Successions in words and compositions

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Abstract

We consider words over the alphabet $[k] = \{1, 2, \ldots, k\}$, $k \geq 2$. For a fixed nonnegative integer p, a *p-succession* in a word $w_1w_2\cdots w_n$ consists of two consecutive letters of the form $(w_i, w_i + p)$, $i = 1, 2, \ldots, n-1$. We analyze words with respect to a given number of contained p-successions. First we find the mean and variance of the number of p-successions. We then determine the distribution of the number of p-successions in words of length n as n (and possibly k) tends to infinity; a simple instance of a phase transition (Gaussian-Poisson-degenerate) is encountered. Finally we also investigate successions in compositions of integers.

1 Introduction

We consider words over the alphabet $[k] = \{1, 2, \ldots, k\}$, $k \geq 2$. For a fixed nonnegative integer p, a *p-succession* in a word $w_1w_2\cdots w_n$ consists of two consecutive letters of the form (w_i, w_i+p), $i = 1, 2, \ldots, n-1$. For example the word 1324122243 contains 3 instances of 2-successions: $13, 24, 24$. It is immediate that if $p \geq k$, then no word over $[k]$ can contain a p-succession.

In this paper we analyze words with respect to a given number of contained p-successions. We will also investigate successions in compositions (ordered partitions) of integers.

The subject of enumeration of finite sequences according to the number of p-successions has been much studied in the literature. The classical definition, in which $p = 1$, was first applied by Kaplansky and Riordan, in the 1940's, to the enumeration of subsets of $[k]$ (see [9, 17]). Subsequently, several authors have considered the enumeration of permutations of $[k]$ by the number of 1-successions, in conjunction with other well-known permutation statistics [4, 15, 16, 20]. Extentions of the 1-succession idea in the case of subsets and set partitions have been studied in [11] and [12, 13], respectively. Recently, two of the authors have carried out an interesting enumeration of integer partitions with respect to p-successions in [10].

Patterns in words, of which successions are a special case, have also been studied extensively in the past, also in view of their importance in computer science. A general framework for the analysis of patterns was developed in the late seventies and early eighties, in particular in the works of Goulden and Jackson [6] and Guibas and Odlyzko [7]. Nowadays, there are even software packages available that determine generating functions for the problem of counting occurrences of patterns in words automatically, see [1, 14]. For further information on this rich subject, we refer to the books by Flajolet and Sedgewick [5] and Szpankowski [19] and the references therein.

We are interested in the distribution of the number of p-successions in words of length n as n (and possibly k) tends to infinity. To this end, we first derive a bivariate generating function for the number of words with a given number of p-successions in Section 2. The limiting distribution is obtained in Section 3, see Theorem 3; it is a well known fact that the distibution of the number of certain pattern occurrences is asymptotically Gaussian [2] as $n \to \infty$ if k, the size of the alphabet, is fixed. If n and k are allowed to grow simultaneously, it turns out that this remains true as long as n grows faster than k. If k grows at the same speed as k, however, we encounter a phase transition: the limiting distribution is a Poisson distribution in this case. For even larger k, the distribution becomes degenerate.

In Section 4 we determine asymptotics for words with no p-successions. The enumeration of integer compositions by the number of p-successions is considered in Section 5. The asserted results include the mean and variance of the number of p-successions in a random composition of an integer n. Again, the limiting distribution is found to be Gaussian.

2 Generating functions

We denote the length of a word w by $\ell(w)$, the last letter of w by $t(w)$ and the number of its p-successions by $s(w)$; p is assumed to be fixed throughout the paper, hence we ignore the dependence of $s(w)$ on p. Furthermore, we will also assume that p is nonnegative, since a p-succession in a word w corresponds to a $(-p)$-succession in the reversed word. Finally, we assume that $k > p$, since otherwise there cannot be any p-succession in any word over the alphabet $[k]$. Define the generating function

$$v_j(x,y) = \sum_{w\,:\,t(w)=j} x^{\ell(w)} y^{s(w)},$$

where the summation is over all words whose last letter is j ($j \in [k]$). It is easy to see that the functions $v_1, v_2, \ldots, v_k$ satisfy the functional equation

$$v_j(x,y) = \begin{cases} x + x\sum_{i=1, i\neq j-p}^{k} v_i(x,y) + xyv_{j-p}(x,y) & j > p, \\ x + x\sum_{i=1}^{k} v_i(x,y) & j \leq p. \end{cases}$$

Assume first that $p > 0$; write $k = ap + b$, where $0 \leq b < p$, and set $V(x,y) = 1 + \sum_{i=1}^{k} v_i(x,y)$. Then $V(x,y)$ is the generating function for all words (including the empty word), which is what we are actually interested in. It follows that $v_j(x,y) = xV(x,y)$ for $j \le p$ and

$$v_j(x,y) = xV(x,y) + x(y-1)v_{j-p}(x,y)$$

otherwise. Straightforward induction yields

$$v_j(x,y) = \frac{1 - x^r(y-1)^r}{1 - x(y-1)} \cdot xV(x,y)$$

if $(r-1)p < j \le rp$. Writing $z = x(y-1)$ for convenience, we can rewrite the sum of all v_j as follows:

$$\begin{aligned} V(x,y) &= 1 + \sum_{i=1}^{k} v_i(x,y) = 1 + pxV(x,y)\sum_{r=1}^{a} \frac{1-z^r}{1-z} + bxV(x,y)\frac{1-z^{a+1}}{1-z} \\ &= 1 + \frac{xV(x,y)}{1-z}\left(ap - \frac{pz(1-z^a)}{1-z} + b\left(1-z^{a+1}\right)\right) \\ &= 1 + \frac{xV(x,y)}{1-z}\left(k - \frac{z}{1-z}\left(p(1-z^a) + b(1-z)z^a\right)\right). \end{aligned}$$

Solving for $V(x,y)$ yields

$$\begin{aligned} V(x,y) &= \left(1 - \frac{x}{(1-z)^2}\left(k(1-z) - z\left(p(1-z^a) + b(1-z)z^a\right)\right)\right)^{-1} \\ &= \left(1 - \frac{x}{(1-z)^2}\left(k - (k+p)z + (p-b)z^{a+1} + bz^{a+2}\right)\right)^{-1} \end{aligned}$$

The special case $p = 1$ occurs as Exercise 2.4.14 in [6]. The case $p = 0$ can be treated in a similar way, and indeed one obtains the same formula (with $p = b = 0$, even though a is undefined in this case). Then the generating function simply reduces to

$$V(x,y) = \left(1 - \frac{kx}{1-z}\right)^{-1}.$$

It should also be noted that

$$V(x,1) = \frac{1}{1-kx},$$

as expected. Differentiating the generating function with respect to y and plugging in $y = 1$, one immediately finds explicit formulae for the mean and variance of the number of successions: one has

$$V_y(x,1) = \frac{(k-p)x^2}{(1-kx)^2}$$

and

$$V_{yy}(x,1) + V_y(x,1) = \frac{2(k-p)^2x^4}{(1-kx)^3} + \frac{(k-p)x^2}{(1-kx)^2} + [a>1]\frac{2(k-2p)x^3}{(1-kx)^2}.$$

Here we use Iverson's notation: $[P] = 1$ if P is true and $[P] = 0$ otherwise. Extracting coefficients and noting that $[a > 1] = [k \ge 2p]$, one obtains the following theorem:

Theorem 1 *The average number of p-successions in words of length n is*

$$\frac{(k-p)(n-1)}{k^2}$$

for $n > 0$, while the variance is given by

$$\frac{(k-p)(n-1)}{k^2} - \frac{(k-p)^2(3n-5)}{k^4} + [k \geq 2p]\frac{2(k-2p)(n-2)}{k^3}$$

for $n > 1$.

3 Limiting distribution

Let us now consider the distribution of the number of p-successions in more detail. If k is constant, then it follows easily from general theorems that the limiting distribution is Gaussian. Actually, this is known in more generality for arbitrary patterns in words [2], we also refer to [5, Note IX.33] and the references therein. Therefore, we consider a more general model in which k, the size of the alphabet, grows simultaneously with the length of our random words. It turns out that we have a very simple example of a phase transition: if k grows slowly compared to n (so that $\frac{k}{n} \to 0$), the limiting distribution is still Gaussian. If, on the other hand, $\frac{k}{n} \to \infty$, then Theorem 1, together with the Markov inequality, shows that the number of p-successions is almost surely 0. In the remaining case that k and n are of the same asymptotic order, we will obtain a Poisson distribution in the limit.

In order to prove these results, we return to our bivariate generating function. For the distribution of the number of successions, the behavior around $y = 1$ (and thus $z = 0$) is essential. First we prove the following lemma:

Lemma 2 *If $|y-1| \leq \frac{1}{10}$, then the polynomial*

$$P(x) = (1-x(y-1))^2 - x\left(k - (k+p)x(y-1) + (p-b)x^{a+1}(y-1)^{a+1} + bx^{a+2}(y-1)^{a+2}\right)$$

has exactly one zero $\rho = \rho(u,k)$ such that $|\rho| < \frac{2}{k}$, where $u = y-1$. This zero satisfies the inequality

$$\left|\rho - \frac{1}{k}\right| \leq \frac{13|u|}{k^2}.$$

Proof: We compare the polynomial to the linear polynomial $1 - kx$, which clearly has exactly one zero inside the circle $|x| = \frac{2}{k}$. On this circle, one has $|1-kx| \geq 1$ and on the other hand, writing $u = y-1$ (so that $|u| \leq \frac{1}{10}$),

$$\begin{aligned}
|P(x) - (1-kx)| &= \left|-2xu + x^2u^2 + (k+p)x^2u - (p-b)x^{a+2}u^{a+1} - bx^{a+3}u^{a+2}\right| \\
&\leq 2|x||u| + |x|^2|u|^2 + (k+p)|x|^2|u| + (p-b)|x|^{a+2}|u|^{a+1} + b|x|^{a+3}|u|^{a+2} \\
&\leq 2|x||u| + |x|^2|u|^2 + 2k|x|^2|u| + p|x|^3|u|^2 \\
&\leq \frac{2}{5k} + \frac{1}{25k^2} + \frac{4}{5k} + \frac{2}{25k^2} = \frac{6}{5k} + \frac{3}{25k^2} < 1.
\end{aligned}$$

Hence, by Rouché's Theorem, there must be exactly one zero inside the circle $|x| = \frac{2}{k}$. Furthermore, the above derivation shows that

$$|P(x) - (1 - kx)| \le \frac{12|u|}{k} + \frac{12|u|^2}{k^2} \le \frac{13|u|}{k}$$

holds for $|x| \le \frac{2}{k}$. Hence, if $P(\rho) = 0$, one has

$$|1 - k\rho| \le \frac{13|u|}{k}$$

and thus

$$\left|\rho - \frac{1}{k}\right| \le \frac{13|u|}{k^2},$$

as claimed. ■

Now we can apply the residue theorem to extract the coefficient of x^n from $V(x,y)$: if $|u| = |y-1| \le \frac{1}{10}$, then

$$[x^n]V(x,y) = \frac{1}{2\pi i}\oint_{|z|=r} z^{-n-1}V(z,y)\,dy = \frac{1}{2\pi i}\oint_{|z|=r} z^{-n-1}(1 - z(y-1))^2 P(z)^{-1}\,dy$$

for any $0 < r < |\rho|$. We shift the path of integration to obtain

$$\begin{aligned}
[x^n]V(x,y) &= -\rho^{-n-1}\operatorname{Res}_{z=\rho}(1-uz)^2P(z)^{-1} + \frac{1}{2\pi i}\oint_{|z|=2/k} z^{-n-1}(1-uz)^2P(z)^{-1}\,dy \\
&= -\frac{\rho^{-n}(1-u\rho)^2}{\rho P'(\rho)} + \frac{1}{2\pi i}\oint_{|z|=2/k} z^{-n-1}(1-uz)^2P(z)^{-1}\,dy
\end{aligned}$$

By the inequalities above, $|P(z)|$ is uniformly bounded below on the circle $|z| = \frac{2}{k}$ by an absolute positive constant. Hence,

$$[x^n]V(x,y) = -\frac{\rho^{-n}(1-u\rho)^2}{\rho P'(\rho)} + O((k/2)^n),$$

uniformly for $|u| \le \frac{1}{10}$. Note that $\rho \le \frac{1}{k} + \frac{13|u|}{k^2} \le \frac{33}{20k}$, so the error term is indeed smaller than the main term by an exponential factor. For fixed k, this formula would already imply a central limit theorem by Hwang's Quasi-Power Theorem ([8], see also [5, Theorem IX.8]). If k is allowed to grow with n, we have to do a little more work.

First we need more precise asymptotic information about ρ: we assume that $a > 1$, since the case $a = 1$ can be treated analogously and since it can only occur if k is bounded. Noting that $\rho = O(k^{-1})$, the definition of ρ yields

$$0 = P(\rho) = 1 - (k + 2u)\rho + (u^2 + ku + pu)\rho^2 + O(k^{-4}u^3)$$

and thus

$$\begin{aligned}\rho &= \frac{k+2u-\sqrt{(k+2u)^2-4(u^2+ku+pu)(1+O(k^{-4}u^3))}}{2(u^2+ku+pu)}\\ &= \frac{k+2u-\sqrt{k^2-4pu+O(k^{-3}u^4)}}{2u(k+p+u)}\\ &= \frac{1}{k}-\frac{k-p}{k^3}u+\frac{k^2-2kp+2p^2}{k^5}u^2+O(k^{-4}u^3)\end{aligned}$$

after a few simplifications. Plugging in, one also obtains

$$-\frac{(1-u\rho)^2}{\rho P'(\rho)} = 1+O(k^{-1}u).$$

Let ω_n denote the number of p-successions in a random word of length n. The moment generating function of this random variable is given by

$$\mathbb{E}\left(e^{\omega_n t}\right) = k^{-n}[x^n]V(x,e^t).$$

Instead of dealing with ω_n directly, we consider the normalized random variable $\varpi_n = \frac{\omega_n-\mu_n}{\sigma_n}$, where μ_n and σ_n^2 are the mean and variance of ω_n respectively, as given in Theorem 1. The moment generating function of ϖ_n is given by

$$\mathbb{E}\left(e^{(\omega_n-\mu_n)t/\sigma_n}\right) = k^{-n}e^{-\mu_n t/\sigma_n}[x^n]V(x,e^{t/\sigma_n}).$$

Now we apply the asymptotic formula for $[x^n]V(x,y)$ with $y=e^{t/\sigma_n}$ (and thus $u=y-1=\frac{t}{\sigma_n}+\frac{t^2}{2\sigma_n^2}+O\left(\frac{t^3}{\sigma_n^3}\right)$) to obtain

$$\rho = \frac{1}{k}-\frac{k-p}{k^3}\cdot\frac{t}{\sigma_n}+\left(\frac{k^2-2kp+2p^2}{k^5}-\frac{k-p}{2k^3}\right)\cdot\frac{t^2}{\sigma_n^2}+O\left(\frac{t^3}{k^2\sigma_n^3}\right).$$

It follows that

$$\log(k\rho) = -\frac{k-p}{k^2}\cdot\frac{t}{\sigma_n}-\frac{k^3-k^2p-k^2+2kp-3p^2}{2k^4}\cdot\frac{t^2}{\sigma_n^2}+O\left(\frac{t^3}{k\sigma_n^3}\right)$$

and thus

$$\begin{aligned}&\mathbb{E}\left(e^{(\omega_n-\mu_n)t/\sigma_n}\right)\\ &\quad= k^{-n}e^{-\mu_n t/\sigma_n}[x^n]V(x,e^{t/\sigma_n}) = e^{-\mu_n t/\sigma_n}\cdot\left(-\frac{(1-u\rho)^2}{\rho P'(\rho)}\right)\cdot(k\rho)^{-n}+O\left(e^{-\mu_n t/\sigma_n}2^{-n}\right)\\ &\quad= \exp\left(-\frac{\mu_n t}{\sigma_n}+\frac{(k-p)n}{k^2}\cdot\frac{t}{\sigma_n}+\frac{(k^3-k^2p-k^2+2kp-3p^2)n}{2k^4}\cdot\frac{t^2}{\sigma_n^2}+O\left(\frac{t^3 n}{k\sigma_n^3}\right)\right)\\ &\qquad\cdot\left(1+O\left(\frac{t}{k\sigma_n}\right)\right)+O\left(e^{-\mu_n t/\sigma_n}2^{-n}\right).\end{aligned}$$

Taking into account that

$$\mu_n = \frac{(k-p)n}{k^2} + O(k^{-1}) \qquad \text{and} \qquad \sigma_n^2 = \frac{(k^3 - k^2p - k^2 + 2kp - 3p^2)n}{k^4} + O(k^{-1}),$$

this reduces to

$$\begin{aligned}\mathbb{E}\left(e^{(\omega_n-\mu_n)t/\sigma_n}\right) &= \exp\left(\frac{t^2}{2} + O\left(\frac{t}{k\sigma_n} + \frac{t^3 n}{k\sigma_n^3}\right)\right) + O\left(e^{-\mu_n t/\sigma_n} 2^{-n}\right) \\ &= \exp\left(\frac{t^2}{2} + O\left(\frac{t}{\sqrt{kn}} + \frac{t^3\sqrt{k}}{\sqrt{n}}\right)\right) + O\left(e^{-\mu_n t/\sigma_n} 2^{-n}\right).\end{aligned}$$

Hence, the moment generating function tends to $e^{t^2/2}$ (pointwise and uniformly on compact subsets of $\mathbb{R}$) if $\sigma_n^2 \sim \frac{n}{k} \to \infty$ (or, in other words, $\frac{k}{n} \to 0$), which is the moment generating function of a standard normal distribution. By Curtiss's Theorem [3], this implies that the distribution of ϖ_n tends weakly to a standard normal distribution.

Things are slightly different if k is proportional to n, i.e. $k \sim \frac{n}{c}$ for some positive constant c. In this case, the mean and variance no longer tend to infinity; in fact, both tend to c. However, the proof of convergence to a limiting distribution is actually shorter in this case: one can even work directly with the ordinary probability function of ω_n, namely

$$\sum_{i=0}^{\infty} \mathbb{P}(\omega_n = i) y^i = k^{-n}[x^n]V(x,y).$$

We obtain the following asymptotic formulae:

$$\rho = \frac{1}{k} - \frac{u}{k^2} + O(k^{-3}),$$

thus

$$\log(k\rho) = -\frac{u}{k} + O(k^{-2})$$

and

$$-\frac{(1-u\rho)^2}{\rho P'(\rho)} = 1 + O(k^{-1}).$$

Hence we have

$$\begin{aligned}k^{-n}[x^n]V(x,y) &= (1 + O(k^{-1}))(k\rho)^{-n} + O(2^{-n}) \\ &= (1 + O(k^{-1}))\exp\left(\frac{un}{k} + O(nk^{-2})\right) + O(2^{-n}) \\ &= \exp\left(\frac{un}{k} + O(n^{-1})\right) + O(2^{-n}),\end{aligned}$$

which tends to $\exp(cu) = \exp(c(y-1))$ (at least if $|u| < \frac{1}{10}$), which is exactly the probability generating function of a Poisson distribution with mean and variance c. Using [5, Theorem IX.1], it follows that the distribution of ω_n tends to a Poisson distribution, and we end up with the following theorem that summarizes the results of this section:

Theorem 3 *If $\frac{k}{n} \to 0$, then the distribution of the number of p-successions is asymptotically normal; if k and n are of the same order, i.e., $k \sim \frac{n}{c}$ for some constant c, then the distribution of the number of p-successions tends to a Poisson distribution. Finally, if $\frac{k}{n} \to \infty$, then there are almost surely no p-successions in a random word of length n, so the distribution is degenerate in this case.*

4 Words without p-successions

The generating function for words without p-successions can be found by putting $y = 0$ in $V(x, y)$. One obtains

$$W(x) = \left(1 - \frac{x}{(1+x)^2}\left(k + (k+p)x + (p-b)(-x)^{a+1} + b(-x)^{a+2}\right)\right)^{-1}.$$

The dominant pole of this function must lie between $\frac{1}{k}$ and $\frac{1}{k-1}$: this follows from the observation that there are at least $k(k-1)^{n-1}$ words of length n without p-successions, but at most k^n such words. For large k, this pole (let us denote it by ρ_0) can be approximated quite well: one has

$$\left|(p-b)(-\rho_0)^{a+2} + b(-\rho_0)^{a+3}\right| \leq p\rho_0^{k/p+1} = O\left((k-1)^{-k/p-1}\right)$$

and thus

$$(1+\rho_0)^2 - k\rho_0 - (k+p)\rho_0^2 + O\left((k-1)^{-k/p-1}\right) = 0,$$

from which one deduces

$$\rho_0 = \frac{\sqrt{k^2+4p} - (k-2)}{2(k+p-1)} + O\left((k-1)^{-k/p-2}\right).$$

The coefficient $[x^n]W(x)$ is asymptotically $(-\operatorname{Res}_{z=\rho_0} W(z))\rho_0^{-n-1}$, and so one obtains the following theorem:

Theorem 4 *The number of words of length n without p-successions is asymptotically given by $\alpha_k \beta_k^n$, where*

$$\alpha_k = \frac{(k+p)(k+\sqrt{k^2+4p}) + 2p}{2(k+p-1)\sqrt{k^2+4p}} + O\left((k-1)^{-k/p-1}\right)$$

and

$$\beta_k = \frac{\sqrt{k^2+4p} + k - 2}{2} + O\left((k-1)^{-k/p}\right).$$

Note that the formulae for α_k and β_k are exact (without the error term) if $p = 0$.

5 Successions in compositions

Compositions can be treated in a similar way: in analogy to Section 2, we define the generating function $v_j(x,y)$ for compositions whose last summand is j (this approach is essentially equivalent to the "adding a slice" technique, see [5, Section 3.7]). The functions $v_1, v_2, \ldots$ satisfy the functional equations

$$v_j(x,y) = \begin{cases} x^j + x^j \sum_{i \geq 1, i \neq j-p} v_i(x,y) + x^j y v_{j-p}(x,y) & j > p, \\ x^j + x^j \sum_{i \geq 1} v_i(x,y) & j \leq p. \end{cases}$$

We are interested in the combined generating function $V(x,y) = 1 + \sum_{j \geq 1} v_j(x,y)$ again. In order to find an expression for this function, we first introduce auxiliary functions $U_r(x,y) = \sum_{j \geq 1} x^{rj} v_j(x,y)$. Then $U_0(x,y) = V(x,y) - 1$, and the functional equations stated above imply

$$\begin{aligned} U_r(x,y) &= \sum_{j \geq 1} x^{rj} v_j(x,y) = \sum_{j \geq 1} x^{rj} \cdot x^j V(x,y) + \sum_{j > p} x^{rj} (y-1) x^j v_{j-p}(x,y) \\ &= \frac{x^{r+1}}{1-x^{r+1}} \cdot V(x,y) + x^{(r+1)p}(y-1) \sum_{j \geq 1} x^{(r+1)j} v_j(x,y) \\ &= \frac{x^{r+1}}{1-x^{r+1}} \cdot V(x,y) + x^{(r+1)p}(y-1) U_{r+1}(x,y). \end{aligned}$$

Substituting $x^{r(r+1)p/2}(y-1)^r U_r(x,y) = T_r(x,y)$, one obtains

$$T_r(x,y) = \frac{x^{r(r+1)p/2+(r+1)}(y-1)^r}{1-x^{r+1}} \cdot V(x,y) + T_{r+1}(x,y)$$

with $T_0(x,y) = U_0(x,y) = V(x,y) - 1$ and thus by induction

$$T_r(x,y) = V(x,y) - 1 - \left(\sum_{j=1}^{r} (y-1)^{j-1} \frac{x^{j(j-1)p/2+j}}{1-x^j} \right) V(x,y)$$

As $r \to \infty$, $T_r(x,y) \to 0$ (as a formal power series), and so we have

$$V(x,y) = \left(1 - \sum_{j=1}^{\infty} (y-1)^{j-1} \frac{x^{j(j-1)p/2+j}}{1-x^j} \right)^{-1}.$$

Note that one has $V(x,1) = \frac{1-x}{1-2x}$, as it should be. Furthermore, one can easily determine the first and second derivative in order to find the mean and variance:

$$V_y(x,1) = \frac{(1-x)x^{p+2}}{(1+x)(1-2x)^2}$$

and

$$V_{yy}(x,1) = \frac{2x^{2p+4}(1-x)}{(1+x)^2(1-2x)^3} + \frac{2x^{3p+3}(1-x)}{(1+x+x^2)(1-2x)^2}.$$

Now one can read off the coefficients to obtain the following theorem:

Theorem 5 *The mean and variance of the number of p-successions in a random composition of n are given by*

$$\mu_n = 2^{-p}\left(\frac{n}{6} - \frac{3p-1}{18}\right) + \frac{4}{9}(-1)^p\left(-\frac{1}{2}\right)^n$$

for $n > p$ and

$$\sigma_n^2 = \left(\frac{2^{-p}}{6} - \frac{(6p+7)2^{-2p}}{108} + \frac{2^{-3p}}{7}\right)n$$
$$- \left(\frac{(3p-1)2^{-p}}{18} - \frac{(27p^2+36p-19)2^{-2p}}{324} + \frac{(21p+3)2^{-3p}}{49}\right) + O\left(n2^{-n}\right).$$

Furthermore, the distribution of the number of p-successions is asymptotically normal.

Proof: The mean and variance follow directly from the explicit formulae for the derivatives of $V(x, y)$, so it remains to prove the limit law. This, however, is essentially a consequence of the fact that $V(x, y)$ is the quotient of two analytic functions (within suitable regions); see [5, Theorem IX.9]. ■

It should be noted that an explicit formula for the variance can be given as well; since it is quite lengthy, only the main terms are provided here.

Remark It is interesting to compare the mean number of successions in compositions, which is linear in n, with the mean number of successions in partitions of integers, which is shown in [10] to grow like $\frac{\sqrt{\frac{6}{\pi^2}}}{p(p+1)}n^{1/2}$ as $n \to \infty$.

6 Conclusion

It is quite likely that all our results concerning limiting distributions, in particular the phase transition observed in Theorem 3, hold for more general patterns: if $\mathcal{S}(k)$ is a suitable collection of patterns in words over the alphabet $[k]$, and the size of $\mathcal{S}(k)$ grows linearly with k, then it is probable that the same type of phase transition occurs. The technical details might be intricate, though (in particular the proper definition of "suitable collection of patterns").

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