THE NUMBER OF SPANNING TREES IN SELF-SIMILAR GRAPHS

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Abstract. The number of spanning trees of a graph, also known as the complexity, is investigated for graphs which are constructed by a replacement procedure yielding a self-similar structure. It is shown that exact formulæ for the number of spanning trees can be given for sequences of self-similar graphs under certain symmetry conditions. These formulæ exhibit interesting connections to the theory of electrical networks. Examples include the well-known Sierpiński graphs and their higher-dimensional analoga. Several remarkable auxiliary results are provided on the way—for instance, a property of the number of rooted spanning forests is proven for graphs with a high amount of symmetry. Furthermore, it is shown that the enumeration of spanning trees can be simplified by a procedure similar to the Wye-Delta-transform under certain circumstances.

1. Introduction

The number of spanning trees of a finite graph or multigraph X, also known as the *complexity* $\tau(X)$, is certainly one of the most important graph-theoretical parameters. Its applications range from the theory of networks, where the number of spanning trees is used as a measure for network reliability [14, 36] to statistical physics, where the complexity is of use in the study of lattices [40], and theoretical chemistry, in connection with the enumeration of certain chemical isomers [8].

Of course, counting the number of spanning trees in certain graphs or graph classes is also a prominent problem in combinatorics. Kirchhoff's celebrated matrix tree theorem [26] relates the properties of an electrical network to the number of spanning trees in the underlying graph. There is a large variety of proofs for the matrix tree theorem, see for instance [7, 12, 21], and several extensions and generalizations have been provided in the past. One of them, due to Moon [34], which gives a general formula for *spanning forests*, will be of vital importance within this paper. It is also known that there are connections to other enumeration problems—namely, those for Eulerian cycles [21] and for perfect matchings [23].

In view of the large number of interpretations and applications, it is not surprising that many papers deal with exact formulæ for the number of spanning trees in certain graph classes. Cayley's well-known enumeration of labelled trees [11], which is equivalent to the enumeration of spanning trees in a complete graph K_n, can be seen as the starting point for this path of investigation: Cayley's theorem states that

$$\tau(K_n) = n^{n-2}.$$

This formula has been generalized in many ways. For instance, the complexity of a complete multipartite graph $K_{n_1,\dots,n_d}$ is given by

$$\tau(K_{n_1,\dots,n_d}) = n^{d-2}\prod_{i=1}^{d}(n-n_i)^{n_i-1},$$

where $n = n_1 + \cdots + n_d$ [2, 17]. A nice combinatorial proof for this formula (involving a modified form of Prüfer sequences) has been given by Lewis [29]. Further examples of closed formulæ include those for wheels, fans, ladders, prisms and other special families [4, 6, 35, 44]. A collection of formulæ can also be found in Berge's book [5].

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Most of the graphs for which an exact enumeration of spanning trees is possible are highly symmetric—indeed, there are certain methods which work well for graphs with a large automorphism group, such as the fullerenes investigated in [8]. Of course, regular graphs are of particular interest in this context, see also [1, 13, 15, 32].

Lattices, in particular rectangular and triangular lattices, are of special interest in theoretical physics—here, various graph-theoretical parameters are important, such as the number of perfect matchings [22], but also the number of spanning trees [19]. The quantity

$$h = \lim_{n\to\infty} \frac{\log(\tau(X_n))}{|VX_n|},$$

where X_n is an increasing sequence of graphs (such as finite sections of a lattice) approaching an infinite graph (in some sense), is a useful descriptor in this context. In [30] this quantity is termed *tree entropy* and its relation to the simple random walk is studied. A closed formula for h in terms of return probabilities of the infinite graph in the transitive setting is derived. In the case of the square lattice it is known that $h = \frac{4G}{\pi}$, where

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \approx 0.915965594\ldots$$

is Catalan's constant, and for the regular tree of degree four h is given by $h = 3\log(\frac{3}{2})$, see [9, 32, 40] and the references therein for this and several other examples.

The graphs we are going to investigate in this paper are of a *self-similar* nature, and they are typically related to fractals. Even though these graphs are quite popular in the study of electrical networks and random walks (see the lecture notes of Barlow [3], Kigami's book [24], and the references therein), it seems that the enumeration of spanning trees (mainly exact enumeration) has been somewhat neglected up to now in spite of the obvious connections. The strong relation between the tree counting problem and electrical networks, Laplacians and random walks is exhibited by the main result involving the so-called *spectral dimension* or *resistance scaling factor*, respectively. These notions appear in the study of the Laplace operator on fractals like the Sierpiński gasket. The resistance scaling factor is usually defined by an eigenvalue problem of a non-linear map, called the renormalization map, using energy forms, see for example [33].

Using Kirchhoff's theorem the complexity of a graph is closely related to the spectrum of the combinatorial Laplacian. For self-similar graphs and fractals the Dirichlet- and Neumann-spectrum of the Laplace operator was studied from several points of view, see [31, 37, 39] and the references therein. Sabot [37] has shown that the spectrum is related to the dynamical behavior of a multi-dimensional polynomial. In the case of the Sierpiński gasket and other highly symmetric fractals the dimension of this map reduces to 1, see [18, 39].

The substitution process that is used for defining sequences of self-similar graphs in this paper is essentially a special case of the construction that was defined in the authors' paper [42], where enumeration problems are treated from a more general point of view. It is one of several possibilities to define self-similarity on graphs (see [27, 31, 37] for instance).

The paper will be structured as follows: first, the necessary preliminaries on set partitions and group actions are given; then, we introduce auxiliary tools from the theory of electrical networks, including the relations between electrical networks and spanning forests. Next, we turn to our graph construction process and prove a decomposition property for spanning forests. This can be used to establish a system of polynomial recurrences for the number of spanning trees and certain auxiliary parameters. Finally, it is shown how the dynamical system given by this multi-dimensional polynomial can be reduced and simplified in a step-by-step manner, thus yielding a closed formula (see Theorem 29): if $X_0, X_1, \ldots$ is a sequence of finite self-similar (multi-)graphs satisfying additional assumptions (connectedness and symmetry), the complexity $\tau(X_n)$ of X_n is given by

$$\tau(X_n) = \tau(X_0)\left(\frac{|EX_n|}{|EX_0|}\right)^{c(1-2/d_s)} C^{\frac{|EX_n|}{|EX_0|}-1},$$

where c, C are constants depending on the graph sequence, and d_s denotes the associated spectral dimension. This formula was already obtained by the authors for the special case of finite Sierpiński graphs, see [43]. Finally, several examples for the application of the results are given as well.

2. Preliminaries

We write $\mathbb{N}$ for the positive integers and $\mathbb{N}_0$ for the positive integers with zero. A *multigraph* $X = (VX, EX)$ has a vertex set VX and an edge multiset EX with

$$\Big\{\{x, y\} \,:\, x, y \in VX\Big\}$$

as underlying set of elements and is always supposed to be undirected. Of course, an edge of the form $\{x, x\} = \{x\}$ is a loop at the vertex x. If the multigraph X has no multiple edges and no loops, we regard X as a (simple) graph. An *isomorphism* $\gamma : X \to Y$ between two multigraphs X and Y is a pair (γ_v, γ_e), where $\gamma_v : VX \to VY$ and $\gamma_e : EX \to EY$ are bijections, so that $\gamma_e(\{x, y\}) = \{\gamma_v(x), \gamma_v(y)\}$ holds for all edges $\{x, y\} \in EX$. The *automorphism group* $\mathsf{Aut}(X)$ is the set of all isomorphisms from X to itself and its elements are called automorphisms. If no ambiguity can occur, we write $\gamma(x)$ or γx instead of $\gamma_v(x)$ if $\gamma : X \to Y$ is an isomorphism and $x \in VX$ a vertex.

2.1. **Number partitions and set partitions.** For $n \in \mathbb{N}$ denote by $\mathcal{P}(n)$ the set of number partitions of the integer n, and write $\nu_k(p)$ for the number of occurrences of $k \in \mathbb{N}$ in the partition $p \in \mathcal{P}(n)$, so that

$$n = \sum_{k \in \mathbb{N}} k \cdot \nu_k(p)$$

and $\nu_k(p) = 0$ for $k > n$. In addition, define $|p|$ by

$$|p| = \sum_{k \in \mathbb{N}} \nu_k(p)$$

and set $\mathcal{P}_r(n) = \{p \in \mathcal{P}(n) \,:\, |p| = r\}$ for $r \in \mathbb{N}$. If a number partition has $k_1, \ldots, k_r$ as its distinct addends, we write

$$p = k_1^{\nu_{k_1}(p)} \cdots k_r^{\nu_{k_r}(p)}$$

as a shorthand. Usually, the summands $k_1, \ldots, k_r$ are sorted in descending order. For example $3^1 2^3 1^2$ means the number partition $3 + 2 + 2 + 2 + 1 + 1$.

Let M be a finite set. A *set partition* B of M is a family of non-empty and disjoint subsets of M, so that their union is equal to M. The elements of M are called *blocks.*

The block sizes of B define a number partition p of $|M|$ and the set partition B is said to be of *type* p in this case. For convenience, the type p of B is denoted $\ell(B) = p$. Let $\mathcal{B}(M)$ be the set of all set partitions of M and denote by $\mathcal{B}_p(M) \subseteq \mathcal{B}(M)$ those partitions of type p. Of course,

$$\mathcal{B}(M) = \biguplus_{p \in \mathcal{P}(|M|)} \mathcal{B}_p(M).$$

If K is a subset of M, then the *restriction* $B|_K \in \mathcal{B}(K)$ of $B \in \mathcal{B}(M)$ is given by

$$B|_K = \Big\{b \cap K \,:\, b \in B, b \cap K \neq \varnothing\Big\}.$$

Finally, set $\varphi(B) = \{\varphi(b) \,:\, b \in B\}$ for any $B \in \mathcal{B}(M)$ and any map $\varphi : M \to R$. Of course, $\varphi(B) \in \mathcal{B}(\varphi(M))$ if φ is one-to-one.

Let I be an index set. For $i \in I$ let $M_i \subseteq M$ be a non-empty subset of M, so that the union of all M_i is equal to M. Let B_i be a set partition of M_i and denote by $\mathcal{B} = \{B_i \,:\, i \in I\}$ the family of these partitions. Then define a multigraph $X_{\mathcal{B}}$ as follows:

$$VX_{\mathcal{B}} = \Big\{(B, b) \,:\, B \in \mathcal{B}, b \in B\Big\}$$

and two distinct vertices (B_1, b_1) and (B_2, b_2) are joined by $|b_1 \cap b_2|$ edges in $EX_{\mathcal{B}}$. By definition $X_{\mathcal{B}}$ does not contain any loops. We call $\mathcal{B}$

- *cycle-free*, if the multigraph $X_{\mathcal{B}}$ is cycle-free (and hence simple),
- *connected*, if $X_{\mathcal{B}}$ is so.

Notice that two blocks from distinct partitions of a cycle-free family have at most one point in common. The connected components of $X_{\mathcal{B}}$ naturally define a set partition on M, which is called the *transitive union* $\mathsf{Union}(\mathcal{B}) \in \mathcal{B}(M)$ of $\mathcal{B}$: each block of $\mathsf{Union}(\mathcal{B})$ is given as the union of all blocks of the family $\mathcal{B}$, which are contained in one connected component of $X_{\mathcal{B}}$. In other words, $\mathsf{Union}(\mathcal{B})$ is the finest partition of M, such that each block of the family $\mathcal{B}$ is contained in one block of $\mathsf{Union}(\mathcal{B})$.

The number $b(p)$ of set partitions of M of type p is given by

$$b(p) = |\mathcal{B}_p(M)| = |M|! \left(\prod_{k\in\mathbb{N}} \nu_k(p)!\,(k!)^{\nu_k(p)} \right)^{-1}. \tag{1}$$

Let o, p be number partitions of $|M|$ and fix a set partition O of M with $\ell(O) = o$. Denote by $\mathcal{A}(O)$ the family of set partitions P of M with the property that $\{O, P\}$ forms a cycle-free, connected family, and set $\mathcal{A}(O,p) = \mathcal{A}(O) \cap \mathcal{B}_p(M)$. Then the number $\alpha(O,p) = |\mathcal{A}(O,p)|$ only depends on the type of O and not on O itself. Thus we may define $\alpha(o,p) = \alpha(O,p)$ for any $O \in \mathcal{B}_o(M)$. If O and P are set partitions of M, so that $\mathcal{B} = \{O, P\}$ is cycle-free and connected, then $|O| + |P| = |M| + 1$, since the associated graph $X_{\mathcal{B}}$ is a tree, $|VX_{\mathcal{B}}| = |O| + |P|$, and $|EX_{\mathcal{B}}| = |M|$. This implies

$$\mathcal{A}(O) = \biguplus_{p \in \mathcal{P}_k(|M|)} \mathcal{A}(O,p), \tag{2}$$

where k is given by $k = |M| + 1 - |O|$.

Theorem 1. *Let M be a finite set and $o, p \in \mathcal{P}(|M|)$ with $|o| + |p| = |M| + 1$. Then the formula*

$$\alpha(o,p) = (|o|-1)!\,(|p|-1)! \prod_{k\in\mathbb{N}} \frac{k^{\nu_k(o)}}{\nu_k(p)!\big((k-1)!\big)^{\nu_k(p)}} \tag{3}$$

holds.

Proof. For the moment write $\mathcal{P}_r$ for the set of all number partitions with exactly r terms and set $\nu(p) = n$ if $p \in \mathcal{P}(n)$. Fix some set partition $O \in \mathcal{B}_o(M)$ and some integer $r \geq 1$ with $\nu_r(p) > 0$. Let P be a set partition of type p, such that $\{O, P\}$ is a cycle-free, connected family. Then each element of a block $b \in P$ of size r (there are $\nu_r(p)$ possibilities to choose such a block) is contained in exactly one of r pairwise different blocks $c_1, \ldots, c_r$ of O. Denote by $q \in \mathcal{P}_r$ the type of $\{c_1, \ldots, c_r\}$. Then there are

$$\prod_{k\in\mathbb{N}} \binom{\nu_k(o)}{\nu_k(q)} k^{\nu_k(q)}$$

choices for the r elements of the block b inside the partition O, if the type of the "neighboring" blocks $\{c_1, \ldots, c_r\}$ is given by q. Now consider $P' = P \setminus \{b\}$ and

$$O' = (O \setminus \{o_1, \ldots, o_r\}) \cup \{(c_1 \cup \cdots \cup c_r) \setminus b\}.$$

Both P' and O' are set partitions of $M \setminus b$, and the family $\{O', P'\}$ is cycle-free and connected. The type p' of P' is obtained from p by removing one addend of size r: $\nu_r(p') = \nu_r(p) - 1$. On the other hand, the type o' of O' is given by

$$\nu_k(o') = \begin{cases} \nu_k(o) - \nu_k(q) + 1 & \text{if } k = \nu(q) - r, \\ \nu_k(o) - \nu_k(q) & \text{otherwise.} \end{cases}$$

The partition O' emerges from O by removing $\nu_k(q)$ blocks of size k and adding one block of size

$$\sum_{k\in\mathbb{N}} (k-1)\nu_k(q) = \nu(q) - r. \tag{4}$$

Obviously, p' and o' are both number partitions of $|M| - r$ and $|p'| = |p| - 1$, $|o'| = |o| - r + 1$. Finally, we point out the dependency of o' on q.

The considerations above yield a recursive formula for $\alpha(o,p)$:

$$\alpha(o,p) = \frac{1}{\nu_r(p)} \sum_{q \in \mathcal{P}_r} \alpha(o',p') \prod_{k\in\mathbb{N}} \binom{\nu_k(o)}{\nu_k(q)} k^{\nu_k(q)}. \tag{5}$$

Now, we may proceed by induction. Equation (3) is trivial if $|o| = 1$ or $|p| = 1$. The hypothesis for $\alpha(o',p')$ implies

$$\alpha(o',p') = (|o|-r)!\,(|p|-2)!\,(r-1)!\,\nu_r(p)(\nu(q)-r) \prod_{k\in\mathbb{N}} \frac{k^{\nu_k(o)-\nu_k(q)}}{\nu_k(p)!\big((k-1)!\big)^{\nu_k(p)}}.$$

Inserting this into (5) shows that it suffices to prove

$$(|p|-1)\binom{|o|-1}{r-1} = \sum_{q\in\mathcal{P}_r} (\nu(q)-r) \prod_{k\in\mathbb{N}} \binom{\nu_k(o)}{\nu_k(q)}.$$

The term on the right-hand side, which we denote by A, can be written as

$$A = [y^r]\frac{\partial}{\partial x} \prod_{k\in\mathbb{N}} \sum_{j\in\mathbb{N}_0} \binom{\nu_k(o)}{j} x^{(k-1)j} y^j \Big|_{x=1}$$

bearing the identity (4) in mind. However, some elementary transformations yield

$$\begin{aligned} A &= [y^r]\frac{\partial}{\partial x} \prod_{k\in\mathbb{N}} (1+x^{k-1}y)^{\nu_k(o)} \Big|_{x=1} \\ &= [y^r] \sum_{j\in\mathbb{N}} \frac{\nu_j(o)(j-1)x^{j-2}y}{1+x^{j-1}y} \prod_{k\in\mathbb{N}} (1+x^{k-1}y)^{\nu_k(o)} \Big|_{x=1} \\ &= [y^r] \sum_{j\in\mathbb{N}} \nu_j(o)(j-1)y(1+y)^{|o|-1} \\ &= (|M|-|o|)\binom{|o|-1}{r-1}, \end{aligned}$$

which proves the theorem, since $|o| + |p| = |M| + 1$. ■

Define number partitions $p_k \in \mathcal{P}(|M|)$ for $k \in \{1,\dots,|M|\}$ as follows: For $k = 1$ set $p_1 = 1^{|M|}$ and for $k \geq 2$ set $p_k = k^1\, 1^{|M|-k}$. Thus

$$p_k = k + \underbrace{1+\cdots+1}_{|M|-k \text{ times}}$$

and $|p_k| = |M| + 1 - k$ for all $k \in \{1,\dots,|M|\}$. Let $p \in \mathcal{P}(|M|)$ with $|p| = k$. Then we set $\alpha_p = \alpha(p_k, p)$. Some simplifications lead to

$$\alpha_p = (|M|-|p|)!\,|p|! \left(\prod_{k\in\mathbb{N}} \nu_k(p)!\,\big((k-1)!\big)^{\nu_k(p)} \right)^{-1}.$$

Suppose that $F \subseteq M$ has $|p|$ elements, then α_p counts the number of set partitions $P \in \mathcal{B}_p(M)$, each of whose blocks contains exactly one element from F. Last, but not least, we remark that the quotient

$$\frac{\alpha(o,p)}{\alpha_p} = \frac{1}{|M|+1-|o|} \prod_{k\in\mathbb{N}} k^{\nu_k(o)}$$

is independent of p for all $o \in \mathcal{P}(|M|)$ with $|o| + |p| = |M| + 1$ and will be denoted by β_o. This implies $\alpha(o,p) = \beta_o \alpha_p$.

2.2. **Symmetry.** An action of a group $\mathit{\Gamma}$ on a set M is called *transitive*, if for any $x, y \in M$ there is a $\gamma \in \mathit{\Gamma}$ with $\gamma x = y$. The action of $\mathit{\Gamma}$ naturally extends to tuples and subsets of M: $\gamma(m_1, \dots, m_k) = (\gamma m_1, \dots, \gamma m_k)$ for a tuple $(m_1, \dots, m_k) \in M^k$ and $\gamma K = \{\gamma m : m \in K\}$ for a subset $K \subseteq M$. The action of $\mathit{\Gamma}$ on M is called

- *k-transitive*, if $\mathit{\Gamma}$ acts transitive on the set of k-tuples in M^k with distinct entries and
- *k-homogeneous*, if $\mathit{\Gamma}$ acts transitive on the set of k-subsets of M.

For a subset $K \subseteq M$ we denote by $\mathsf{Stab}_{(K)}(\mathit{\Gamma})$ and $\mathsf{Stab}_{\{K\}}(\mathit{\Gamma})$ the pointwise and setwise stabilizer of K, respectively:

$$\mathsf{Stab}_{(K)}(\mathit{\Gamma}) = \Big\{\gamma \in \mathit{\Gamma} \,:\, \gamma m = m \text{ for all } m \in K\Big\}$$

and

$$\mathsf{Stab}_{\{K\}}(\mathit{\Gamma}) = \Big\{\gamma \in \mathit{\Gamma} \,:\, \gamma K = K\Big\}.$$

Then the restriction $\gamma \mapsto \gamma|_K$ defines a homomorphism from $\mathsf{Stab}_{\{K\}}(\mathit{\Gamma})$ to the symmetric group $\mathsf{Sym}(K)$ of K, whose kernel is $\mathsf{Stab}_{(K)}(\mathit{\Gamma})$. The image of this homomorphism is denoted by $\mathsf{Action}(\mathit{\Gamma}, K) \le \mathsf{Sym}(K)$.

Let us mention some facts about group actions (see for example [10, 16]): Obviously k-transitivity implies k-homogeneity and $(k-1)$-transitivity if $2 \le k \le |M|$. On the other hand, k-homogeneity implies $(k-1)$-transitivity if $2 \le k \le \frac{1}{2}|M|$. Using the classification of finite simple groups [20], $\mathsf{Action}(\mathit{\Gamma}, M)$ is equal to the alternating group $\mathsf{Alt}(M)$ or the symmetric group $\mathsf{Sym}(M)$ of M, if $\mathit{\Gamma}$ acts 6-transitive on M (see [10, 16]).

We say that $\mathit{\Gamma}$ acts *partition-homogeneous* on a finite set M, if for any number partition $p \in \mathcal{P}(|M|)$ the action of $\mathit{\Gamma}$ is transitive on $\mathcal{B}_p(M)$, where $\gamma B = \{\gamma b \,:\, b \in B\}$ for $B \in \mathcal{B}(M)$.

Lemma 2. *If $\mathit{\Gamma}$ acts partition-homogeneous on M and $|M| > 2$, then $\mathit{\Gamma}$ acts k-homogeneous for all $k \in \{1, \dots, |M|\}$.*

Proof. Let $k \le \frac{1}{2}|M|$ and K_1, K_2 be two k-subsets of M. Consider the set partitions

$$B_i = \{M \setminus K_i\} \cup \Big\{\{x\} : x \in K_i\Big\}$$

for $i \in \{1, 2\}$. Obviously, B_1 and B_2 have the same type, so there is a $\gamma \in \mathit{\Gamma}$ with $\gamma B_1 = B_2$. This implies $\gamma K_1 = K_2$. For $k > \frac{1}{2}|M|$, we note that k-homogeneity is equivalent to $(|M| - k)$-homogeneity. ■

Lemma 3. *If $\mathit{\Gamma}$ acts partition-homogeneous on M and $R \subseteq M$, then the setwise stabilizer $\mathsf{Stab}_{\{R\}}(\mathit{\Gamma})$ of R acts 2-homogeneous on R.*

Proof. Let $A_1, A_2 \subseteq R$ be two 2-sets. First, assume that $|M \setminus R| > 2$: Set

$$B_i = \{M \setminus R, A_i\} \cup \Big\{\{x\} \,:\, x \in R \setminus A_i\Big\}$$

for $i \in \{1, 2\}$. Then there exists a $\gamma \in \mathit{\Gamma}$ with $\gamma B_1 = B_2$. It follows, that $\gamma R = R$ and $\gamma A_1 = A_2$. Secondly, if $|M \setminus R| \le 2$ and $|M| > 6$, set

$$B_i = \{A_i, R \setminus A_i\} \cup \Big\{\{x\} \,:\, x \in M \setminus R\Big\}$$

for $i \in \{1, 2\}$. Then there is a $\gamma \in \mathit{\Gamma}$ with $\gamma B_1 = B_2$, which implies $\gamma R = R$ and $\gamma A_1 = A_2$. Finally, the remaining case $|M \setminus R| \le 2$ and $|M| \le 6$ follows from individual discussions depending on $|R|$ and $|M|$. ■

Proposition 4. *If $\mathit{\Gamma}$ acts partition-homogeneous on M then*

$$\mathsf{Action}(\mathit{\Gamma}, M) = \mathsf{Alt}(M) \qquad \text{or} \qquad \mathsf{Action}(\mathit{\Gamma}, M) = \mathsf{Sym}(M).$$

Proof. If $|M| \ge 14$, then $\mathit{\Gamma}$ is 7-homogeneous and thus 6-transitive, which implies the assertion using the classification of finite simple groups. The remaining cases follow from a case-by-case study. (Using the fact that with some exceptions k-homogeneity implies k-transitivity the previous argument can be refined, so that only a few cases remain.) ■

Let X be a multigraph and $\mathsf{Aut}(X)$ be its automorphism group. Furthermore, let $D \subseteq VX$ be a vertex subset. We say that X is k-homogeneous with respect to D, if $\mathsf{Stab}_{\{D\}}(\mathsf{Aut}(X))$ acts k-homogeneous on D. Similarly, we say that X is partition-homogeneous with respect to D, if the action of $\mathsf{Stab}_{\{D\}}(\mathsf{Aut}(X))$ on D is so.

2.3. **Electrical networks.** Let F be a finite non-empty set and X a multigraph with vertex set F. In addition, let $c : EX \to (0, \infty)$ be conductances on the edges of X. Then the pair (F, c) is called an *electrical network* (this notation suppresses the dependence on X, since X is implicitly defined by c). The network (F, c) is called *irreducible*, if the multigraph X is connected. Furthermore, the (positive semidefinite) *Laplace operator* (or *Laplacian*) $\varDelta : \mathbb{R}^F \to \mathbb{R}^F$ of a network (F, c) is defined by

$$\varDelta(f)(x) = \sum_{\substack{e \in EX \\ e=\{x,y\}}} \big(f(x) - f(y)\big)c(e).$$

For a non-empty subset $B \subseteq F$ and a function $g : B \to \mathbb{R}$ there exist solutions $f : F \to \mathbb{R}$ of the Dirichlet problem: $f|_B = g$ and $(\varDelta f)(x) = 0$ for all $x \in F \setminus B$. Any solution is unique on connected components of the multigraph X containing elements of B. The harmonic extension $H_B^F g$ of g is defined to be the unique solution of the Dirichlet problem, which is identically zero on components disjoint from B. This defines a linear operator $H_B^F : \mathbb{R}^B \to \mathbb{R}^F$.

Two networks (F, c_F) and (G, c_G) with $\varnothing \neq B \subseteq F \cap G$ are called *electrically equivalent* with respect to B, if they cannot be distinguished by applying voltages to B and measuring the resulting currents on B. In terms of the associated Laplace operators $\varDelta_F$ and $\varDelta_G$ electrical equivalence means

$$\varPi_B \varDelta_F H_B^F = \varPi_B \varDelta_G H_B^G,$$

where $\varPi_B : \mathbb{R}^F \to \mathbb{R}^B$, $f \mapsto f|_B$ is the canonical projection.

Let (F, c) be a network, $\varDelta$ be the associated Laplace operator, and $B \subseteq F$ a non-empty set. Define the *trace* $\mathsf{Tr}(\varDelta|B) : \mathbb{R}^B \to \mathbb{R}^B$ of $\varDelta$ on B by $\mathsf{Tr}(\varDelta|B) = \varPi_B \varDelta H_B^F$ and denote by $\mathsf{Tr}(c|B)$ the conductances on the complete graph with vertex set B associated with $\mathsf{Tr}(\varDelta|B)$. Then (F, c) and $(B, \mathsf{Tr}(c, B))$ are equivalent with respect to B. Note that the Dirichlet principle implies

$$\langle \mathsf{Tr}(\varDelta|B)g, g\rangle = \min\{\langle \varDelta f, f\rangle : f \in \mathbb{R}^F, f|_B = g\}, \tag{6}$$

where the minimum is attained if f is the harmonic extension of g.

Lemma 5. *Let (F, c) be a network with $c : EX \to (0, \infty)$ and $\varGamma \leq \mathsf{Aut}(X)$, such that $c(e) = c(\gamma_e(e))$ for all $e \in EX$ and $\gamma \in \varGamma$. If B is a non-empty subset of F and $c_B = \mathsf{Tr}(c|B)$, then $c_B(\{x, y\}) = c_B(\{\gamma x, \gamma y\})$ for all $x, y \in B$ and $\gamma \in \mathsf{Stab}_{\{B\}}(\varGamma)$.*

Proof. Set $\varDelta_B = \mathsf{Tr}(\varDelta|B)$. If $g \in \mathbb{R}^B$ and $\gamma \in \mathsf{Stab}_{\{B\}}(\varGamma)$, then

$$\begin{aligned}\langle \varDelta_B g \circ \gamma, g \circ \gamma\rangle &= \inf\{\langle \varDelta h, h\rangle : h \in \mathbb{R}^F, h|_B = g \circ \gamma\} \\ &= \inf\{\langle \varDelta f \circ \gamma, f \circ \gamma\rangle : f \in \mathbb{R}^F, f|_B = g\} = \langle \varDelta_B g, g\rangle\end{aligned}$$

by virtue of (6). Now the polarization equation implies

$$\begin{aligned}c_B(\{x, y\}) &= \langle \varDelta_B 1_{\{x\}}, 1_{\{y\}}\rangle = \tfrac{1}{2}\big(\langle \varDelta_B 1_{\{x,y\}}, 1_{\{x,y\}}\rangle - \langle \varDelta_B 1_{\{x\}}, 1_{\{x\}}\rangle - \langle \varDelta_B 1_{\{y\}}, 1_{\{y\}}\rangle\big) \\ &= \tfrac{1}{2}\big(\langle \varDelta_B 1_{\{\gamma x,\gamma y\}}, 1_{\{\gamma x,\gamma y\}}\rangle - \langle \varDelta_B 1_{\{\gamma x\}}, 1_{\{\gamma x\}}\rangle - \langle \varDelta_B 1_{\{\gamma y\}}, 1_{\{\gamma y\}}\rangle\big) \\ &= \langle \varDelta_B 1_{\{\gamma x\}}, 1_{\{\gamma y\}}\rangle = c_B(\{\gamma x, \gamma y\})\end{aligned}$$

for $x, y \in B$ and $\gamma \in \mathsf{Stab}_{\{B\}}(\varGamma)$, where 1_A denotes the characteristic function of a set A. ∎

The unit conductances $c : EX \to (0, \infty)$ on a finite multigraph X are defined by $c(e) = 1$ for all edges $e \in EX$. In this case, the Laplace operator $\varDelta$ of c corresponds to the combinatorial Laplace matrix of X. Note that $c(e) = c(\gamma_e(e))$ for $e \in EX$ and $\gamma \in \mathsf{Aut}(X)$ in this case.

Corollary 6. *Let X be a finite, connected multigraph and c be the unit conductances on X. If D is a non-empty subset of VX, so that X is 2-homogeneous with respect to D, then $\mathsf{Tr}(c|D)$ is a multiple of the unit conductances c_D on the complete graph with vertex set D.*

In the setting of the previous corollary, the factor ρ for which $\mathsf{Tr}(c|D) = \rho^{-1}c_D$ holds is called the *resistance scaling factor* of X with respect to D, see [3, 24, 33] for similar notions.

Lemma 7. *The resistance scaling factor of the star $K_{1,\theta}$ with respect to the set D consisting of the θ leaves is given by $\rho = \theta$.*

Proof. Denote by $u \in VK_{1,\theta}$ the center of star $K_{1,\theta}$, fix some leaf $v \in D$, and let K_θ be the complete graph with vertex set D ($VK_\theta = D$). Let c and c_D be the unit conductances on $K_{1,\theta}$ and on K_θ, respectively, and denote by $\varDelta$ and $\varDelta_D$ the associated Laplace operators. Finally, set $g : VK_\theta \to \mathbb{R}, w \mapsto 1_{\{v\}}(w)$ and let $h : VK_{1,\theta} \to \mathbb{R}$ be the harmonic extension of g: this implies $h(u) = \theta^{-1}$. Then we have

$$\rho = \frac{\langle \varDelta_D g, g\rangle}{\langle \varDelta h, h\rangle} = \frac{\theta - 1}{\frac{\theta-1}{\theta}} = \theta,$$

which proves the statement. ■

2.4. **Spanning forests.** Let X be a multigraph with θ distinguished vertices $D \subseteq VX$. Every spanning forest F of X induces a set partition B on D: the distinguished vertices in one connected component of F form a block of B. Let $\mathcal{S}_X$ be the set of non-empty spanning forests of X, which only have components containing at least one distinguished vertex each. For $B \in \mathcal{B}(D)$ write $\mathcal{S}_X(B)$ for the set of those forests in $\mathcal{S}_X$, whose induced set partition is B. If $p \in \mathcal{P}(\theta)$,

$$\mathcal{S}_X(p) = \biguplus_{B \in \mathcal{B}_p(D)} \mathcal{S}_X(B)$$

denotes the set of spanning forests in $\mathcal{S}_X$ defining a set partition of type p. Then

$$\mathcal{S}_X = \biguplus_{p \in \mathcal{P}(\theta)} \mathcal{S}_X(p).$$

The number of spanning trees in a finite multigraph X is often called the *complexity* of X and denoted by $\tau(X)$.

A *rooted spanning forest* (F, R) of a multigraph X is a spanning forest F of X together with a collection $R \subseteq VX$ of roots, such that F has exactly $|R|$ components and each component contains exactly one element of R. We denote by $\mathcal{R}_X(R)$ the set of all rooted spanning forests of X with roots $R \subseteq VX$. Let $\varDelta$ be the Laplace operator associated with the unit conductances on X. An extension of Kirchhoff's famous matrix tree theorem states that the number of rooted spanning forests (F, R) of a finite multigraph X with given roots $\varnothing \neq R \subseteq VX$ is

$$|\mathcal{R}_X(R)| = \mathsf{det}(\varPi_H \varDelta \varPi_H^*),$$

where $H = VX \setminus R$, see [34]. (If $H = \varnothing$ the above determinant is defined to be 1.) Note that $\varPi_H \varDelta \varPi_H^*$ is the Dirichlet-Laplace operator with respect to the boundary R.

Let X be a finite multigraph and $D \subseteq VX$ be a θ-set. Then $\tau(X) = |\mathcal{S}_X(\{D\})| = |\mathcal{R}_X(\{v\})|$ for any vertex $v \in VX$. Similarly, we have $|\mathcal{S}_X(B)| = |\mathcal{R}_X(D)|$ for $B = \{\{v\} : v \in D\} \in \mathcal{B}(D)$. If $k \in \{2, \dots, \theta - 1\}$ and X is k-homogeneous with respect to D, then $|\mathcal{R}_X(R_1)| = |\mathcal{R}_X(R_2)|$ for any two k-sets $R_1, R_2 \subseteq D$.

Theorem 8. *Let X be a connected, finite multigraph and let $D \subseteq VX$ be a vertex subset with θ vertices. Suppose that X is partition-homogeneous with respect to D. Then*

$$|\mathcal{R}_X(R)| = k\rho^{k-1}\theta^{1-k}\tau(X)$$

for all k-sets $R \subseteq D$, where ρ is the resistance scaling factor of X with respect to D.

Proof. Since X is partition-homogeneous with respect to D, X is also k-homogeneous with respect to D for $k \in \{2, \dots, \theta - 1\}$ by Lemma 2. Hence

$$|\mathcal{R}_X(R_1)| = |\mathcal{R}_X(R_2)| \tag{7}$$

for all $R_1, R_2 \subseteq D$ of equal size. Now let B, C be non-empty subsets of D with $B \uplus \{w\} = C$. We prove that

$$|\mathcal{R}_X(C)| = \frac{\rho|C|}{\theta|B|}\, |\mathcal{R}_X(B)|$$

holds, which implies the statement by an easy induction. As before, let Δ be the Laplace operator associated with the unit conductances on X. For convenience, set $\Delta_A = \Pi_{VX\setminus A} \Delta \Pi^*_{VX\setminus A}$ for any non-empty set $A \subseteq D$. Then

$$\frac{|\mathcal{R}_X(B \cup \{x\})|}{|\mathcal{R}_X(B)|} = \frac{\det \Delta_{B\cup\{x\}}}{\det \Delta_B} = \langle 1_{\{x\}}, \Delta_B^{-1} 1_{\{x\}} \rangle$$

for all $x \in D \setminus B$. Thus (7) yields

$$\langle 1_{\{x\}}, \Delta_B^{-1} 1_{\{x\}} \rangle = \langle 1_{\{y\}}, \Delta_B^{-1} 1_{\{y\}} \rangle$$

for all $x, y \in D \setminus B$. Furthermore, if $v, w, x, y \in D \setminus B$ with $v \neq w$ and $x \neq y$, then there is an automorphism γ of X due to Lemma 3, which stabilizes the set B and satisfies $\{\gamma v, \gamma w\} = \{x, y\}$. This implies

$$\langle 1_{\{v\}}, \Delta_B^{-1} 1_{\{w\}} \rangle = \langle 1_{\{x\}}, \Delta_B^{-1} 1_{\{y\}} \rangle,$$

since Δ_B is symmetric. Hence all diagonal entries, as well as all non-diagonal entries of Δ_B^{-1} corresponding to indices from $D \setminus B$ are equal: there are numbers a and b, so that

$$\langle 1_{\{x\}}, \Delta_B^{-1} 1_{\{x\}} \rangle = a \qquad \text{and} \qquad \langle 1_{\{x\}}, \Delta_B^{-1} 1_{\{y\}} \rangle = b$$

for all distinct $x, y \in D \setminus B$. Set

$$h = H_D^{VX} 1_{\{w\}}, \qquad g = \Delta h = \Delta H_D^{VX} 1_{\{w\}},$$

where $H_D^{VX} f$ is the harmonic extension of a function $f : D \to \mathbb{R}$. Note that $\Pi_D\, g = \mathsf{Tr}(\Delta|D)\, 1_{\{w\}}$ and $\Pi_{VX\setminus D}\, g = 0$. Using the symmetry condition once again, $g(w) = (\theta - 1)\rho^{-1}$ and $g(x) = -\rho^{-1}$ for $x \in D \setminus \{w\}$. The definition of h implies $\Pi_B\, h = 0$ and therefore

$$\Delta_B(\Pi_{VX\setminus B}\, h) = \Pi_{VX\setminus B}\, g \qquad \text{and} \qquad \Pi_{VX\setminus B}\, h = \Delta_B^{-1}(\Pi_{VX\setminus B}\, g).$$

For $x \in D \setminus B$ a short computation yields

$$h(x) = \big(\Delta_B^{-1}(\Pi_{VX\setminus B}\, g)\big)(x) = \sum_{y \in D\setminus B} \langle 1_{\{x\}}, \Delta_B^{-1} 1_{\{y\}} \rangle\, g(y),$$

since $\Pi_{VX\setminus D}\, g = 0$. If $x = w$ and $x \neq w$, respectively, we obtain a simple linear system of equations from the last identity:

$$\begin{aligned} 1 &= (\theta - 1)\rho^{-1} a - (\theta - |C|)\rho^{-1} b, \\ 0 &= -\rho^{-1} a + |C|\rho^{-1} b, \end{aligned}$$

with the solution

$$a = \frac{\rho|C|}{\theta|B|} \qquad \text{and} \qquad b = \frac{\rho}{\theta|B|}$$

using $|C| = |B| + 1$, which finishes the proof. ■

Finally, we remark the following connection between the complexity and the spectrum of the combinatorial Laplacian by virtue of Kirchhoff's theorem. The complexity $\tau(X)$ of a graph X with $v = |VX|$ vertices is given by $v\,\tau(X) = \lambda_1 \cdots \lambda_{v-1}$, where $\lambda_1, \dots, \lambda_{v-1}$ are the nonzero eigenvalues of the combinatorial Laplacian Δ on X (counted with multiplicity). Denote by P the characteristic polynomial of Δ, then the product of the nonzero eigenvalues is equal to the coefficient of the linear term of P (up to the sign): $[x]P(-x) = v\,\tau(X)$. Hence the quantity $[x]P(-x)$ is given by the number of rooted spanning trees of X. Similarly, the coefficient $[x^k]P(-x)$ is equal to the number of rooted spanning forests with exactly k components, see for instance [34].

Now, if D is a vertex subset with θ elements, so that X is partition-homogeneous with respect to D, then the previous theorem relates the spectrum of the combinatorial Laplacian Δ with the spectrum of the Dirichlet-Laplace operator with boundary $R \subseteq D$.

3. Self-similar graphs

3.1. **Construction.** Let G be an edgeless graph with $\theta \geq 2$ distinguished vertices given by $\eta : \Theta \to VG$ ($\Theta = \{1, \ldots, \theta\}$). Let $s \geq 2$ substitutions be defined by injective maps $\sigma_i : \Theta \to VG$ for $i \in S = \{1, \ldots, s\}$. For any multigraph X and any injective map $\varphi : \Theta \to VX$ a new multigraph Y together with an injective map $\psi : \Theta \to VY$ is constructed as follows:

For each $i \in S$ let Z_i be an isomorphic copy of the multigraph X, so that the vertex sets $VZ_1, \ldots, VZ_s$, and VG are mutually disjoint. The isomorphism between X and Z_i is denoted by $\zeta_i : VX \to VZ_i$. Let Z be the disjoint union of G and $Z_1, \ldots, Z_s$ and define the relation $\sim$ on VZ as the reflexive, symmetric, and transitive hull of

$$\bigcup_{i=0}^{s} \Big\{ (\sigma_i(j), \zeta_i(\varphi(j))) \, : \, j \in \Theta \Big\} \subseteq VZ \times VZ.$$

Then the multigraph Y is defined by its vertex set $VY = VZ/{\sim}$ and edge multiset

$$EY = \Big\{ \{[v], [w]\} \, : \, \{v, w\} \in EZ \Big\},$$

where $[v]$ denotes the equivalence class of a vertex v. The map $\psi : \Theta \to VY$ is defined by $\psi(i) = [\eta(i)] \in VY$.

If the pair (Y, ψ) is constructed as above from (X, φ), we write $(Y, \psi) = \mathsf{Copy}(X, \varphi)$. Since we fix G, η, and $\{\sigma_i : i \in S\}$, the dependence on these items is suppressed. Note that Y is the amalgamation of s isomorphic copies of X: for $i \in S$ define $\bar{Z}_i$ by

$$V\bar{Z}_i = \Big\{ [v] \, : \, v \in VZ_i \Big\} \qquad \text{and} \qquad E\bar{Z}_i = \Big\{ \{[v], [w]\} \, : \, \{v, w\} \in EZ_i \Big\},$$

then $\bar{Z}_i$ is isomorphic to X and the isomorphism is given by

$$\bar{\zeta}_i : VX \to V\bar{Z}_i, \quad v \mapsto [\zeta_i(v)].$$

The subgraph $\bar{Z}_i$ is called the *i-th part* of Y. On the i-th part of Y distinguished vertices are given by $\Theta \to V\bar{Z}_i$, $j \mapsto \bar{\zeta}_i(\varphi(j)) = [\sigma_i(j)]$. We say, that the initial data G, η and σ_i satisfy

- the *connectedness condition*, if the union of $\sigma_i(\Theta)$ for $i \in S$ covers VG and if the family $\{\{\sigma_i(\Theta)\} : i \in S\}$ is connected.
- the *separation condition*, if, for distinct $i, j \in S$, the intersection $\sigma_i(\Theta) \cap \sigma_j(\Theta)$ contains at most one vertex of G,

The following lemmata collect immediate consequences of the construction:

Lemma 9. *Let X be a connected multigraph, $\varphi : \Theta \to VX$ be an injective map, and set $(Y, \psi) = \mathsf{Copy}(X, \varphi)$.*

- *If the initial data satisfy the connectedness condition and if X is connected, then Y is connected, too.*
- *If the initial data satisfy the separation condition and if X is a graph, then Y is also a graph (i. e. there are no parallel edges or loops).*
- *If connectedness holds, then $|VG| \leq s(\theta - 1) + 1$.*

Define the constant κ by $\kappa = s(\theta - 1) + 1 - |VG|$. If the connectedness condition is satisfied, then $\kappa \geq 0$ has a geometrical interpretation: Suppose that X is a connected and $(Y, \psi) = \mathsf{Copy}(X, \varphi)$. If H is a subgraph of Y, so that the restriction of H on each part of Y is a spanning tree, then the cyclomatic number of H is κ.

Lemma 10. *The cardinalities of VY and EY satisfy*

$$|VY| = s(|VX| - \theta) + |VG| = s(|VX| - 1) - \kappa + 1 \qquad \text{and} \qquad |EY| = s\,|EX|.$$

Thus, if $c(X)$ and $c(Y)$ are the cyclomatic numbers of X and Y, respectively, then $c(Y) = s\,c(X) + \kappa$.

3.2. **Examples.** It occurs frequently that the above substitution procedure is applied to the θ-complete graph $X = K_\theta$. In this case it does not matter which specific injective map $\varphi : \varTheta \to VX$ is chosen, since all of them yield isomorphic results $(Y, \psi) = \mathsf{Copy}(X, \varphi)$. Similarly, if X is equal to the star $K_{1,\theta}$, φ will always be some injective map from $\varTheta$ to the leaves of X, and the result $(Y, \psi) = \mathsf{Copy}(X, \varphi)$ does not depend on the specific choice of φ. In these two cases φ will not be explained any further.

3.2.1. *Sierpiński graphs.* Fix some $d \in \mathbb{N}_0$ and let $s = \theta = d + 1$. Define the edgeless graph G by

$$VG = \left\{ \boldsymbol{x} \in \mathbb{N}_0^{d+1} \,:\, \|\boldsymbol{x}\|_1 = 2 \right\}$$

and the map $\eta : \varTheta \to VG$ by $\eta(i) = 2\mathbf{e}_i$, where $\mathbf{e}_i$ is the i-th canonical basis vector of $\mathbb{R}^{d+1}$. In addition, set $\sigma_i(j) = \mathbf{e}_i + \mathbf{e}_j \in VG$ for $i \in S$ and $j \in \varTheta$. Note that $\varTheta = S = \{1, \ldots, d+1\}$. It is easy to see that

$$|VG| = \tfrac{1}{2}(d+2)(d+1) \qquad \text{and} \qquad \kappa = d(d+1) + 1 - \tfrac{1}{2}(d+2)(d+1) = \tfrac{1}{2}d(d-1).$$

The usual finite d-dimensional Sierpiński graphs are then constructed as follows: Let $X_0 = K_{d+1}$ and inductively define (X_n, φ_n) by $(X_n, \varphi_n) = \mathsf{Copy}(X_{n-1}, \varphi_{n-1})$ for $n \in \mathbb{N}$. See Figure 1 for the case $d = 2$. The resistance scaling factor $\rho(X_1)$ of X_1 with respect to $\varphi_1(\varTheta)$ is given by $\rho(X_1) = \frac{d+3}{d+1}$, which can be seen from a successive application of Lemma 7 together with the rule for resistors in series.

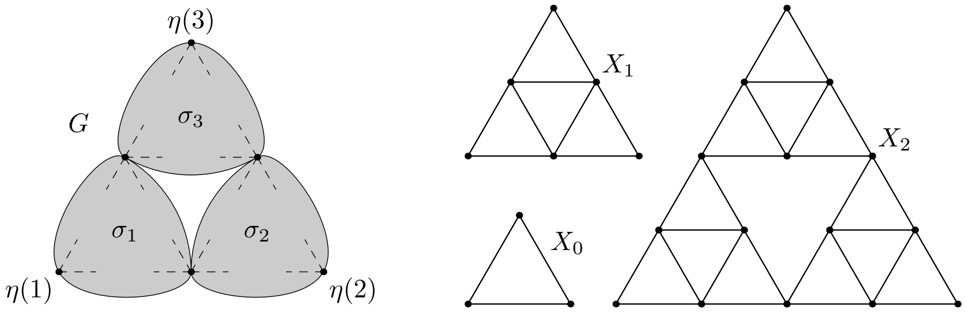

FIGURE 1. Initial data and finite 2-dimensional Sierpiński graphs.

3.2.2. *Austria graphs.* The "Austria" graphs are studied in [28] (their shape resembles a map of Austria). Let $\theta = 2$, $s = 4$, and $VG = \{1, 2, 3, 4\}$. Define η and $\sigma_1, \ldots, \sigma_4$ as follows:

i	$\eta(i)$	$\sigma_1(i)$	$\sigma_2(i)$	$\sigma_3(i)$	$\sigma_4(i)$
1	1	1	2	4	4
2	4	2	3	2	3

Obviously, we have $\kappa = 1$. The finite Austria graphs are inductively constructed by $X_0 = K_2$ and $(X_n, \varphi_n) = \mathsf{Copy}(X_{n-1}, \varphi_{n-1})$ for $n \in \mathbb{N}$, see Figure 2 for an illustration of the initial data and some finite Austria graphs. Note that the resistance scaling factor $\rho(X_1)$ of X_1 with respect to $\varphi_1(\varTheta)$ is given by $\rho(X_1) = \frac{5}{3}$. The orientation of each of the four substitutions (defined by $\sigma_1, \ldots, \sigma_4$) can be flipped. For example, σ_1 could also be defined by $\sigma_1(1) = 2$ and $\sigma_1(2) = 1$. Note that two distinct choices yield different graph sequences $X_0, X_1, \ldots$ (the specific configuration can be identified in X_2). Here the substitutions $\sigma_1, \ldots, \sigma_4$ are chosen, so that the vertex degrees in $X_0, X_1, \ldots$ are uniformly bounded.

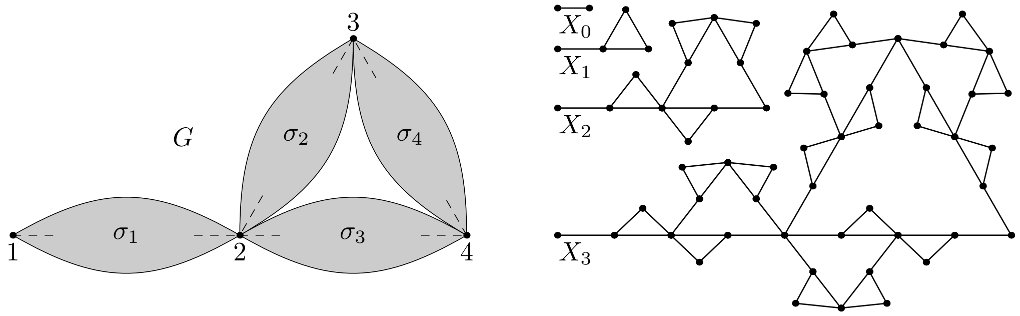

FIGURE 2. Initial data and finite Austria graphs.

3.2.3. *A multigraph example.* Set $\theta = 3$, $s = 4$, $VG = \{1, \dots, 6\}$, and define η and $\sigma_1, \dots, \sigma_4$ as follows (see Figure 3):

i	$\eta(i)$	$\sigma_1(i)$	$\sigma_2(i)$	$\sigma_3(i)$	$\sigma_4(i)$
1	1	1	1	1	2
2	2	2	2	3	3
3	3	3	4	5	6

In addition, it is easy to see that $\kappa = 3$. The multigraph sequence $X_0, X_1, \dots$ is defined by $(X_n, \varphi_n) = \mathsf{Copy}(X_{n-1}, \varphi_{n-1})$, where $X_0 = K_3$. As a consequence of the construction principle, several parallel edges appear, and the number of vertex pairs connected by more than one edge is unbounded as well as the number of edges connecting certain pairs. Finally, a short computation yields the resistance scaling factor $\rho(X_1)$ of X_1 with respect to $\varphi_1(\Theta)$: $\rho(X_1) = \frac{2}{5}$.

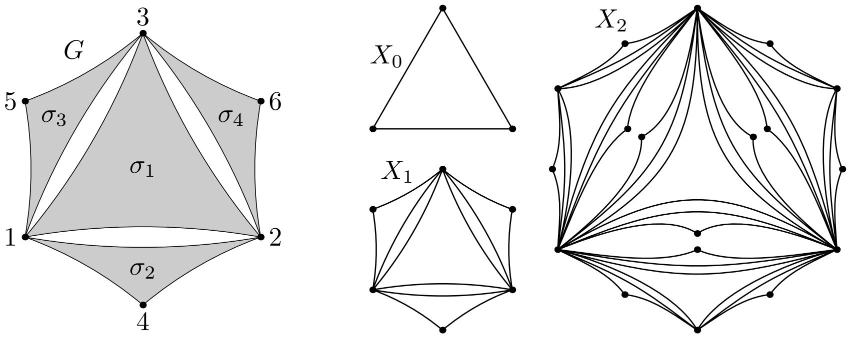

FIGURE 3. Initial data and the graphs X_0, X_1, X_2.

3.2.4. *An example without full symmetry.* Let $\theta = 3$, $s = 7$, $VG = \{1, \dots, 12\}$, and define η and $\sigma_1, \dots, \sigma_7$ as follows:

i	$\eta(i)$	$\sigma_1(i)$	$\sigma_2(i)$	$\sigma_3(i)$	$\sigma_4(i)$	$\sigma_5(i)$	$\sigma_6(i)$	$\sigma_7(i)$
1	1	1	2	3	4	5	6	7
2	3	7	8	9	10	11	12	9
3	5	12	7	8	9	10	11	11

It is then easy to compute κ: $\kappa = 3$. Now set $X_0 = K_3$ and define $(X_n, \varphi_n) = \mathsf{Copy}(X_{n-1}, \varphi_{n-1})$. Note that, for $n \geq 1$, the action of the automorphism group $\mathsf{Aut}(X_n)$ on the set $\varphi_n(\Theta)$ is given by the alternating group of degree 3:

$$\mathsf{Action}(\mathsf{Aut}(X_n), \varphi_n(\Theta)) = \mathsf{Alt}(\varphi_n(\Theta)).$$

See Figure 4 for an illustration of the initial data and the graphs X_0, X_1, and X_2. Quite surprisingly, the value of the resistance scaling factor $\rho(X_1)$ of X_1 with respect to $\varphi_1(\Theta)$ is a very simple one, namely $\rho(X_1) = 2$, which can be seen from successive applications of the Wye-Delta-transform or by calculating the energy form.

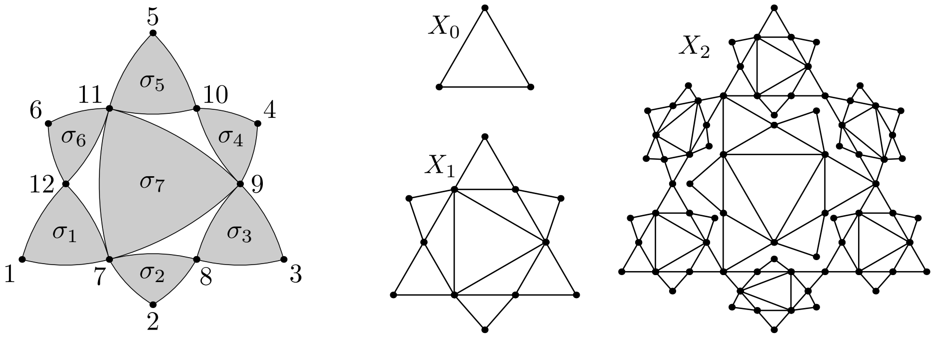

FIGURE 4. Initial data and the graphs X_0, X_1, X_2.

3.3. **Symmetry.** A group $\varGamma \leq \mathsf{Sym}(\varTheta)$ is *invariant* with respect to the above construction, if the following holds: for each $\gamma \in \varGamma$ there are $\xi \in \mathsf{Sym}(VG)$, $\pi \in \mathsf{Sym}(S)$, and $\gamma_1, \ldots, \gamma_s \in \varGamma$, so that $\xi \circ \eta = \eta \circ \gamma$ and $\xi \circ \sigma_i = \sigma_{\pi(i)} \circ \gamma_i$ for all $i \in S$. The following lemma explains the relevance of invariant groups:

Lemma 11. *Let X be a multigraph, $\varphi : \varTheta \to VX$ an injective map, and set $(Y, \psi) = \mathsf{Copy}(X, \varphi)$. Let $\varGamma$ be an invariant group, set*

$$\varGamma^{\varphi} = \{\varphi \circ \gamma \circ \varphi^{-1} \, : \, \gamma \in \varGamma\} \leq \mathsf{Sym}(\varphi(\varTheta))$$

and analogously define $\varGamma^{\psi} \leq \mathsf{Sym}(\psi(\varTheta))$. If $\varGamma^{\varphi}$ is a subgroup of $\mathsf{Action}(\mathsf{Aut}(X), \varphi(\varTheta))$, then $\varGamma^{\psi}$ is a subgroup of $\mathsf{Action}(\mathsf{Aut}(Y), \psi(\varTheta))$

Proof. We have to show that for each $\gamma \in \varGamma$ there is a $\bar{\gamma} \in \mathsf{Aut}(Y)$ with $\psi \circ \gamma = \bar{\gamma} \circ \psi$. By definition, there are $\xi \in \mathsf{Sym}(VG)$, $\pi \in \mathsf{Sym}(S)$, and $\gamma_1, \ldots, \gamma_s \in \varGamma$, so that $\xi \circ \eta = \eta \circ \gamma$ and $\xi \circ \sigma_i = \sigma_{\pi(i)} \circ \gamma_i$ for all $i \in S$. By assumption, there are $\bar{\gamma}_1, \ldots, \bar{\gamma}_s \in \mathsf{Aut}(X)$, so that $\varphi \circ \gamma_i = \bar{\gamma}_i \circ \varphi$ for $i \in S$. Now, if x is a vertex in the i-th part of Y, set

$$\bar{\gamma}(x) = \bar{\zeta}_{\pi(i)} \circ \bar{\gamma}_i \circ \bar{\zeta}_i^{-1}(x).$$

(Here $\bar{\zeta}_i : VX \to V\bar{Z}_i$ is the isomorphism from X to the i-th part $\bar{Z}_i$ of Y.) It is easy to check that $\bar{\gamma}$ is a well-defined automorphism of Y, which satisfies $\psi \circ \gamma = \bar{\gamma} \circ \psi$: notice that $\bar{\gamma}([v]) = [\xi(v)]$ for all $v \in VG$. ■

Note that the join $\varGamma_1 \vee \varGamma_2$ of two invariant groups $\varGamma_1$ and $\varGamma_2$ is also an invariant group. Hence there exists a maximal invariant group. If this maximal invariant group acts k-homogeneous on $\varTheta$, and if $|\eta(\varTheta) \cap \sigma_i(\varTheta)| = k$ for some $i \in S$, then $s \geq \binom{\theta}{k}$: for any k-subset $K \subseteq \varTheta$ there must be an index $j \in S$ with $\eta(\varTheta) \cap \sigma_j(\varTheta) = \eta(K)$.

Corollary 12. *If separation and connectedness hold, and if the maximal invariant group acts k-homogeneous on $\varTheta$ for all $k \in \varTheta$, then $s \geq \theta$.*

Proof. Using $s \geq 2$, separation, and connectedness there exists an index $i \in S$, so that $|\eta(\varTheta) \cap \sigma_i(\varTheta)| = k$ for some $k \in \{1, \ldots, \theta - 1\}$. The k-homogeneity of the maximal invariant group implies $s \geq \binom{\theta}{k} \geq \theta$. ■

We say that the initial data satisfies the *symmetry condition*, if the maximal invariant group acts partition-homogeneous on $\varTheta$. Proposition 4 and Lemma 11 imply the following:

Corollary 13. *Suppose that X is a partition-homogeneous multigraph with respect to $\varphi(\varTheta)$, where $\varphi : \varTheta \to VX$ is injective, and set $(Y, \psi) = \mathsf{Copy}(X, \varphi)$. If the symmetry condition holds, then Y is partition-homogeneous with respect to $\psi(\varTheta)$.*

It is easy to see, that all examples of Section 3.2 satisfy the symmetry condition. In fact, the maximal invariant group of Example 3.2.1 and 3.2.3 is the symmetric group, whereas for Example 3.2.2 and 3.2.4 the maximal invariant group is given by the alternating group.

3.4. **Decomposition of set partitions and spanning forests.** Consider an element $\boldsymbol{\omega} = (\omega_1, \dots, \omega_s)$ in the Cartesian product $\prod_{i \in S} \mathcal{B}(\Theta)$ and denote by $\sigma(\boldsymbol{\omega})$ the family

$$\sigma(\boldsymbol{\omega}) = \Big\{ \sigma_i(\omega_i) \,:\, i \in S \Big\}.$$

Then $\mathsf{Union}(\sigma(\boldsymbol{\omega}))$ is a set partition of VG. Moreover, define the following counting functions:

$$\chi_p(\boldsymbol{\omega}) = \Big| \Big\{ i \in S \,:\, \omega_i \in \mathcal{B}_p(\Theta) \Big\} \Big| \qquad \text{and} \qquad \chi(\boldsymbol{\omega}) = \sum_{i \in S} |\omega_i| = \sum_{p \in \mathcal{P}(\theta)} |p| \, \chi_p(\boldsymbol{\omega})$$

for $p \in \mathcal{P}(\theta)$ and $\boldsymbol{\omega} \in \prod_{i \in S} \mathcal{B}(\Theta)$.

Now let X be a multigraph, $\varphi : \Theta \to VX$ be an injective map, and set $(Y, \psi) = \mathsf{Copy}(X, \varphi)$. For the sake of notation, define $\psi_i : \Theta \to VY$ by $\psi_i(j) = \bar{\zeta}_i(\varphi(j))$ $(i \in S)$. For $P \in \mathcal{B}(VY)$ consider the restriction $P|_{\psi_i(\Theta)}$ of P on the distinguished vertices of the i-th part of Y $(i \in S)$. Thus the assignment

$$\mathsf{Tr}(P) = \big(\psi_i^{-1}(P|_{\psi_i(\Theta)}) \big)_{i \in S}$$

defines a map $\mathsf{Tr} : \mathcal{B}(VY) \to \prod_{i \in S} \mathcal{B}(\Theta)$, the *trace* of a set partition.

Let F be a spanning forest in $\mathcal{S}_Y(\psi(B))$ for some $B \in \mathcal{B}(\Theta)$ and denote by F_i the restriction of F on the i-th part of Y $(i \in S)$. Then, for each $i \in S$, there exists exactly one spanning forest L_i of X, so that

$$F_i = \bar{\zeta}_i(L_i). \tag{8}$$

Furthermore, F induces a set partition P of VY, where the blocks of P are the vertex sets of the connected components of F. Define the *trace* $\boldsymbol{\omega} = \mathsf{Tr}(F)$ of F to be equal to $\mathsf{Tr}(P)$. Then the forest L_i is contained in the set $\mathcal{S}_X(\varphi(\omega_i))$ for $i \in S$, and we can draw some conclusions from this setup:

- For each $b \in \mathsf{Union}(\sigma(\boldsymbol{\omega}))$, the intersection $b \cap \eta(\Theta)$ is not empty.
- The restriction $\mathsf{Union}(\sigma(\boldsymbol{\omega}))|_{\eta(\Theta)}$ equals $\eta(B)$.
- The family $\sigma(\boldsymbol{\omega})$ is cycle-free.

The above discussion motivates the definition of $\Omega(B)$ for $B \in \mathcal{B}(\Theta)$: $\Omega(B)$ is the set of all $\boldsymbol{\omega} \in \prod_{i \in S} \mathcal{B}(\Theta)$, such that $b \cap \eta(\Theta) \neq \varnothing$ for $b \in \mathsf{Union}(\sigma(\boldsymbol{\omega}))$, $\mathsf{Union}(\sigma(\boldsymbol{\omega}))|_{\eta(\Theta)} = \eta(B)$, and $\sigma(\boldsymbol{\omega})$ is a cycle-free family. Then $\mathsf{Tr}(\mathcal{S}_Y(\psi(B))) = \Omega(B)$ and there is a bijective correspondence between

$$\mathcal{S}_Y(\psi(B)) \qquad \text{and} \qquad \biguplus_{\boldsymbol{\omega} \in \Omega(B)} \prod_{i \in S} \mathcal{S}_X(\varphi(\omega_i)). \tag{9}$$

for $B \in \mathcal{B}(\Theta)$, which is determined by (8).

With the symmetry condition in mind let us define the set $\Omega(p)$ for a number partition $p \in \mathcal{P}(\theta)$ by

$$\Omega(p) = \biguplus_{B \in \mathcal{B}_p(\Theta)} \Omega(B).$$

It is remarkable that, for any tuple $\boldsymbol{\omega} \in \Omega(p)$, the number of blocks $\chi(\boldsymbol{\omega})$ in $\boldsymbol{\omega}$ satisfies an identity, which only involves $|p|$:

Lemma 14. *Suppose that the connectedness condition is satisfied. Then, for $p \in \mathcal{P}(\theta)$, we have*

$$\chi(\boldsymbol{\omega}) = \kappa + s + |p| - 1$$

for all $\boldsymbol{\omega} \in \Omega(p)$.

Proof. Suppose that $X = K_\theta$ and $(Y, \psi) = \mathsf{Copy}(X, \varphi)$. We prove that $\chi(\mathsf{Tr}(F)) = \kappa + s + |p| - 1$ holds for all spanning forests $F \in \mathcal{S}_Y(p)$, which implies the statement, since each $\boldsymbol{\omega} \in \Omega(p)$ has a representation as a spanning forest in $\mathcal{S}_Y(p)$.

Let $\boldsymbol{\omega} \in \Omega(p)$ and let F be a spanning forest with $\boldsymbol{\omega} = \mathsf{Tr}(F)$. If $\omega_i \in \mathcal{B}_q(\Theta)$, F has exactly $\theta - |q|$ edges in the i-th part of Y (since F induces a spanning forest with $|q|$ components). Similarly,

F has a total of exactly $|VY| - |p|$ edges. Therefore, we have two expressions for the number of edges of F:

$$|VY| - |p| = \sum_{q\in\mathcal{P}(\Theta)} (\theta - |q|)\chi_q(\boldsymbol{\omega}) = \theta \sum_{q\in\mathcal{P}(\Theta)} \chi_q(\boldsymbol{\omega}) - \chi(\boldsymbol{\omega}) = \theta s - \chi(\boldsymbol{\omega}).$$

From Lemma 10 we know that $|VY| = s(\theta - 1) - \kappa + 1$. Now, solving the equation for $\chi(\boldsymbol{\omega})$ yields the lemma. ■

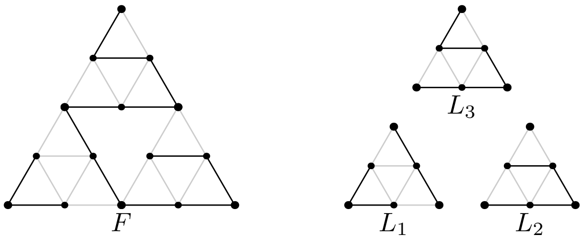

Figure 5. Decomposition of spanning forests.

Finally, we illustrate the correspondence given in (9) for the case of finite 2-dimensional Sierpiński graphs. Figure 5 depicts the decomposition of a spanning forest F of X_1 into a triple (L_1, L_2, L_3), so that the relation (8) holds. It is readily seen that $F \in \mathcal{S}_{X_1}(\varphi_1(\{1, 23\}))$ and

$$L_1 \in \mathcal{S}_{X_0}(\varphi_0(\{1, 23\})), \quad L_2 \in \mathcal{S}_{X_0}(\varphi_0(\{12, 3\})), \quad L_3 \in \mathcal{S}_{X_0}(\varphi_0(\{123\})).$$

Thus the trace of F is given by

$$\mathsf{Tr}(F) = (\{1, 23\}, \{12, 3\}, \{123\}) \in \Omega(\{1, 23\}).$$

Here and in the following we sometimes write $\{1, 23\}$ as a shorthand for the partition $\{\{1\}, \{2, 3\}\}$ and analogously for other partitions, if no ambiguity can occur.

4. Results

In the rest of this paper, we always assume that the initial data satisfy the connectedness and symmetry condition.

4.1. A recursion for spanning forests.

Let X be a connected multigraph and $\varphi : \Theta \to VX$ be an injective map. Suppose that X is partition-homogeneous with respect to $\varphi(\Theta)$. By virtue of symmetry

$$|\mathcal{S}_X(B_1)| = |\mathcal{S}_X(B_2)|$$

for all $B_1, B_2 \in \mathcal{B}(\varphi(\Theta))$ of the same type. Thus define $\tau_p(X)$ by

$$\tau_p(X) = \frac{|\mathcal{S}_X(p)|}{b(p)}$$

for $p \in \mathcal{P}(\theta)$, where $b(p) = |\mathcal{B}_p(\Theta)|$ is the number of set partitions of Θ of type p given by (1). Furthermore, write $\boldsymbol{\tau}(X)$ for the vector $(\tau_p(X))_{p\in\mathcal{P}(\theta)}$. Note that $\tau_p(X) = |\mathcal{S}_X(B)|$ for any $B \in \mathcal{B}_p(\varphi(\Theta))$ and $\tau_p(X)$ is equal to the complexity $\tau(X)$ of X if p is the trivial partition with one summand given by $p = \theta$. No confusion should occur between the complexity $\tau(X)$ and the vector $\boldsymbol{\tau}(X)$.

If $(Y, \psi) = \mathsf{Copy}(X, \varphi)$ then Equation (9) implies

$$b(p)\,\tau_p(Y) = \sum_{B\in\mathcal{B}_p(\Theta)} |\mathcal{S}_Y(\psi(B))| = \sum_{\boldsymbol{\omega}\in\Omega(p)} \prod_{i\in S} |\mathcal{S}_X(\varphi(\omega_i))| = \sum_{\boldsymbol{\omega}\in\Omega(p)} \prod_{q\in\mathcal{P}(\theta)} \tau_q(X)^{\chi_q(\boldsymbol{\omega})}$$

for all $p \in \mathcal{P}(\theta)$. For a subset $\Omega \subseteq \prod_{i\in S} \mathcal{B}(\Theta)$ define the generating function $\mathsf{GF}(\Omega \,|\, \boldsymbol{x})$ by

$$\mathsf{GF}(\Omega \,|\, \boldsymbol{x}) = \sum_{\boldsymbol{\omega}\in\Omega} \prod_{i\in S} x_{\ell(\omega_i)} = \sum_{\boldsymbol{\omega}\in\Omega} \prod_{q\in\mathcal{P}(\theta)} x_q^{\chi_q(\boldsymbol{\omega})},$$

where $\ell(\omega_i)$ is the type of the set partition ω_i. Now the following proposition is immediate:

Proposition 15. *The vectors* $\boldsymbol{\tau}(X)$ *and* $\boldsymbol{\tau}(Y)$ *satisfy the following identity:*

$$\boldsymbol{\tau}(Y) = \boldsymbol{Q}(\boldsymbol{\tau}(X)),$$

where the s*-homogeneous polynomial function* $\boldsymbol{Q} : \mathbb{R}^{\mathcal{P}(\theta)} \to \mathbb{R}^{\mathcal{P}(\theta)}$ *is given by its coordinates*

$$Q_p(\boldsymbol{x}) = \frac{1}{b(p)} \,\mathsf{GF}(\mathit{\Omega}(p) \,|\, \boldsymbol{x})$$

for $p \in \mathcal{P}(\theta)$*. Additionally, the symmetry condition implies that*

$$Q_p(\boldsymbol{x}) = \mathsf{GF}(\mathit{\Omega}(B) \,|\, \boldsymbol{x})$$

for $p \in \mathcal{P}(\theta)$ *and any* $B \in \mathcal{B}_p(\Theta)$*.*

Lemma 14 implies a constraint for the monomials of $\boldsymbol{Q}$: Let $p \in \mathcal{P}(\theta)$ and let

$$\prod_{q \in \mathcal{P}(\theta)} x_q^{n_q}$$

be a monomial with nonzero coefficient in Q_p, then there is some $\boldsymbol{\omega} \in \mathit{\Omega}(p)$ with $\chi_q(\boldsymbol{\omega}) = n_q$ for all $q \in \mathcal{P}(\theta)$ and the relation

$$\sum_{q \in \mathcal{P}(\theta)} |q| n_q = \chi(\boldsymbol{\omega}) = \kappa + s + |p| - 1$$

holds.

Let $p \in \mathcal{P}(\theta)$ and $F \in \mathcal{S}_Y(p)$, then $\mathsf{Tr}(F) \in \mathit{\Omega}(p)$. On the other hand, given a tuple $\boldsymbol{\omega}$ in $\mathit{\Omega}(p)$, there are

$$\prod_{q \in \mathcal{P}(\theta)} \tau_q(X)^{\chi_q(\boldsymbol{\omega})}$$

spanning forests F in $\mathcal{S}_Y(p)$ with $\mathsf{Tr}(F) = \boldsymbol{\omega}$. Note that this number may be zero, since $\tau_p(X) = 0$ for some $p \in \mathcal{P}(\theta)$. (This happens for example if $X = K_{1,\theta}$ is the star.) However, if X is given by the θ-complete graph K_θ, then, for $p \in \mathcal{P}(\theta)$, $\tau_p(X)$ is given by

$$\tau_p(X) = \prod_{k \in \mathbb{N}} (k^{k-2})^{\nu_k(p)},$$

which is always strictly larger than zero. Thus, if $X = K_\theta$ and $(Y, \psi) = \mathsf{Copy}(X, \varphi)$, each $\boldsymbol{\omega} \in \mathit{\Omega}(p)$ has a representation as spanning forest F in $\mathcal{S}_Y(p)$. This can be used to obtain another representation of $\boldsymbol{Q}$:

Corollary 16. *Let* $X = K_\theta$ *the complete graph and* $(Y, \psi) = \mathsf{Copy}(X, \varphi)$*. Then the coordinates of the polynomial* $\boldsymbol{Q}$ *satisfy*

$$Q_p(\boldsymbol{x}) = \frac{1}{b(p)} \sum_{F \in \mathcal{S}_Y(p)} \prod_{q \in \mathcal{P}(\theta)} \left(\frac{x_q}{\tau_q(X)} \right)^{\chi_q(\mathsf{Tr}(F))}$$

for all $p \in \mathcal{P}(\theta)$*.*

As an exemplification, we study these results in the case of finite 2-dimensional Sierpiński graphs in more detail: Recall that $\theta = s = 3$ and note that $\mathcal{P}(3) = \{3^1, 2^1 1^1, 1^3\}$. Furthermore, $\mathcal{B}_p(\{1,2,3\})$ for $p \in \mathcal{P}(3)$ is given by

$$\mathcal{B}_{3^1}(\{1,2,3\}) = \{\{123\}\}, \qquad \mathcal{B}_{1^3}(\{1,2,3\}) = \{\{1,2,3\}\},$$
$$\mathcal{B}_{2^1 1^1}(\{1,2,3\}) = \{\{12,3\},\{13,2\},\{23,1\}\}.$$

The left part of Figure 6 shows the initial data with complete labelling, whereas the right part yields a table of all arrangements for the construction of spanning forests. (The shaded area indicates connected pieces.) For example, up to symmetry, there is one way to construct a spanning tree F of X_{n+1} from a triple (L_1, L_2, L_3) of certain spanning forests of X_n, so that the relation (8) holds. This arrangement is illustrated in the first row and first line of this table. Therefore,

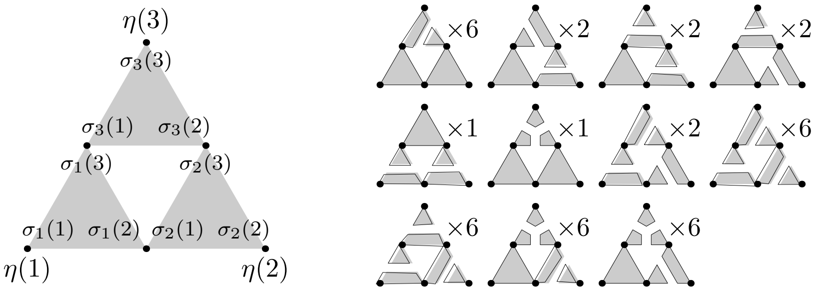

FIGURE 6. Initial data with complete labelling and (up to symmetry) all arrangements for the construction of spanning forests.

$\mathit{\Omega}(3^1) = \mathit{\Omega}(\{123\})$ consists of the six tuples

$$(\{123\}, \{123\}, \{13, 2\}), (\{123\}, \{123\}, \{23, 1\}), (\{123\}, \{12, 3\}, \{123\}),$$
$$(\{123\}, \{23, 1\}, \{123\}), (\{12, 3\}, \{123\}, \{123\}), (\{13, 2\}, \{123\}, \{123\}).$$

The next five arrangements of the table belong to $\mathit{\Omega}(2^1 1^1)$ and the last five to $\mathit{\Omega}(1^3) = \mathit{\Omega}(\{1, 2, 3\})$. Altogether, we get

$$\boldsymbol{Q}\begin{pmatrix} x_{3^1} \\ x_{2^1 1^1} \\ x_{1^3} \end{pmatrix} = \begin{pmatrix} 6\, x_{3^1}^2\, x_{2^1 1^1} \\ 7\, x_{3^1}\, x_{2^1 1^1}^2 + x_{3^1}^2\, x_{1^3} \\ 14\, x_{2^1 1^1}^3 + 12\, x_{3^1}\, x_{2^1 1^1}\, x_{1^3} \end{pmatrix}.$$

It is easy to see that the initial values are $\boldsymbol{\tau}(X_0) = (3, 1, 1)$. Therefore

$$\boldsymbol{\tau}(X_1) = \boldsymbol{Q}(\boldsymbol{\tau}(X_0)) = (54, 30, 50),$$

$\boldsymbol{\tau}(X_2) = (524880, 486000, 1350000)$ and so forth. Notice that the second and third component of $\boldsymbol{\tau}(X_n)$ are prescribed by the first using Theorem 8, since there is a correspondence between spanning forests and rooted spanning forests in this case:

$$\mathcal{R}_{X_n}(\varphi_n(\{1\})) = \mathcal{S}_{X_n}(3^1), \qquad \mathcal{R}_{X_n}(\varphi_n(\{1, 2, 3\})) = \mathcal{S}_{X_n}(1^3),$$
$$\mathcal{R}_{X_n}(\varphi_n(\{1, 2\})) = \mathcal{S}_{X_n}(\varphi_n(\{13, 2\})) \uplus \mathcal{S}_{X_n}(\varphi_n(\{23, 1\})).$$

This will be studied in the next chapter.

A similar enumeration yields the polynomial $\boldsymbol{Q}$ in the 3-dimensional case, see Table 1. The initial values are $\boldsymbol{\tau}(X_0) = (16, 3, 1, 1, 1)$ and thus $\boldsymbol{\tau}(X_1) = (131072, 42996, 6156, 18432, 27648), \dots$ Note that there are five partitions of $\theta = 4$:

$$\mathcal{P}(5) = \{4^1, 3^1 1^1, 2^2, 2^1 1^2, 1^4\};$$

and $3^1 1^1$ and 2^2 both have two terms. As a consequence, the aforementioned correspondence does not hold in this case. However, a carefully weighted sum of terms $\tau_p(X_n)$ with $p \in \mathcal{P}_k(\theta)$ for some $k \in \mathit{\Theta}$ gives the number of rooted spanning forests with k roots fixed in the set $\varphi_n(\mathit{\Theta})$. (Recall that $\mathcal{P}_k(\theta)$ is the set of number partitions with k terms.)

4.2. **A recursion for rooted spanning forests.** Let X be a connected multigraph, which is partition-homogeneous with respect to $\varphi(\mathit{\Theta})$, where $\varphi : \mathit{\Theta} \to VX$ is injective. Using Theorem 8 the number $|\mathcal{R}_X(W)|$ of rooted spanning forests with roots $W \subseteq \varphi(\mathit{\Theta})$ depends only on the size of W. Hence define

$$r_k(X) = |\mathcal{R}_X(W)|$$

for some $W \subseteq \varphi(\mathit{\Theta})$ with $k \in \mathit{\Theta}$ elements. Therefore $\boldsymbol{r}(X) = (r_1(X), \dots, r_\theta(X))$ is a vector in $\mathbb{R}^\theta$. We remark that $r_1(X)$ is precisely the complexity $\tau(X)$ of X and $r_\theta(X) = \tau_p(X)$, where the number partition p is given by $p = 1^\theta$.

Let $W \subseteq \varphi(\mathit{\Theta})$ be a k-set for $k \in \mathit{\Theta}$ and $p \in \mathcal{P}_k(\theta)$, then by Theorem 1 there are α_p set partitions $B \in \mathcal{B}_p(\varphi(\mathit{\Theta}))$, so that $|b \cap W| = 1$ for all $b \in B$. If B is such a set partition and F is a spanning forest in $\mathcal{S}_X(B)$, then (F, W) is a rooted spanning forest in $\mathcal{R}_X(W)$.

$$\begin{aligned}
Q_{4^1}(\boldsymbol{x}) &= 56x_{4^1}x_{3^1 1^1}^3 + 168x_{4^1}x_{3^1 1^1}^2 x_{2^2} + 168x_{4^1}x_{3^1 1^1}x_{2^2}^2 + 56x_{4^1}x_{2^2}^3 \\
&\quad + 72x_{4^1}^2 x_{3^1 1^1}x_{2^1 1^2} + 72x_{4^1}^2 x_{2^2}x_{2^1 1^2} \\
Q_{3^1 1^1}(\boldsymbol{x}) &= 20x_{3^1 1^1}^4 + 96x_{3^1 1^1}^2 x_{2^2}^2 + 108x_{4^1}x_{3^1 1^1}^2 x_{2^1 1^2} + 192x_{4^1}x_{3^1 1^1}x_{2^2}x_{2^1 1^2} \\
&\quad + 72x_{3^1 1^1}^3 x_{2^2} + 56x_{3^1 1^1}x_{2^2}^3 + 84x_{4^1}x_{2^2}^2 x_{2^1 1^2} + 24x_{4^1}^2 x_{2^1 1^2}^2 \\
&\quad + 6x_{4^1}^2 x_{3^1 1^1}x_{1^4} + 6x_{4^1}^2 x_{2^2}x_{1^4} \\
Q_{2^2}(\boldsymbol{x}) &= 2x_{3^1 1^1}^4 + 16x_{3^1 1^1}^3 x_{2^2} + 36x_{3^1 1^1}^2 x_{2^2}^2 + 32x_{3^1 1^1}x_{2^2}^3 + 12x_{4^1}x_{3^1 1^1}^2 x_{2^1 1^2} \\
&\quad + 22x_{2^2}^4 + 48x_{4^1}x_{3^1 1^1}x_{2^2}x_{2^1 1^2} + 36x_{4^1}x_{2^2}^2 x_{2^1 1^2} + 2x_{4^1}^2 x_{2^1 1^2}^2 \\
Q_{2^1 1^1}(\boldsymbol{x}) &= 88x_{3^1 1^1}^3 x_{2^1 1^2} + 264x_{3^1 1^1}^2 x_{2^2}x_{2^1 1^2} + 264x_{3^1 1^1}x_{2^2}^2 x_{2^1 1^2} + 88x_{2^2}^3 x_{2^1 1^2} \\
&\quad + 120x_{4^1}x_{3^1 1^1}x_{2^1 1^2}^2 + 120x_{4^1}x_{2^2}x_{2^1 1^2}^2 + 14x_{4^1}x_{3^1 1^1}^2 x_{1^4} \\
&\quad + 28x_{4^1}x_{3^1 1^1}x_{2^2}x_{1^4} + 14x_{4^1}x_{2^2}^2 x_{1^4} + 6x_{4^1}^2 x_{2^1 1^2}x_{1^4} \\
Q_{1^4}(\boldsymbol{x}) &= 720x_{3^1 1^1}^2 x_{2^1 1^2}^2 + 1440x_{3^1 1^1}x_{2^2}x_{2^1 1^2}^2 + 720x_{2^2}^2 x_{2^1 1^2}^2 + 208x_{4^1}x_{2^1 1^2}^3 \\
&\quad + 56x_{3^1 1^1}^3 x_{1^4} + 168x_{3^1 1^1}^2 x_{2^2}x_{1^4} + 168x_{3^1 1^1}x_{2^2}^2 x_{1^4} + 56x_{2^2}^3 x_{1^4} \\
&\quad + 144x_{4^1}x_{3^1 1^1}x_{2^1 1^2}x_{1^4} + 144x_{4^1}x_{2^2}x_{2^1 1^2}x_{1^4}
\end{aligned}$$

TABLE 1. The polynomial $\boldsymbol{Q}$ for finite 3-dimensional Sierpiński graphs.

This motivates the following definitions: For a set $K \subseteq \Theta$ with $k \in \Theta$ elements define the set partition P_K of Θ by

$$P_K = \{K\} \uplus \Big\{\{j\} \,:\, j \in \Theta \setminus K\Big\}$$

Notice that the type of P_K is given by the number partition

$$p_k = k + \underbrace{1 + \cdots + 1}_{\theta - k \text{ times}}$$

and $|P_K| = |p_k| = \theta + 1 - k$. Then, for a spanning forest $F \in \mathcal{S}_X(\varphi(B))$ with $B \in \mathcal{A}(P_K)$, the tuple $(F, \varphi(K))$ is a rooted spanning forest in $\mathcal{R}_X(\varphi(K))$. Hence

$$\mathcal{R}_X(\varphi(K)) = \biguplus_{B \in \mathcal{A}(P_K)} \Big\{(F, \varphi(K)) \,:\, F \in \mathcal{S}_X(\varphi(B))\Big\}$$

and

$$r_k(X) = \sum_{p \in \mathcal{P}_k(\theta)} \sum_{B \in \mathcal{A}(P_K, p)} |\mathcal{S}_X(\varphi(B))| = \sum_{p \in \mathcal{P}_k(\theta)} \alpha_p\, \tau_p(X),$$

using the decomposition (2) and $|\mathcal{A}(P_K, p)| = \alpha_p$. Thus define the map $\boldsymbol{\Sigma} : \mathbb{R}^{\mathcal{P}(\theta)} \to \mathbb{R}^\theta$ by its coordinates

$$\Sigma_k(\boldsymbol{x}) = \sum_{p \in \mathcal{P}_k(\theta)} \alpha_p\, x_p.$$

Corollary 17. *Suppose X is a connected multigraph and $\varphi : \Theta \to VX$ is an injective map. If X is partition-homogeneous with respect to $\varphi(X)$, then $\boldsymbol{r}(X) = \boldsymbol{\Sigma}(\boldsymbol{\tau}(X))$.*

For a k-set $K \subseteq \Theta$ define $\mathcal{O}(K)$ to be the set of all $\boldsymbol{\omega}$, such that the family $\sigma(\boldsymbol{\omega}) \cup \{\eta(P_K)\}$ is connected and cycle-free. Then the following partition is immediate:

$$\mathcal{O}(K) = \biguplus_{B \in \mathcal{A}(P_K)} \Omega(B).$$

This implies

$$\begin{aligned}
\Sigma_k(\boldsymbol{Q}(\boldsymbol{x})) &= \sum_{p \in \mathcal{P}_k(\theta)} \alpha_p Q_p(\boldsymbol{x}) = \sum_{p \in \mathcal{P}_k(\theta)} \sum_{B \in \mathcal{A}(P_K, p)} Q_p(\boldsymbol{x}) \\
&= \sum_{B \in \mathcal{A}(P_K)} Q_p(\boldsymbol{x}) = \sum_{B \in \mathcal{A}(P_K)} \mathsf{GF}(\Omega(B) \,|\, \boldsymbol{x}) = \mathsf{GF}(\mathcal{O}(K) \,|\, \boldsymbol{x})
\end{aligned}$$

using Equation (2), Proposition 15 and $\alpha_p = |\mathcal{A}(P_K, p)|$.

Lemma 18. *Let $j \in S$ and fix partitions $B_i \in \mathcal{B}(\Theta)$ for $i \in S \setminus \{j\}$, so that*

$$(B_1, \dots, B_{j-1}, B_j, B_{j+1}, \dots, B_s) \in \mathcal{O}(K)$$

for some $B_j \in \mathcal{B}(\Theta)$. Now consider the cycle-free family

$$\mathcal{B} = \Big\{\sigma_i(B_i) \,:\, i \in S \setminus \{j\}\Big\} \cup \{\eta(P_K)\}.$$

Then the s-tuple

$$\boldsymbol{\omega} = (B_1, \dots, B_{j-1}, \omega_j, B_{j+1}, \dots, B_s)$$

is contained in $\mathcal{O}(K)$ for every $\omega_j \in \mathcal{A}(O)$, where $O = \sigma_j^{-1}(\mathsf{Union}(\mathcal{B})|_{\sigma_j(\Theta)})$.

Proof. We have to prove that $\sigma(\boldsymbol{\omega}) \cup \{\eta(P_K)\}$ is connected and cycle-free for every $\omega_j \in \mathcal{A}(O)$. However, $\omega_j \in \mathcal{A}(O)$ implies that $\{\omega_j, O\}$ is connected and cycle-free. Note that $\mathcal{B}$ is cycle-free and $\sigma_j(O)$ reflects the connected components of $\mathcal{B}$ on $\sigma_j(\Theta)$. Therefore

$$\sigma(\boldsymbol{\omega}) \cup \{\eta(P_K)\} = \mathcal{B} \cup \{\sigma_j(\omega_j)\}$$

is also connected and cycle-free. ■

Corollary 19. *Let $\boldsymbol{B} = (B_1, \dots, B_s) \in \mathcal{O}(K)$ be an s-tuple of set partitions. For $j \in S$ define $\mathcal{O}(K, \boldsymbol{B}, j)$ to be the set of all $\boldsymbol{\omega} \in \mathcal{O}(K)$, such that $\omega_i = B_i$ for $i \in S \setminus \{j\}$. Then, for each $j \in S$, there exists a constant $c_{\boldsymbol{B},j}$, so that*

$$\mathsf{GF}(\mathcal{O}(K, \boldsymbol{B}, j) \,|\, \boldsymbol{x}) = c_{\boldsymbol{B},j}\, \Sigma_m(\boldsymbol{x}) \prod_{i \in S \setminus \{j\}} x_{\ell(B_i)},$$

where $m = |B_j|$.

Proof. Consider the family $\mathcal{B}$ defined in Lemma 18 and set $O = \sigma_j^{-1}(\mathsf{Union}(\mathcal{B})|_{\sigma_j(\Theta)})$. Notice that $m + |O| = \theta + 1$. Lemma 18 states that

$$\mathcal{O}(K, \boldsymbol{B}, j) = \{B_1\} \times \cdots \times \{B_{j-1}\} \times \mathcal{A}(O) \times \{B_{j+1}\} \times \cdots \times \{B_s\}.$$

Using Equation (2) and $|\mathcal{A}(O,p)| = \alpha(\ell(O), p) = \beta_{\ell(O)}\alpha_p$ for $p \in \mathcal{P}_m(\theta)$ we obtain

$$\sum_{\omega \in \mathcal{A}(O)} x_{\ell(\omega)} = \sum_{p \in \mathcal{P}_m(\theta)} \sum_{\omega \in \mathcal{A}(O,p)} x_p = \sum_{p \in \mathcal{P}_m(\theta)} \beta_{\ell(O)}\alpha_p x_p = \beta_{\ell(O)}\Sigma_m(\boldsymbol{x}),$$

which implies the statement for $c_{\boldsymbol{B},j} = \beta_{\ell(O)}$. ■

Proposition 20. *Let I be an index set and $\{x_\iota : \iota \in I\}$ be a set of variables. Define the polynomial P by*

$$P = \sum_{\boldsymbol{u} \in U} a(\boldsymbol{u}) \prod_{j \in S} x_{u_j},$$

where $U \subseteq I^s$ is a finite set of s-tuples, and $a(\boldsymbol{u})$ is a real number for $\boldsymbol{u} \in U$. If $\boldsymbol{v} \in U$ and $j \in S$ are given, we denote by $U(\boldsymbol{v}, j)$ the set of all $\boldsymbol{u} \in U$ with $u_i = v_i$ for $i \in S \setminus \{j\}$. Now suppose that there are finite-dimensional subspaces $\mathcal{L}_1, \dots, \mathcal{L}_s$ of the vector space $\sum_{\iota \in I} \mathbb{R}\, x_\iota$ such that

$$\sum_{\boldsymbol{u} \in U(\boldsymbol{v},j)} a(\boldsymbol{u}) \prod_{i \in S} x_{u_i} = L_{\boldsymbol{v},j} \prod_{i \in S \setminus \{j\}} x_{v_i}$$

holds with $L_{\boldsymbol{v},j} \in \mathcal{L}_j$ for all $\boldsymbol{v} \in U$ and all $j \in S$. Then P can be written in the form

$$P = \sum_{m=1}^{M} \prod_{i \in S} L'_{m,i}$$

for some $M \in \mathbb{N}$ and some $L'_{m,i} \in \mathcal{L}_i$.

Proof. We use simultaneous induction on s and $d = \dim \mathcal{L}_s$. The claim is trivial for $s = 1$ as well as for $d = 0$ (P is identically 0 in the latter case). Choose a basis $\Lambda_1, \dots, \Lambda_d$ of $\mathcal{L}_s$ in reduced echelon form. Hence Λ_1 contains a variable x_c for some $c \in I$ that is not contained in $\Lambda_2, \dots, \Lambda_d$. We may suppose that the coefficient of x_c in Λ_1 is 1.

Now, consider those tuples $\boldsymbol{v} \in U$, so that the coefficient of Λ_1 in $L_{\boldsymbol{v},s}$ with respect to the basis $\Lambda_1, \dots, \Lambda_d$ is nonzero. By the choice of Λ_1, such tuples $\boldsymbol{v}$ are characterized by the property, that $L_{\boldsymbol{v},s}$ has a nonzero coefficient with respect to x_c, which is equivalent to $\bar{\boldsymbol{v}} = (v_1, \dots, v_{s-1}, c) \in U$ and $a(\bar{\boldsymbol{v}}) \neq 0$. As a consequence, the nonzero coefficient of Λ_1 in $L_{\boldsymbol{v},s}$ is given by $a(\bar{\boldsymbol{v}})$. This motivates the following definition: Let $W \subseteq I^{s-1}$ be the set of all tuples $\boldsymbol{w} = (w_1, \dots, w_{s-1})$, so that $\bar{\boldsymbol{w}} = (w_1, \dots, w_{s-1}, c) \in U$ and $a(\bar{\boldsymbol{w}}) \neq 0$, and set

$$P^* = \sum_{\boldsymbol{w} \in W} a(\bar{\boldsymbol{w}}) \prod_{i=1}^{s-1} x_{w_i} = \sum_{\substack{\boldsymbol{u} \in U \\ u_s = c}} a(\boldsymbol{u}) \prod_{i=1}^{s-1} x_{u_i}.$$

Then, we have $P = (P - P^* \cdot \Lambda_1) + P^* \cdot \Lambda_1$. The second representation of P^* shows that P^* satisfies the condition of the proposition (with s replaced by $s - 1$). Therefore, by induction hypothesis, P^* can be written in the claimed form. Furthermore, $P - P^* \cdot \Lambda_1$ also satisfies the condition of the proposition, but instead of $\mathcal{L}_s$, we can take $\mathcal{L}_s^*$, the space spanned by $\Lambda_2, \dots, \Lambda_d$. Since $\dim \mathcal{L}_s^* = \dim \mathcal{L}_s - 1$, we may employ the induction hypothesis again, which shows that $P - P^* \cdot \Lambda_1$ can also be written in the desired form. Altogether, we obtain a representation for $P = (P - P^* \cdot \Lambda_1) + P^* \cdot \Lambda_1$ of the form

$$P = \sum_{m=1}^{M} \prod_{i \in S} L'_{m,i},$$

which finishes the proof. ∎

Theorem 21. *There exists an s-homogeneous polynomial $\boldsymbol{R} : \mathbb{R}^\theta \to \mathbb{R}^\theta$ satisfying $\boldsymbol{\Sigma} \circ \boldsymbol{Q} = \boldsymbol{R} \circ \boldsymbol{\Sigma}$, i. e.*

$$\sum_{p \in \mathcal{P}_k(\theta)} \alpha_p Q_p(\boldsymbol{x}) = R_k\biggl(\sum_{p \in \mathcal{P}_1(\theta)} \alpha_p x_p, \dots, \sum_{p \in \mathcal{P}_\theta(\theta)} \alpha_p x_p \biggr)$$

for $k \in \Theta$.

Proof. For each $k \in \Theta$ apply Proposition 20 to the polynomial $\Sigma_k \circ \boldsymbol{Q}$: For $i \in S$ let $\mathcal{L}_i$ be spanned by the linear combinations $\Sigma_1, \dots, \Sigma_\theta$. Then Corollary 19 yields exactly the required condition of Proposition 20. Hence, for each $k \in \Theta$, there exists an s-homogeneous polynomial $R_k : \mathbb{R}^\theta \to \mathbb{R}$, so that $\Sigma_k \circ \boldsymbol{Q} = R_k \circ \boldsymbol{\Sigma}$ holds. ∎

Corollary 22. *Let $k \in \Theta$ and $z_1^{n_1} \cdots z_\theta^{n_\theta}$ be a monomial, which occurs in the polynomial $R_k(\boldsymbol{z})$, then*

$$\sum_{i \in \Theta} i\, n_i = \kappa + s + k - 1.$$

Proof. The monomial $z_1^{n_1} \cdots z_\theta^{n_\theta}$ in $R_k(\boldsymbol{z})$ corresponds to the term

$$\bigl(\Sigma_1(\boldsymbol{x})\bigr)^{n_1} \cdots \bigl(\Sigma_\theta(\boldsymbol{x})\bigr)^{n_\theta}$$

in $\Sigma_k(\boldsymbol{Q}(\boldsymbol{x}))$. Hence for some number partitions $p_1 \in \mathcal{P}_1(\theta), \dots, p_\theta \in \mathcal{P}_\theta(\theta)$ and $p \in \mathcal{P}_k(\theta)$, the monomial $x_{p_1}^{n_1} \cdots x_{p_\theta}^{n_\theta}$ occurs in $Q_p(\boldsymbol{x})$. However, all monomials in $Q_p(\boldsymbol{x})$ are of the form

$$\prod_{q \in \mathcal{P}(\theta)} x_q^{\chi_q(\boldsymbol{\omega})}$$

for some $\boldsymbol{\omega} \in \Omega(p)$. Now the assertion follows from Lemma 14. ∎

Corollary 23. *Let $(Y, \psi) = \mathsf{Copy}(X, \varphi)$, then the following relation between $\boldsymbol{r}(X)$ and $\boldsymbol{r}(Y)$ holds:*

$$\boldsymbol{r}(Y) = \boldsymbol{R}(\boldsymbol{r}(X)).$$

Proof. Proposition 15 implies $\boldsymbol{\tau}(Y) = \boldsymbol{Q}(\boldsymbol{\tau}(X))$. Using Corollary 17 and Theorem 21 we get

$$\boldsymbol{r}(Y) = \boldsymbol{\Sigma}(\boldsymbol{\tau}(Y)) = \boldsymbol{\Sigma}(\boldsymbol{Q}(\boldsymbol{\tau}(X))) = \boldsymbol{R}(\boldsymbol{\Sigma}(\boldsymbol{\tau}(X))) = \boldsymbol{R}(\boldsymbol{r}(X)),$$

which proves the statement. ■

Note that Theorem 21 is trivial for $\theta \leq 3$, since $|\mathcal{P}_k(\theta)| = 1$ for $k \in \Theta$ in this case. For the sake of completeness, we give the map $\boldsymbol{\Sigma}$ for $\theta = 2$ and $\theta = 3$:

$$\boldsymbol{\Sigma}(x_{2^1}, x_{1^2}) = (x_{2^1}, x_{1^2}) \qquad \text{and} \qquad \boldsymbol{\Sigma}(x_{3^1}, x_{2^1 1^1}, x_{1^3}) = (x_{3^1}, 2x_{2^1 1^1}, x_{1^3})$$

However, if $\theta \geq 4$ the situation is more complicated. As an example, we consider 3-dimensional Sierpiński graphs $X_0, X_1, \ldots$ Here $\theta = s = 4$ and the polynomial $\boldsymbol{Q}$ is given in Table 1. A simple computation yields

$$\boldsymbol{\Sigma}(x_{4^1}, x_{3^1 1^1}, x_{2^2}, x_{2^1 1^2}, x_{1^4}) = (x_{4^1}, 2x_{3^1 1^1} + 2x_{2^2}, 3x_{2^1 1^2}, x_{1^4})$$

and therefore we obtain the polynomial

$$\boldsymbol{R}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 7z_1 z_2^3 + 12 z_1^2 z_2 z_3 \\ \frac{11}{4} z_2^4 + 20 z_1 z_2^2 z_3 + \frac{52}{9} z_1^2 z_3^2 + 6 z_1^2 z_2 z_4 \\ 11 z_2^3 z_3 + 20 z_1 z_2 z_3^2 + \frac{21}{2} z_1 z_2^2 z_4 + 6 z_1^2 z_3 z_4 \\ 20 z_2^2 z_3^2 + \frac{208}{27} z_1 z_3^3 + 7 z_2^3 z_4 + 24 z_1 z_2 z_3 z_4 \end{pmatrix}$$

satisfying $\boldsymbol{\Sigma} \circ \boldsymbol{Q} = \boldsymbol{R} \circ \boldsymbol{\Sigma}$. This is a considerable simplification compared to the polynomial $\boldsymbol{Q}$ given in Table 1.

4.3. **A simplified recursion.** Besides the obvious parameters s, θ, and κ there are two further intrinsic parameters of the initial data, which we are going to introduce now:

Let X be a connected multigraph and $\varphi : \Theta \to VX$ be an injective map, so that X is partition-homogeneous with respect to $\varphi(\Theta)$. Additionally, set $(Y, \psi) = \mathsf{Copy}(X, \varphi)$. Denote by $\rho(X)$ and $\rho(Y)$ the resistance scaling factor of X with respect to $\varphi(\Theta)$ and of Y with respect to $\psi(\Theta)$, respectively.

Lemma 24. *With the above notation the quotient $\rho(Y)/\rho(X)$ is independent of the specific choice of the multigraph X and will be denoted by λ, called the* resistance scaling factor *of the initial data.*

Proof. We define λ by $\lambda = \rho(Y)$, where $(Y, \psi) = \mathsf{Copy}(X, \varphi)$ and X is the complete graph with θ vertices. For general multigraphs X we have to prove that $\rho(Y) = \lambda\, \rho(X)$:

Let c_Y be the unit conductances on Y. Furthermore, let X_K be the complete graph with vertex set $\varphi(\Theta)$, set $(Y_K, \psi) = \mathsf{Copy}(X_K, \varphi)$ and let c_K be the unit conductances on Y_K. Finally, let c_D be the unit conductances on the complete graph with vertex set $\psi(\Theta)$.

By definition, we have $\mathsf{Tr}(c_Y | \psi(\Theta)) = \rho(Y)^{-1} c_D$ and $\mathsf{Tr}(c_K | \psi(\Theta)) = \lambda^{-1} c_D$. In addition, (VY, c_Y) and $(VY_K, \rho(X)^{-1} c_K)$ are electrically equivalent with respect to $\psi(\Theta)$: By construction Y consists of s edge-disjoint parts, which are isomorphic to X. Since $\rho(X)$ is the resistance scaling factor of X, each copy of X in Y can be replaced by a complete graph with constant conductances $\rho(X)^{-1}$ without a change of the trace. Hence the two networks are equivalent, which implies the statement. ■

Lemma 25. *Let R_1 be given by*

$$R_1(\boldsymbol{z}) = \sum_{\boldsymbol{n}} a_{\boldsymbol{n}} \boldsymbol{z}^{\boldsymbol{n}}$$

using multi-index notation and define μ by

$$\mu = \theta^{-\kappa} \sum_{\boldsymbol{n}} a_{\boldsymbol{n}} \prod_{k \in \Theta} k^{n_k}.$$

Then the complexity of Y is given by $\tau(Y) = \mu\, \rho(X)^{\kappa}\, \tau(X)^{s}$ and μ is called the tree scaling factor *of the initial data.*

Proof. Note that $\tau(X) = r_1(X)$ and $\tau(Y) = r_1(Y)$ by definition. Theorem 8 yields the relation $r_k(X) = k\rho(X)^{k-1}\theta^{1-k}\tau(X)$ for $k \in \Theta$. Inserting this into the recursion $\tau(Y) = R_1(\boldsymbol{r}(X))$ (see Corollary 23) implies

$$\tau(Y) = \sum_{\boldsymbol{n}} a_{\boldsymbol{n}} \prod_{k\in\Theta} \left(k\rho(X)^{k-1}\theta^{1-k}\tau(X)\right)^{n_k}.$$

By the s-homogeneity of $\boldsymbol{R}$ and Corollary 22 the identities

$$\sum_{k\in\Theta} n_k = s \qquad \text{and} \qquad \sum_{k\in\Theta}(k-1)n_k = \kappa$$

hold. Therefore we obtain

$$\tau(Y) = \rho(X)^{\kappa}\,\tau(X)^{s}\,\theta^{-\kappa} \sum_{\boldsymbol{n}} a_{\boldsymbol{n}} \prod_{k\in\Theta} k^{n_k} = \mu\,\rho(X)^{\kappa}\,\tau(X)^{s},$$

finishing the proof. ■

The two quantities λ and μ completely describe the evolution of the complexity and the resistance. Let us combine the last two lemmata:

Theorem 26. *Define* $\boldsymbol{T} : \mathbb{R}^2 \to \mathbb{R}^2$ *by* $\boldsymbol{T}(a,b) = (\lambda a, \mu a^{\kappa} b^{s})$. *Then*

$$(\rho(Y), \tau(Y)) = \boldsymbol{T}(\rho(X), \tau(X)).$$

Theorem 27. *In general the following estimate for* μ *holds:*

$$\mu \le \frac{1}{\theta^{\kappa}}\binom{s\theta}{\kappa} \le \frac{s^{\kappa}}{\kappa!}.$$

In the special case of $\kappa = 0$ *equality holds, i. e.* $\mu = 1$.

Proof. Let $X = K_{1,\theta}$ be the star and $(Y,\psi) = \mathsf{Copy}(X,\varphi)$. Then Y has $s\theta$ edges and cyclomatic number κ; a spanning tree is thus obtained by deleting κ edges in such a way that no cycle remains. This implies

$$\tau(Y) \le \binom{s\theta}{\kappa}.$$

Note that equality holds in the above estimate if $\kappa = 0$, which implies $\mu = 1$ in this case. Obviously, $\tau(X) = 1$ and $\rho(X) = \theta$ due to Lemma 7. Therefore, we have

$$\mu = \frac{\tau(Y)}{\tau(X)^{s}\rho(X)^{\kappa}} \le \frac{1}{\theta^{\kappa}}\binom{s\theta}{\kappa},$$

which proves the theorem. ■

4.4. **Sequences of self-similar graphs.** Let X_0 be a connected multigraph and $\varphi_0 : \Theta \to VX_0$ be an injective map, so that X_0 is partition-homogeneous with respect to $\varphi_0(\Theta)$. Iteratively define the multigraphs $X_1, X_2, \dots$ and the maps $\varphi_1, \varphi_2, \dots$ by

$$(X_n, \varphi_n) = \mathsf{Copy}(X_{n-1}, \varphi_{n-1})$$

for $n \in \mathbb{N}$. Then X_n is a connected multigraph, which is partition-homogeneous with respect to $\varphi_n(\Theta)$. Denote by $\rho(X_n)$ the resistance scaling factor of X_n with respect to $\varphi_n(\Theta)$; then $\rho(X_n) = \lambda^n \rho(X_0)$ for all $n \in \mathbb{N}_0$. Now Lemma 10 implies the following:

Corollary 28. *The cardinalities of* VX_n *and* EX_n *are given by*

$$|VX_n| = s^n(|VX_0| - 1) - \kappa\tfrac{s^n-1}{s-1} + 1 \qquad \textit{and} \qquad |EX_n| = s^n|EX_0|.$$

Theorem 29. *The complexity* $\tau(X_n)$ *of* X_n *is given by*

$$\tau(X_n) = \lambda^{\kappa\frac{s^n-1-n(s-1)}{(s-1)^2}} \left(\mu\rho(X_0)^{\kappa}\right)^{\frac{s^n-1}{s-1}} \tau(X_0)^{s^n}.$$

Proof. The result follows from Theorem 26 by induction. ■

Since every spanning tree is a subset of the edge set, it is natural to rewrite the formula for $\tau(X_n)$ in terms of $|EX_n|$:

$$\tau(X_n) = \tau(X_0)\left(\frac{|EX_n|}{|EX_0|}\right)^{\frac{\kappa}{s-1}(1-2/d_s)} C^{\frac{|EX_n|}{|EX_0|}-1},$$

where $C = \lambda^{\frac{\kappa}{(s-1)^2}}\mu^{\frac{1}{s-1}}\rho(X_0)^{\frac{\kappa}{s-1}}\tau(X_0)$ and

$$d_s = 2\,\frac{\log(s)}{\log(s\lambda)}$$

is the so-called *spectral dimension.* This quantity appears in the asymptotic behavior of the Dirichlet- or Neumann-eigenvalue statistics of the Laplacian on fractals or on infinite self-similar graphs, as well as in transition density estimates for Brownian motion on fractals and its discrete counterpart, see for instance [3, 18, 24, 25].

Furthermore, there are s possibilities to embed X_n in X_{n+1} as a part of X_{n+1}. Hence for each infinite sequence $\boldsymbol{\iota} = (\iota_1, \iota_2, \dots) \in S^{\mathbb{N}}$, there exists an infinite limit graph $X_\infty(\boldsymbol{\iota})$, so that the embeddings

$$X_0 \xhookrightarrow{\iota_1} X_1 \xhookrightarrow{\iota_2} X_2 \cdots \xhookrightarrow{\iota_n} X_n \cdots \hookrightarrow X_\infty(\boldsymbol{\iota}).$$

hold. In this sense the multigraph sequence $X_0, X_1, \dots$ approaches the infinite multigraph $X_\infty(\boldsymbol{\iota})$. The *tree entropy* h (see [30]) is then given by

$$h = \lim_{n\to\infty}\frac{\log(\tau(X_n))}{|VX_n|} = \frac{\frac{\kappa}{s-1}\log(\lambda) + \log(\mu) + \kappa\log(\rho(X_0)) + (s-1)\log(\tau(X_0))}{(s-1)(|VX_0|-1)-\kappa}.$$

4.5. **Examples.**

In the following we continue studying the examples from Section 3.2: Using Theorem 29 closed formulæ for the complexity are derived. For these examples the parameters θ, s, and κ are mentioned before. In addition, $\rho(X_0)$ and $\rho(X_1)$ is already computed, so that the resistance scaling factor λ is given by $\lambda = \rho(X_1)/\rho(X_0)$. It remains to compute the tree scaling factor μ, which is done by means of Lemma 25.

4.5.1. *Sierpiński graphs.* As a first example, we derive a formula for the complexity of d-dimensional Sierpiński graphs, see Section 3.2.1 for their definition. In order to apply Theorem 29, we have to determine the tree scaling factor μ first. To this end, apply the substitution procedure to $X = K_{1,d+1}$, the star; this method can be seen as an analogon to the Wye-Delta-transform for electrical networks. Then, the resulting graph Y is bipartite and its vertices can be divided into the following categories:

- the centers of the parts $\bar{Z}_i \simeq X$,
- the corner vertices, each of which is attached to exactly one of the centers, and
- linking vertices between the centers: each of these vertices has exactly two neighbors (which are center vertices), and for each pair of center vertices, there is exactly one vertex linking them.

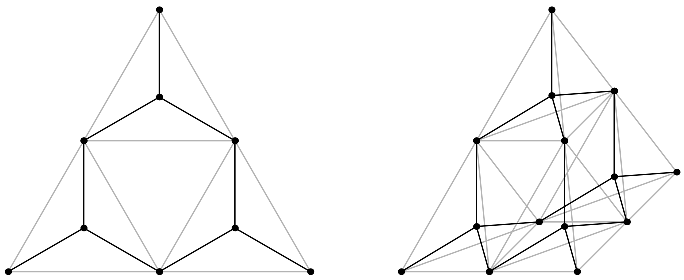

FIGURE 7. The graph Y for $d = 2$ and $d = 3$.

One can regard Y as a complete graph with $d + 1$ vertices whose edges are subdivided, with an additional pendant vertex attached to each of the $d + 1$ vertices (see Figure 7).

Obviously, $\tau(X) = 1$, since X is a tree. Now, the main task is to calculate $\tau(Y)$: A spanning tree of Y has to contain each of the $d+1$ edges incident to the pendant vertices. Furthermore, we can choose any of the $(d+1)^{d-1}$ spanning trees of the complete graphs K_{d+1} (each of the d edges is represented by two edges in view of the subdivisions), and add one of the two possible edges for each of the remaining $\binom{d+1}{2} - d = \frac{d(d-1)}{2}$ linking vertices. Therefore, we have

$$\tau(Y) = (d+1)^{d-1} \cdot 2^{d(d-1)/2}.$$

Lemma 7 yields $\rho(X) = \rho(K_{1,d+1}) = d+1$. Using Lemma 25 and the formula for κ we obtain

$$\mu = \frac{\tau(Y)}{\tau(X)^s \rho(X)^\kappa} = \left(2^d\,(d+1)^{2-d}\right)^{\frac{d-1}{2}}.$$

Since $\rho(X_0) = 1$ and $\rho(X_1) = \frac{d+3}{d+1}$, the parameter λ is given by $\lambda = \frac{d+3}{d+1}$. Now, Theorem 29 can be applied: It is well-known that $\tau(X_0) = \tau(K_{d+1}) = (d+1)^{d-1}$, which gives

$$\tau(X_n) = \left(2^{d((d+1)^n-1)}\,(d+1)^{(d+1)^{n+1}+d(n+1)-1}\,(d+3)^{(d+1)^n-dn-1}\right)^{\frac{d-1}{2d}}. \tag{10}$$

Note that this is a generalization of the formula for spanning trees of 2-dimensional Sierpiński graphs obtained by the authors in [43].

Finally, we remark that the spectrum of the Dirichlet-Laplace operator $\varDelta_n^0 = \varPi_{H_n}\varDelta_n\varPi_{H_n}^*$ with boundary $\varphi_n(\varTheta)$ can be described exactly using the so-called method of spectral decimation, see [18, 38, 41]. Here $\varDelta_n$ is the combinatorial Laplacian of X_n and $H_n = VX_n \setminus \varphi_n(\varTheta)$. Let us quickly state this result: Set $p(x) = x(d+3-x)$ and

$$m^{\pm}(i) = \tfrac{d+1}{2}\left((d-1)(d+1)^{i-1} \pm 1\right).$$

Then the spectrum of $\varDelta_n^0$ is given by the following table:

Eigenvalue x	Multiplicity of x
$x = 2(d+1)$	$m^-(n)$
$x \in p^{-n+1}(2)$	1
$x \in p^{-i}(d+1)$ for $i \in \{0,\dots,n-2\}$	$m^-(n-i-1)$
$x \in p^{-i}(d+3)$ for $i \in \{0,\dots,n-1\}$	$m^+(n-i-1)$

Here $p^{-k}(u)$ is the set of all k-fold backward iterates of u. The multiplicity of some values x might be zero, in which case x is of course not an eigenvalue of $\varDelta_n^0$. Note that, by Vieta's theorem, we have

$$\prod_{x \in p^{-i}(y)} x = y,$$

so that we also obtain an explicit formula for $\mathsf{det}(\varDelta_n^0)$, which is equal to the product of all eigenvalues of $\varDelta_n^0$. Theorem 8 implies

$$\mathsf{det}(\varDelta_n^0) = \mathcal{R}_{X_n}(\varphi_n(\varTheta)) = (d+3)^{dn}(d+1)^{1-d-dn}\tau(X_n).$$

Thus, the above description of the eigenvalues provides a different way to derive formula (10) for the complexity $\tau(X_n)$.

This procedure is always applicable if spectral decimation works (generally, p is a rational function, which does not change too much). Unfortunately, spectral decimation is a rather restricted tool, see [31, 39] for further details. For instance, it does not work for the sequence of Austria graphs, which we are going to investigate next. Even if the method of spectral decimation applies, it needs a little more work depending on the graph sequence to obtain the explicit description of the complete spectrum.

4.5.2. *Austria graphs.* The Austria graphs of Section 3.2.2 provide an example for the fact that no symmetry at all is needed in the case $\theta = 2$. Furthermore, two distinct orientations of the substitutions $\sigma_1, \dots, \sigma_4$ yield different graph sequences, but this does not alter the complexity by our considerations. It is not difficult to determine the polynomial $\boldsymbol{Q}$:

$$\boldsymbol{Q}\begin{pmatrix} x_{2^1} \\ x_{1^2} \end{pmatrix} = \begin{pmatrix} 3\,x_{2^1}^3\,x_{1^2} \\ 5\,x_{2^1}^2\,x_{1^2}^2 \end{pmatrix}.$$

This leads to the closed formula

$$\tau(X_n) = 3^{\frac{1}{9}(2\cdot 4^n+3n-2)}\cdot 5^{\frac{1}{9}(4^n-3n-1)},$$

which also follows from Theorem 29 using the parameters $\theta = 2$, $s = 4$, $\kappa = 1$, $\lambda = \frac{5}{3}$, $\mu = 3$, and $\rho(X_0) = \tau(X_0) = 1$.

4.5.3. *A multigraph example.* This example (Section 3.2.3) shows that all our calculations are valid even in the case of multigraphs. Here the polynomial $\boldsymbol{Q}$ is given by

$$\boldsymbol{Q}\begin{pmatrix} x_{3^1} \\ x_{2^1 1^1} \\ x_{1^3} \end{pmatrix} = \begin{pmatrix} 32\,x_{3^1}\,x_{2^1 1^1}^3 + 6\,x_{3^1}^2\,x_{2^1 1^1}\,x_{1^3} \\ 8\,x_{2^1 1^1}^4 + 4\,x_{3^1}\,x_{2^1 1^1}^2\,x_{1^3} \\ 8\,x_{2^1 1^1}^3\,x_{1^3} \end{pmatrix}.$$

A short computation gives $\mu = \frac{50}{27}$, yielding the formula

$$\tau(X_n) = 3\cdot 2^{\frac{2}{3}(4^n-1)-n}\cdot 5^{\frac{1}{3}(4^n-1)+n}.$$

4.5.4. *An example without full symmetry.* The maximal invariant group of this example is the alternating group $\mathsf{Alt}(\{1,2,3\})$ of degree 3, see Section 3.2.4. (It is not difficult, however, to construct similar examples yielding the maximal invariant group $\mathsf{Alt}(\{1,\dots,\theta\})$ for arbitrary θ.)

We obtain the following expression for the polynomial $\boldsymbol{Q}$:

$$\boldsymbol{Q}\begin{pmatrix} x_{3^1} \\ x_{2^1 1^1} \\ x_{1^3} \end{pmatrix} = \begin{pmatrix} 160\,x_{3^1}^4\,x_{2^1 1^1}^3 + 12\,x_{3^1}^5\,x_{2^1 1^1}\,x_{1^3} \\ 212\,x_{3^1}^3\,x_{2^1 1^1}^4 + 57\,x_{3^1}^4\,x_{2^1 1^1}^2\,x_{1^3} + x_{3^1}^5\,x_{1^3}^2 \\ 792\,x_{3^1}^2\,x_{2^1 1^1}^5 + 412\,x_{3^1}^3\,x_{2^1 1^1}^3\,x_{1^3} + 36\,x_{3^1}^4\,x_{2^1 1^1}\,x_{1^3}^2 \end{pmatrix}.$$

Now, it is easy to determine the value of μ, which is $\frac{196}{27}$ in this case. Together with $\theta = 3$, $s = 7$, and $\kappa = 3$, we obtain the formula

$$\tau(X_n) = 2^{\frac{1}{12}(5\cdot 7^n-6n-5)}\cdot 3^{\frac{1}{2}(7^n+1)}\cdot 7^{\frac{1}{3}(7^n-1)}$$

from Theorem 29.

5. CONCLUSIONS

Our main result, Theorem 29, reveals strong connections between the complexity on finite self-similar graphs and the study of Laplace operators. Polynomials $\boldsymbol{Q}$ and $\boldsymbol{R}$ both cover the information of the resistance scaling factor. Hence it is likely that these polynomials are closely related to the renormalization map, which is usually used in the definition for the resistance scaling factor (see [33]). The Dirichlet- or Neumann-spectrum of Laplace operators on self-similar graphs are well understood and described by the dynamics of a multi-dimensional polynomial, see [37]. Likewise, the complexity is governed by the polynomial $\boldsymbol{Q}$. It is plausible that these two dynamical systems are linked.

Let $X_0, X_1, \dots$ be a sequence of finite self-similar graphs, denote by $\varDelta_n$ the combinatorial Laplacian on X_n and by P_n its characteristic polynomial. Then Theorem 29 yields a closed formula for the coefficient $[x]P_n(x)$ of the linear term of P_n due to the formula $[x]P_n(x) = -|VX_n|\,\tau(X_n)$. Using similar considerations the computation of further coefficients seems to be possible. In the case of 2-dimensional Sierpiński graphs a lengthy calculation shows that the coefficient $[x^2]P_n(x)$ of the quadratic term is given by

$$\Big(\tfrac{1}{40}(14\cdot 15^n - 3\cdot 9^n + 39\cdot 5^n + 6\cdot 3^n + 4) + \tfrac{1}{2}\big(\tfrac{5}{3}\big)^n\Big)\tau(X_n).$$

Note that Proposition 15 generalizes readily to the case when no symmetry condition is satisfied at all. But it seems to be difficult to generalize the further analysis too (especially Corollary 19), although similar results on the complexity are expected to hold more general. However, there is some evidence that Theorem 8 holds under less restricted symmetry assumptions.

Finally, we remark that in [42] it is conjectured that the number of connected subgraphs of X_n asymptotically involves the resistance scaling factor. Our main result proves this conjecture for the number of spanning trees.

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