $\mathcal{A}_{m,m}$
   - The positivity condition for $\mathcal{A}_{m,m+1}$
Refinement pairs

- \( C(\mathbb{R}) := \) the linear space of continuous functions on \( \mathbb{R} \)
- \( C_0(\mathbb{R}) := \{f \in C(\mathbb{R}) : \text{the function } f \text{ is finitely supported}\} \)

**Definition**

If a sequence \( a = \{a_j\} \in M_0(\mathbb{Z}) \) and a function \( \phi \in C_0(\mathbb{R}) \) are such that the refinement equation

\[
\phi(x) = \sum_j a_j \phi(2x - j), \quad x \in \mathbb{R},
\]

is satisfied, we say that \((a, \phi)\) is a refinement pair.

- \( a \) is called the refinement mask
- \( \phi \) is the refinable function corresponding to the mask \( a \)

If moreover,

\[
\phi(j) = \delta_j, \quad j \in \mathbb{Z}
\]

\[\Rightarrow \quad a_{2j} = \delta_j, \quad j \in \mathbb{Z}\]

we say that \((a, \phi)\) is an interpolatory refinement pair.
Example of interpolatory refinable function (IRF)

**Example (The “hat” function)**

Consider the function $\phi = h$,

$$
\begin{align*}
\phi(x) &= \begin{cases} 
1 + x, & -1 < x \leq 0; \\
1 - x, & 0 < x < 1; \\
0, & x \in \mathbb{R} \setminus (-1, 1);
\end{cases}
\end{align*}
$$

which satisfies

$$
\phi(x) = \frac{1}{2} \phi(2x + 1) + \phi(2x) + \frac{1}{2} \phi(2x - 1), \quad x \in \mathbb{R}.
$$

Since $\phi(j) = \delta_j$, $j \in \mathbb{Z}$, we see that $\phi = h$ is IRF with $\{a_j\}$ given by

$$
\{a_{-1}, a_0, a_1\} = \left\{\frac{1}{2}, 1, \frac{1}{2}\right\}, \quad a_j = 0, \quad |j| > 1.
$$
Example of IRF Cont.

\[ h(x) = \frac{1}{2} h(2x + 1) + h(2x) + \frac{1}{2} h(2x - 1), \quad x \in \mathbb{R}. \]
The existence of interpolatory refinable function

**Theorem (JdV, Hunter & Herbst 2003)**

For $m, n \in \mathbb{N}$, with $n \geq m$, suppose $A \in \mathcal{A}_{m,n}$. If $\exists \, \phi \in C_0(\mathbb{R}) \ni (a, \phi)$ is an interpolatory refinement pair, then $\phi$ is the unique solution in $C_0(\mathbb{R})$ of the refinement equation (5), and $\phi$ satisfies the following properties:

\[
\begin{align*}
\phi(x) &= 0, \quad x \in (-2n+1, 2n-1) \\
\sum_j \phi(x-j) &= 1, \quad x \in \mathbb{R} \\
\phi(-x) &= \phi(x), \quad x \in \mathbb{R} \\
\sum_j p(j)\phi(x-j) &= p(x), \quad x \in \mathbb{R}, \quad p \in \pi_{2m-1} \\
\phi\left(j + \frac{1}{2}\right) &= a_{2j+1}, \quad j \in \mathbb{Z}.
\end{align*}
\]

Moreover, the corresponding ISS $(S_a, c)$ is convergent on $\mathcal{M}(\mathbb{Z})$ for every initial sequence $c \in \mathcal{M}(\mathbb{Z})$, in the sense that the function $\Phi_c \in C(\mathbb{R})$ defined by

\[\Phi_c(x) = \sum_j c_j \phi(x-j), \quad x \in \mathbb{R}\]

satisfies

\[c_j^{(r)} = \Phi_c\left(\frac{j}{2^r}\right), \quad j \in \mathbb{Z}, \quad r \in \mathbb{Z}_+.
\]

$\Phi_c$ is the limit function of ISS.
We present a sufficient condition on a mask symbol $A$ for the existence of corresponding IRF $\phi$, and thus for the convergence of the associated ISS, as was proved by [Micchelli, 1996].

**Theorem (Micchelli’s theorem)**

For $m, n \in \mathbb{N}$, with $n \geq m$, suppose $A \in \mathcal{A}_{m,n}$. If, moreover,

$$A(e^{ix}) > 0, \quad x \in (-\pi, \pi)$$

then there exists a refinable function $\phi \in \mathcal{C}_0(\mathbb{R})$ such that $(a, \phi)$ is an interpolatory refinement pair.
Motivation

Outline of today’s talk

Introduction

A class of symmetric interpolatory subdivision scheme (ISS)

An explicit characterisation of the class $\mathcal{A}_{m,m}$

Positivity on the unit circle in $\mathbb{C}$ for $\mathcal{A}_{m,m}$

Recall:

\[
\mathcal{A}_{m,m} = \{A_m\} = \{D_m\}
\]

where

\[
D_m(z) = \frac{1}{2^{m-1}} \left(1 + \frac{z+z^{-1}}{2}\right)^{m-1} \sum_{j=0}^{m-1} \binom{m+j-1}{j} \left[\frac{1}{2} \left(1 - \frac{z+z^{-1}}{2}\right)\right]^j, \quad z \in \mathbb{C}\setminus\{0\}.
\]

Theorem (5)

For $m \in \mathbb{N}$, the DD mask symbol $D_m$, as defined by (6) and (7), satisfies

\[
D_m(e^{ix}) > 0, \quad x \in (-\pi, \pi).
\]

Combining the results of Theorem 5 and Micchelli’s theorem, we thus have the following result.

Corollary (Convergence of DD subdivision)

For $m \in \mathbb{N}$, there exists a unique refinable function $\phi = \phi^D_m \in \mathcal{C}_0(\mathbb{R})$ such that $(d_m, \phi^D_m)$ is an interpolatory refinement pair. Moreover, the DD subdivision scheme $(S_{d_m}, c)$ converges for every initial sequence $c \in \mathcal{M}(\mathbb{Z})$.

- $d_m \in \mathcal{M}_0(\mathbb{Z}) = \text{DD mask of order } m$
- $\phi^D_m = \text{DD refinable function of order } m$
Positivity on the unit circle in $\mathbb{C}$ for $A_{m,m}$

Recall:

\[ A_{m,m} = \{A_m\} = \{D_m\} \]  \hspace{1cm} (6)

where

\[ D_m(z) = \frac{1}{2^{m-1}} \left( 1 + \frac{z + z^{-1}}{2} \right)^m \sum_{j=0}^{m-1} \left( \begin{array}{c} m+j-1 \\ j \end{array} \right) \left[ \frac{1}{2} \left( 1 - \frac{z + z^{-1}}{2} \right) \right]^j, \quad z \in \mathbb{C} \setminus \{0\}. \]  \hspace{1cm} (7)

Theorem (5)

For $m \in \mathbb{N}$, the DD mask symbol $D_m$, as defined by (6) and (7), satisfies

\[ D_m(e^{ix}) > 0, \quad x \in (-\pi, \pi). \]

Combining the results of Theorem 5 and Micchelli’s theorem, we thus have the following result.

Corollary (Convergence of DD subdivision)

For $m \in \mathbb{N}$, there exists a unique refinable function $\phi = \phi^D_m \in C_0(\mathbb{R})$ such that $(d_m, \phi^D_m)$ is an interpolatory refinement pair. Moreover, the DD subdivision scheme $(S_{d_m}, c)$ converges for every initial sequence $c \in M(\mathbb{Z})$.

- $d_m \in M_0(\mathbb{Z}) = \text{DD mask of order } m$
- $\phi^D_m = \text{DD refinable function of order } m$
The DD refinable function $\phi^D_m$ for $m = 2$

$$D_2(z) = -\frac{1}{16}z^{-3} + \frac{9}{16}z^{-1} + 1 + \frac{9}{16}z - \frac{1}{16}z^3, \quad z \in \mathbb{C}\setminus\{0\}$$

- The DD refinable function $\phi^D_2$
- The convergence of the ISS with $D_2$
- 1st iteration
- 2nd iteration
- 3rd iteration
- 4th iteration
The DD refinable function $\phi^D_m$ for $m = 2$

$$D_2(z) = -\frac{1}{16}z^{-3} + \frac{9}{16}z^{-1} + 1 + \frac{9}{16}z - \frac{1}{16}z^3, \ z \in \mathbb{C}\{0\}$$

- The DD refinable function $\phi^D_2$
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  - 2\textsuperscript{nd} iteration
  - 3\textsuperscript{rd} iteration
  - 4\textsuperscript{th} iteration
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- 2\textsuperscript{nd} iteration
- 3\textsuperscript{rd} iteration
- 4\textsuperscript{th} iteration
The DD refinable function $\phi_{D}^{2}$ for $m = 2$

$$D_2(z) = -\frac{1}{16}z^{-3} + \frac{9}{16}z^{-1} + 1 + \frac{9}{16}z - \frac{1}{16}z^3, \quad z \in \mathbb{C}\{0\}$$

- The DD refinable function $\phi_{D}^{2}$
- The convergence of the ISS with $D_2$
- 1st iteration
- 2nd iteration
- 3rd iteration
- 4th iteration
The DD refinable function $\phi^D_m$ for $m = 2$

$$D_2(z) = -\frac{1}{16}z^{-3} + \frac{9}{16}z^{-1} + 1 + \frac{9}{16}z - \frac{1}{16}z^3, \quad z \in \mathbb{C}\{0\}$$

- The DD refinable function $\phi^D_2$
- The convergence of the ISS with $D_2$
- 1st iteration
- 2nd iteration
- 3rd iteration
- 4th iteration
The DD refinable function $\phi^D_m$ for $m = 2$

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- The DD refinable function $\phi^D_2$
- The convergence of the ISS with $D_2$
- 1\textsuperscript{st} iteration
- 2\textsuperscript{nd} iteration
- 3\textsuperscript{rd} iteration
- 4\textsuperscript{th} iteration
The positivity condition for $A_{m,m+1}$

Here, we investigate the positivity condition of Micchelli’s theorem as it applies, for $m \in \mathbb{N}$, to the mask symbol class

$$\mathcal{A}_{m,m+1} = \left\{ A_m(t \cdot) : t \in \mathbb{R} \setminus \left\{ 0, \frac{1}{2^{m-1}} \left( \frac{2m-1}{m-1} \right) \right\} \right\}$$

as was explicitly calculated in Example 2.

Recall:

$$A \in \mathcal{A}_{m,m+1} \iff A(z) = A_m(t | z) = \frac{1}{2^{m-1}} (1 + \zeta)^m \left[ \sum_{j=0}^{m-1} \binom{m+j-1}{j} \left[ \frac{1}{2} (1 - \zeta) \right]^j + 2^{m-1} t \zeta (1 - \zeta)^m \right]$$

where

$$\zeta = \frac{z + z^{-1}}{2}, \quad z \in \mathbb{C} \setminus \{0\}.$$
The positivity condition for $\mathcal{A}_{m,m+1}$ (cont.)

Proposition

For $m \in \mathbb{N}$ and $t \in \mathbb{R} \setminus \left\{0, \frac{1}{2^{m-1}} \binom{2m-1}{m-1}\right\}$, the mask symbol $A = A_m(t|\cdot) \in \mathcal{A}_{m,m+1}$ satisfies

$$A_m(t|e^{ix}) > 0, \quad x \in (-\pi, \pi)$$

if and only if the polynomial $p_m(t|\cdot)$ of degree $(m+1)$, as defined by

$$p_m(t|x) = p_m(x) + 2^{m-1}tx^m(1-2x), \quad x \in \mathbb{R}$$

with $p_m$ denoting the polynomial of degree $(m-1)$ defined by

$$p_m(x) = \sum_{j=0}^{m-1} \binom{m+j-1}{j} x^j, \quad x \in \mathbb{R}$$

satisfies the positivity condition

$$p_m(t|x) > 0, \quad x \in [0,1).$$
The positivity condition for $A_{m,m+1}$ (cont.)

Theorem (De Villiers & Gavhi (2007))

The positivity condition $p_m(t | x) > 0$, $x \in [0,1)$ holds if and only if

$$t \in (-t_m, \tilde{t}_m]$$

where $t_m$ is the positive number defined by

$$t_m = \frac{p_m(x_m)}{2^{2m-1}x_m(1-2x_m)},$$

with $x_m$ denoting the unique zero in $(0,1/2)$ of the polynomial $q_m$ defined by

$$q_m(x) = [m - 2(m+1)x]p_m(x) - x(1-2x)p'_m(x), \quad x \in \mathbb{R}$$

and where $\tilde{t}_m$ is the positive number defined by

$$\tilde{t}_m = \frac{1}{2^{2m-1}} \binom{2m-1}{m-1}.$$
The existence of a refinable function $\phi_m(t|\cdot)$

Combine the results of Theorem 3 and Micchelli’s theorem.

**Corollary**

For $m \in \mathbb{N}$, $\exists$ a function $\phi_m(t|\cdot) \in \mathcal{C}_0(\mathbb{R})$ such that $\{a^{(m)}(t), \phi_m(t|\cdot)\}$ is an interpolatory refinement pair if $t \in (-t_m, \tilde{t}_m]$. Moreover, the ISS $\left(S_{a^{(m)}(t)}, c\right)$ converges for every initial sequence $c \in \mathcal{M}(\mathbb{Z})$.

**Example (The case $m = 1$)**

The class of masks $\{A_1(t|\cdot) : t \in \mathbb{R}\}$, as given by

$$A_1(t|z) = \frac{-t}{8}z^{-3} + \frac{4+t}{8}z^{-1} + 1 + \frac{4+t}{8}z - \frac{t}{8}z^3, \quad z \in \mathbb{C}\backslash\{0\}$$

yields refinable function existence and resulting subdivision convergence if

$$t \in (-4, 1/2].$$

The corresponding refinable function $\phi_1(t|\cdot)$ and the convergence of the corresponding ISS for the values $t = -3.9, -2, 0.25, 0.5$ is shown.
The existence of a refinable function $\phi_m(t|\cdot)$

Combine the results of Theorem 3 and Micchelli’s theorem.

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For $m \in \mathbb{N}$, $\exists$ a function $\phi_m(t|\cdot) \in C_0(\mathbb{R})$ such that $\left(a^{(m)}(t), \phi_m(t|\cdot)\right)$ is an interpolatory refinement pair if $t \in (-t_m, \tilde{t}_m]$. Moreover, the ISS $\left(S_{a^{(m)}(t)}, c\right)$ converges for every initial sequence $c \in \mathcal{M}(\mathbb{Z})$.

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Motivation Outline of today’s talk

Introduction
A class of symmetric interpolatory subdivision scheme (ISS)

An explicit characterisation of the class $\mathbf{A}_{m,n}$

Interpolator

$$A_1(t|z) = \frac{1}{8}(-tz^{-3} + (4+t)z^{-1} + 8 + (4+t)z - tz^3), \quad z \in \mathbb{C}\setminus\{0\}$$

(a) $\phi_1(-3.9|\cdot)$

(b) $\phi_1(-3|\cdot)$

(c) $\phi_1(-2|\cdot)$

(d) $\phi_1(-1|\cdot)$
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(f) $\phi_1(0) = \phi_1^D$

(g) $\phi_1(0.25)$

(a) $\phi_1(0.5) = \phi_2^D$
The convergence of the ISS with $A_1(t|\cdot)$

**Original control points**

- **(a) $t = -3.9$**
- **(b) $t = -3$**
- **(c) $t = -2$**
The convergence of the ISS with $A_1(t|\cdot)$
A general class of symmetric interpolatory subdivision schemes (ISS) with the property of polynomial filling up to a degree $2m - 1$ was introduced.

We gave an explicit characterization for the above class, in the process also showing that its optimally local member is the well-known DD mask symbol $D_m$ of order $m$.

For the case of one degree of freedom in the form of a shape parameter $t$, we establish an interval for $t$ in which refinable function existence, and therefore also subdivision convergence, are guaranteed.
Conclusion

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Current work

- The **regularity (or minimum degree of smoothness)** of $\phi_m(t|\cdot) \in \mathcal{C}_0(\mathbb{R})$: How regular (smooth) is $\phi_m(t|\cdot)$? For instance, how many continuous derivates does $\phi_m(t|\cdot)$ possess?

- A modified subdivision scheme for finite sequences: The algorithms for bi-infinite sequences, as described in this talk, are applied mainly in the case of periodic sequences. For finite sequences these algorithms must be modified to accommodate the boundaries. How do we adapt the one-parameter ISS to the situation where the initial sequence $c$ is finite.

- The role of shape parameter $t$: How to analyzed the fractal properties of one-parameter interpolatory subdivision schemes?
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THANK YOU