Logic and Algebra Day 2010

2 October 2010

Mathematics Division, Department of Mathematical Sciences
University of Stellenbosch

09h00 - 09h45	Zurab Janelidze *Algebraic importance of "modus ponens"*
09h50 - 10h20	Gareth Boxall *NIP* (*Not the Independence Property*)
10h25 - 10h55	James Gray *Algebraic exponentiation*
11h00 - 11h15	Tea/coffee Mathematics Tea Room
11h15 - 12h15	Alessandra Palmigiano *Groupoid quantales beyond the étale setting*
12h20 - 12h50	Willem Conradie *Algorithmic canonicity and correspondence for* (*non-distributive*) *lattice-based modal logic*
12h55 - 14h25	Lunch Stellenbosch Botanical Garden
14h30 - 15h15	James Raftery *Idempotent residuated structures and finiteness conditions*
15h20 - 15h50	Tamar Janelidze *Relative Goursat categories*
15h55 - 16h25	Clint van Alten *Representable Ideal-determined Varieties*
16h30 - 16h45	Tea/Coffee Mathematics Tea Room
16h45 - 17h15	Marcel Wild *Enumerating all models of a Horn formula, e.g. all closed sets of a closure system*
17h20 - 17h50	George Janelidze *What shall be 2-dimensional topology of first order logic*?

Venue: Van der Sterr building, Room 3021.

Contact: Prof Ingrid Rewitzky (rewitzky@sun.ac.za)

Logic and Algebra Day: Abstracts

NIP (Not the Independence Property)

Gareth Boxall
Department of Mathematical Sciences, University of Stellenbosch

NIP (Not the Independence Property) is a property which some first order theories possess. It was defined by Shelah in a paper of 1971 but many of the big discoveries belong to recent work. It is a topic of current interest within model theory. I shall describe some of the main results and also some open problems.

Algorithmic canonicity and correspondence for (non-distributive) lattice-based modal logic

Willem Conradie (Joint work with Alessandra Palmigiano)
Department of Mathematics, University of Johannesburg

In this talk we discuss correspondence and canonicity results - both corner stones of the classical theory of modal logic – in the greatly generalized setting of lattice-based modal logic. The semantics of this logic is obtained by generalizing the algebraic duals of Kripke frames, namely boolean algebras with operators, to arbitrary (not necessarily distributive) lattices with operators.

We develop a non-distributive version of the recently introduced ALBA algorithm. This algorithm computes first-order frame correspondents for inequalities by eliminating the occurring propositional variables in favour of special variables, called nominals and co-nominals, which range, respectively, over the join- and meet-irreducible elements of perfect lattices. We show that all inequalities which are reducible in this way are moreover canonical.

We develop the non-distributive analogues of the well known classes of Sahlqvist and inductive inequalities, and prove the accompanying first-order correspondence and canonicity results by showing that all members are successfully reducible by ALBA.

Algebraic exponentiation

James Gray
Department of Mathematics and Applied Mathematics, University of Cape Town

The concept of centralizer of a subgroup has been given a general categorical definition (see [1], [2] and [3]). In any pointed variety with binary + satisfying the axioms $x + 0 = x = 0 + x$, we can check whether these categorical centralizers exist. It turns out, that in most classical varieties, they do (e.g. for groups, rings, Lie algebras, etc). One could also generalize the concept of centralizer by considering a pair of group homomorphisms $f : A \to C$ and $g : B \to C$ and defining the centralizer as the largest subgroup Z of B with elements of $g(Z)$ commuting with elements of $f(A)$. It can be seen that zeroth non-abelian group cohomology is a special case of this type of centralizer. This leads to a new categorical approach to cohomology of algebraic structures, with Lie algebra cohomology as another example. We will also explain that zeroth non-abelian cohomology can be seen as a kind of algebraic exponent, and consider such algebraic exponents for general universal algebras.

References

[1] D. Bourn, Commutator theory in strongly protomodular categories, Theory and Applications of Categories 13 (2), 2004, 27-40.
[2] S. A. Huq, Commutator, nilpotency and solvability in categories, Quart. J. Math. Oxford (2) 19, 1968, 363-389.
[3] S. A. Huq, Upper central series in a category, J. Reine Angew. Math. 252, 1972, 209-214.

What shall be 2-dimensional topology of first order logic?

George Janelidze
Department of Mathematics and Applied Mathematics, University of Cape Town

This talk is a brief description of a far-in-the-future project based on recent developments in the theory of Barr's relational algebras and Makkai's approach to first order logic. Its aim is to develop a kind of 2-dimensional topology where the role of ultrafilter convergence will be played by constructions involving ultraproducts of models of first order theories.

Relative Goursat categories

Tamar Janelidze
Department of Mathematical Sciences, University of Stellenbosch

The n-permutability of congruences in universal algebras has been already studied in the past in a categorical setting: see [1] and references there. In particular, according to [1], a Goursat category can be defined as a regular category C satisfying the 3-permutability of equivalence relations on the same object. The aim of this talk is to extend the results known for Goursat categories to a relative setting, where a category C is replaced with a pair (C, E), in which E is a class of regular epimorphisms in E satisfying suitable conditions. This reduces to the known case when E is the class of all regular epimorphisms in C.

References

[1] A. Carboni, G.M. Kelly, and M.C. Pedicchio, Some remarks on Maltsev and Goursat categories, Applied Categorical Structures 1, 385-421 (1993)

Algebraic importance of "modus ponens"

Zurab Janelidze
Department of Mathematical Sciences, University of Stellenbosch

Modus ponens states that from $x \to y$ and x we can infer x. In other words, if $x \to y$ and x belong to the collection of statements which are true, then so must x. We interpret this principle in a universal-algebraic setting, where "$\to$" is replaced with an abstract binary term $d(x, y)$ in an algebraic theory satisfying $d(x, x) = d(y, y)$, and where the collection of true statements is replaced by any subalgebra of a universal algebra which is a model of the given theory. This leads to important axiomatic contexts in modern general algebra, developed for the unified study of properties of group-like algebraic structures.

Groupoid quantales beyond the étale setting
Alessandra Palmigiano
ILLC, University of Amsterdam

Quantales are ordered algebras which can be thought of as pointfree noncommutative topologies. In recent years, their connections have been studied with fundamental notions in noncommutative geometry such as groupoids and C^*-algebras. In particular, the setting of quantales corresponding to étale groupoids has been very well understood: a bijective correspondence has been defined between localic étale groupoids and inverse quantale frames. We present an equivalent but independent way of defining this correspondence for topological tale groupoids and we extend this correspondence to a non-étale setting.

Idempotent residuated structures and finiteness conditions
James Raftery and Ai-Ni Hsieh (to be presented by Raftery)
School of Mathematical Sciences, University of KwaZulu-Natal

A class K of similar algebras is said to have the *finite embeddability property* (briefly, the *FEP*) if every finite subset of an algebra in K can be extended to a finite algebra in K, with preservation of all partial operations. If a finitely axiomatized variety or quasivariety of finite type has the FEP, then its universal first order theory is decidable, hence its equational and quasi-equational theories are decidable as well. Where the algebras are residuated ordered groupoids, these theories are often interchangeable with logical systems of independent interest. Partly for this reason, there has been much recent investigation of finiteness properties such as the FEP in varieties of residuated structures.

A residuated partially ordered monoid is said to be *idempotent* if its monoid operation is idempotent. In this case, the partial order is equationally definable, so the structures can be treated as pure algebras. Such an algebra is said to be *conic* if each of its elements lies above or below the monoid identity $\mathtt{t}$; it is *semiconic* if it is a subdirect product of conic algebras. We prove that

> *the class* SCIP *of all semiconic idempotent commutative residuated po-monoids is locally finite,*

i.e., every finitely generated member of this class is a finite algebra. It turns out that SCIP is a quasivariety; it is not a variety.

The lattice-ordered members of SCIP form a variety SCIL, provided that we add the lattice operations $\wedge, \vee$ to the similarity type. This variety is not locally finite, but the local finiteness of SCIP facilitates a proof that SCIL has the FEP. In fact, we show that

> *for every relative subvariety* K *of* SCIP*, the lattice-ordered members of* K *form a variety with the FEP.*

(A *relative subvariety* of SCIP is a subclass axiomatized, relative to SCIP, by some set of equations.) It is also shown that

> SCIL *has a continuum of semisimple subvarieties.*

Note that SCIL contains all Brouwerian lattices, i.e., the algebraic models of positive intuitionistic logic. It also includes all positive Sugihara monoids (cf. [3]); these algebras model the positive fragment of the system **R**–mingle. The results here give a unified explanation of the strong finite model property for many extensions of these and other systems. They generalize Diego's Theorem, as well as the main theorem of [5], which showed that the variety generated by all idempotent commutative residuated *chains* is locally finite. Another generalization of the latter result, in a different direction, has been obtained in [4]. Further, we show that

> *the involutive algebras in* SCIL *are subdirect products of chains.*

Although SCIL is finitely axiomatized, it is not clear whether SCIP has a finite basis. Motivated in part by this question, we consider the larger quasivariety IP of *all* idempotent commutative residuated po-monoids. It is proved that

> *a relative subvariety of* IP *consists of semiconic algebras if and only if it satisfies* $x \approx (x \to \mathsf{t}) \to x$.

It follows that SCIP is not itself a relative subvariety of IP. The result also has corollaries for the logical system $\mathbf{RMO}^*$, which adds fusion and the Ackermann truth constant to Anderson and Belnap's $\mathbf{RMO}_{\to}$ in a natural (and conservative) manner. Because the axiomatic extensions of $\mathbf{RMO}^*$ are in one-to-one correspondence with the relative subvarieties of IP, we can infer the following:

> *If an axiomatic extension of* $\mathbf{RMO}^*$ *has* $((p \to \mathsf{t}) \to p) \to p$ *among its theorems, then it is locally tabular*

(i.e., it has only finitely many inequivalent n–variable formulas, for every finite n). In particular, such an extension is strongly decidable, provided that it is finitely axiomatized.

Most of the results reported here have been written up in [1, 2].

References

[1] A. Hsieh, *Some locally tabular logics with contraction and mingle*, Rep. Math. Logic **45** (2010), 143–159.

[2] A. Hsieh, J.G. Raftery, *Semiconic idempotent residuated structures*, Algebra Universalis **61** (2009), 413–430.

[3] J.S. Olson, J.G. Raftery, *Positive Sugihara monoids*, Algebra Universalis **57** (2007), 75–99.

[4] J.S. Olson, J.G. Raftery, *Residuated structures, concentric sums and finiteness conditions*, Communications in Algebra **36** (2008), 3632–3670.

[5] J.G. Raftery, *Representable idempotent commutative residuated lattices*, Trans. Amer. Math. Soc. **359** (2007), 4405–4427.

Representable Ideal-determined Varieties

Clint van Alten
School of Computer Science, University of Witwatersrand, Johannesburg

For many classes of ordered algebraic structures, the subclass consisting of all linearly ordered algebras has special interest. A general problem in this regard is to axiomatize the variety generated by the linearly ordered algebras, and thereby also the equational theory. The members of the variety generated by linearly ordered algebras are often called 'representable' or 'prelinear'. Classes of algebras where such axiomatizations have been obtained include lattice-ordered groups, lattice-ordered rings, Heyting algebras and other ordered algebraic structures arising from logic such as residuated lattices.

In this talk, we give a general method for axiomatizing representable subvarieties of 'ideal-determined' varieties of ordered algebras. Ideal-determined varieties are such that there is an isomorphism between the congruence lattice and the lattice of 1-classes of each algebra, for some distinguished constant 1 in the language. The 1-classes are called 'ideals'. We require a slight strengthening of the ideal-determined condition, however, all above-mentioned examples satisfy this condition, and thus we offer a uniform method that can be applied in all these cases and is easily extended to related classes.

Enumerating all models of a Horn formula, e.g. all closed sets of a closure system

Marcel Wild
Department of Mathematical Sciences, University of Stellenbosch

The multiplication table of any universal algebra, say a semigroup $(S, *)$, completely determines the closure system C of all subsemigroups of S. Specifically, let F be the set of all "implications" $\{a, b\} \to \{a * b\}$ with a, b ranging over S. Then a subset T of S belongs to C iff it satisfies all implications of F in the sense that whenever $\{a, b\}$ is contained in T then so must be $a * b$. In this case one also calls T a model of F. The conjunction of implications (viewed as a propositional formula) is an example of a so called Horn formula. I present an algorithm to enumerate (or generate) all models of a Horn formula.