THE NUMBER OF CONES GENERATED BY A MULTIRESOLUTION ANALYSIS WITH A SEQUENCE OF *LULU* OPERATORS

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Abstract. LULU operators provide a simple yet effective nonlinear algorithm, with many desired attributes, to decompose a sequence into unit pulses at ascending resolution levels. This decomposition is however linear on the cone generated by the constituent pulses, with important consequences for image analysis. It is therefore a natural problem to study the number of possible cones that can arise for sequences of a given length. We provide an answer to this combinatorial question, proving that the number of different cones associated with sequences of length n is the central binomial coefficient $\binom{2(n-2)}{n-2}$.

1. Introduction

There is adequate evidence that, whereas linear filters form smoothing and linear Multiresolution Analysis is appropriate and sufficient for the processing of auditory signals, this is not the case for image smoothing. Consensus is that Nonlinear Multiresolution Analysis, for instance with Median Smoothers, is better on images even than the, originally promising, Wavelet based decompositions. Edge preservation is no problem, and the introduction of artificial artefacts associated with linear decompositions seems absent. Lack of theory is a long observed problem of Nonlinear Smoothers.

The so-called *LULU*-theory, based on a specific, selected class of Morphological Filters, has provided an alternative in Nonlinear Multiresolution Analysis (see [3] and the references therein). Not only can this be related to Median Decomposition and shown to be at least as good, but the theory proves and explains most of the important advantages observed. Computational complexity is vastly improved, and allows extensive parallelisation. There even is a Parseval-type identity that proportionally divides the total variation $T(x)$ of a given sequence x to its resolution levels $y^i(x)$, so that $T(x) = \Sigma T(y^i(x))$. All this occurs with a clear, strong theory allowing creative design of smoothers for specific purposes.

The *LULU* operators, acting on sequences $x_1, x_2, \dots, x_n$, are defined as follows:

$$(L_w x)_j = \max_{j-w \le i \le j} \min_{i \le k \le i+w} x_k,$$
$$(U_w x)_j = \min_{j \le i \le j+w} \max_{i-w \le k \le i} x_k,$$

where the sequence x is extended by defining $x_k = x_1$ if $k < 1$ and $x_k = x_n$ if $k > n$. The effect of the *LULU* operator L_w on a sequence x is that all "bumps" of width at most w are removed; such a bump is defined as a subsequence $x_j, x_{j+1}, \dots, x_{j+w-1}$ with

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the property that $x_{j-1}, x_{j+w} < x_j, x_{j+1}, \dots, x_{j+w-1}$. Likewise, the operator U_w removes "pits" (defined analogously) of width at most w. If the *LULU* operators L_1, U_1, L_2, U_2, ... are applied to a sequence x in this order, the bumps and pits are gradually removed until a monotone sequence is obtained. This procedure also results in a decomposition that is known as the *discrete pulse transform*: set $C_n = L_n U_n C_{n-1}$ and $C_0 = I$, and define $y^i(x) = C_{i-1}(x) - C_i(x)$. Then one has

$$x = x' + \sum_i y^i(x), \tag{1}$$

where x' is a monotone sequence. One of the most important results is that the *LULU*-decompositions act linearly in the cone generated by the vectors $y^i(x)$. The linearity on the cone generated by individual resolution layers $y^i(x)$ can be stated as an equality $y^i(z) = \alpha_i y^i(x)$ when $z = \sum_i \alpha_i y^i(x)$, for arbitrary non-negative constants α_i.

This result was expanded to the larger cone generated by the vectors $y^i_+(x)$ and $y^i_-(x)$, which are the positive and negative parts of $y^i(x)$. It was even observed to be expandable to the cone generated by the vectors defined by individual pulses in $y^i_+(x)$ and $y^i_-(x)$. This was stated as the Highlight Conjecture [5], and has subsequently been proved [2, 4], and been used in practice in highlighting an individual, newly arrived golf-ball in a television image on a green from others already there without distorting the rest of the image in any way.

The total number of pulses in all the resolution levels of a *LULU*-decomposition can easily be argued to be not more than the number of elements of x, or the dimension of the vector space from which x comes: Starting with the original n values of the sequence x, every pulse results from subtraction of one nearby value from a particular value, and the elimination of the latter value. This leaves one less of the original n values. Clearly the cone generated by the individual pulses of a particular *LULU*-decomposition lies inside this vector space, but depends on the original choice of x. This raises a natural combinatorial problem: how many different cones can arise from such a Multiresolution Analysis?

Clearly, any such cone is characterised by the set of pulses in all resolution levels; we can write the elements of the cone as

$$\sum_i \alpha_i p^i,$$

where p^i is a unit pulse, given by $p^i_k = 1$ if $j \leq k < j + w$ for some j and w (or $p^i_k = -1$ if $j \leq k < j + w$), and $p^i_k = 0$ otherwise. We call the set of all p^i a *pulse basis*; counting cones is thus equivalent to counting pulse bases. We assume $x_1 = x_n = 0$ (resulting in a constant sequence $x' = 0$ at the end of the process), since we are only interested in the second summand in the decomposition (1). Assuming that the elements of the sequence x are otherwise all distinct, it can be shown easily that the resulting pulse basis must consist of $n - 2$ pulses. This assumption will be made throughout the rest of the paper. In [2], the following properties of pulse bases are stated:

- No two pulses overlap partially. Either the support of the smaller pulse is contained in the support of the larger pulse, or the pulses have disjoint support.

- Any two positive pulses of width $\geq w$ are separated by ranges of width $\geq w$ containing only negative pulses and shorter positive pulses.
- Any two negative pulses of width $\geq w$ are separated by ranges of width $> w$ containing only positive pulses and shorter negative pulses.

For example, the following six pulse bases are possible for $n = 4$:

j	2	3
p_j^1	1	0
p_j^2	1	1

j	2	3
p_j^1	0	1
p_j^2	1	1

j	2	3
p_j^1	1	0
p_j^2	0	-1

j	2	3
p_j^1	0	1
p_j^2	-1	0

j	2	3
p_j^1	-1	0
p_j^2	-1	-1

j	2	3
p_j^1	0	-1
p_j^2	-1	-1

The apparent symmetry with respect to signs is misleading and does not persist for larger values of n. For instance,

j	2	3	4
p_j^1	1	0	0
p_j^2	0	0	1
p_j^3	1	1	1

is a valid pulse basis for $n = 5$, while

j	2	3	4
p_j^1	-1	0	0
p_j^2	0	0	-1
p_j^3	-1	-1	-1

is not (since p^1 and p^2 are not separated by a large enough range). The aim of this paper is to show that there are in general exactly $\binom{2(n-2)}{n-2}$ different pulse bases (and thus cones generated) for sequences of length n. The following section exhibits an approach known as the *roadmaker's algorithm* that turns out to be equivalent to the described *LULU* smoothing process. This restatement will actually allow us to prove a more general result, from which the stated formula for the number of cones will follow as a simple corollary. Our main results can be stated as follows:

Theorem 1. *The number of cones that are generated by sequences $x_1, x_2, \ldots, x_n$ with $x_1 = x_n$ and precisely h* ascents *(i.e., indices j for which $x_j > x_{j-1}$) is $\binom{n-1}{h}\binom{n-3}{h-1}$.*

Theorem 2. *The total number of cones is the central binomial coefficient $\binom{2(n-2)}{n-2}$.*

2. The roadmaker's algorithm: restatement of the problem

Instead of applying the $LULU$ operators, it is possible to use slightly modified operators, as explained in [2]: the roadmaker's algorithm makes use of the bump-razing operator B_w and the pit-filling operator A_w that are defined by

$$(B_w x)_k = \max(x_{j-1}, x_{j+w})$$

if $x_j = x_{j+1} = \ldots = x_{j+w-1} > \max(x_{j-1}, x_{j+w})$, and $(B_w x)_k = x_k$ otherwise, and analogously

$$(A_w x)_k = \min(x_{j-1}, x_{j+w})$$

if $x_j = x_{j+1} = \ldots = x_{j+w-1} < \min(x_{j-1}, x_{j+w})$, and $(A_w x)_k = x_k$ otherwise.

Application of the $LULU$ operators $L_1, U_1, L_2, U_2, \ldots$ to a sequence in this order is in fact equivalent to applying $B_1, A_1, B_2, A_2, \ldots$: whereas the $LULU$ operator L_w removes bumps of all widths $\leq w$, the bump-razing operator only affects those of width exactly w. However, in the algorithm both operators are only applied once all bumps of width $< w$ have been removed, so that the two have the same effect.

In order to treat our combinatorial problem, it is useful to introduce an encoding for sequences as follows: a sequence $x = x_1, x_2, \ldots, x_n$ is uniquely defined by a sequence $v_1, v_2, \ldots, v_k$ of values and a sequence $q_1, q_2, \ldots, q_k$ of associated multiplicities. For instance, the sequence

$$x = (5, 3, 1, 1, 1, 2, 2, 5, 5, 4, 2, 3, 1, 1, 1, 1)$$

is represented by $v = (5, 3, 1, 2, 5, 4, 2, 3, 1)$ and $q = (1, 1, 3, 2, 2, 1, 1, 1, 4)$. If the roadmaker's algorithm is applied to a sequence $x = x^0$ to yield a sequence $x^1, x^2, \ldots$ of sequences, where $x^{2k-1} = B_k x^{2k-2}$ and $x^{2k} = A_k x^{2k-1}$, then the associated sequences v^m and q^m change as follows:

- If $m = 2l - 1$ is odd, $v_j^{m-1} > \max(v_{j-1}^{m-1}, v_{j+1}^{m-1})$ and $q_j^{m-1} = l$, then v_j^{m-1} and q_j^{m-1} are removed; further, if $v_{j-1}^{m-1} > v_{j+1}^{m-1}$, then q_j^{m-1} is added to q_{j-1}^{m-1}. Otherwise, q_j^{m-1} is added to q_{j+1}^{m-1}. If, on the other hand, $v_j^{m-1} < \max(v_{j-1}^{m-1}, v_{j+1}^{m-1})$, v_j^{m-1} and q_j^{m-1} remain as they are.
- If $m = 2l$ is even, $v_j^{m-1} < \min(v_{j-1}^{m-1}, v_{j+1}^{m-1})$ and $q_j^{m-1} = l$, then v_j^{m-1} and q_j^{m-1} are removed; all other changes are analogous to the previous case.

This is exhibited in the following example:

k	x^k
0	$(0, 1, 4, 9, 6, -2, 5, -3, 2, 0)$
1	$(0, 1, 4, 6, 6, -2, -2, -3, 0, 0)$
2	$(0, 1, 4, 6, 6, -2, -2, -2, 0, 0)$
3	$(0, 1, 4, 4, 4, -2, -2, -2, 0, 0)$
4	$(0, 1, 4, 4, 4, -2, -2, -2, 0, 0)$
5	$(0, 1, 1, 1, 1, -2, -2, -2, 0, 0)$
6	$(0, 1, 1, 1, 1, 0, 0, 0, 0, 0)$
7	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$

The associated sequences v^k and q^k are given by

k	v^k
0	$(0, 1, 4, 9, 6, -2, 5, -3, 2, 0)$
1	$(0, 1, 4, 6, -2, -3, 0)$
2	$(0, 1, 4, 6, -2, 0)$
3	$(0, 1, 4, -2, 0)$
4	$(0, 1, 4, -2, 0)$
5	$(0, 1, -2, 0)$
6	$(0, 1, 0)$
7	(0)

and

k	q^k
0	$(1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$
1	$(1, 1, 1, 2, 2, 1, 2)$
2	$(1, 1, 1, 2, 3, 2)$
3	$(1, 1, 3, 3, 2)$
4	$(1, 1, 3, 3, 2)$
5	$(1, 4, 3, 2)$
6	$(1, 4, 5)$
7	(10)

and the pulse basis associated with this example would be

j	2	3	4	5	6	7	8	9
p^1_j	0	0	1	0	0	0	0	0
p^2_j	0	0	0	0	0	1	0	0
p^3_j	0	0	0	0	0	0	0	1
p^4_j	0	0	0	0	0	0	-1	0
p^5_j	0	0	1	1	0	0	0	0
p^6_j	0	1	1	1	0	0	0	0
p^7_j	0	0	0	0	-1	-1	-1	0
p^8_j	1	1	1	1	0	0	0	0

We introduce yet another sequence of auxiliary sequences, denoted r^k; it is defined by $r^k_j = \operatorname{sgn}(v^k_{j+1} - v^k_j)$. This describes the "rough" shape, the ups and downs of x^k. Note that r^k and q^k alone are sufficient to locate all bumps and pits (a bump occurs wherever $r^k_j > \max(r^k_{j-1}, r^k_{j+1})$, and q^k_j gives the width). Therefore, the pulse basis can be uniquely determined from the sequences r^k and q^k ($k \geq 0$).

How do r^k and q^k change through the course of the algorithm? The changes can be described as follows:

(1) At the beginning, r^0 is a sequence of length $n-1$ whose elements are ± 1; furthermore, the assumption that $x_1 = x_n = 0$ implies that r^0 is not constant (since x can not be strictly monotone); likewise, r^k is a nonconstant sequence of ± 1 for any k.

(2) $q^0 = (1, 1, \ldots, 1)$, and the length of q^0 is n.

(3) (a) If $k = 2l - 1$ is odd, r^k and q^k are obtained from r^{k-1} and q^{k-1} in the following manner: for each j with $r^k_{j-1} = 1$, $r^k_j = -1$ and $q^k_j = l$, we remove q^k_j and either r^k_{j-1} (this occurs if $x^k_{j-1} > x^k_{j+1}$) or r^k_j (otherwise), and add q^k_j to q^k_{j-1} in the former case and to q^k_{j+1} in the latter.
(b) If $k = 2l$ is even, r^k and q^k are obtained from r^{k-1} and q^{k-1} in the following manner: for each j with $r^k_{j-1} = -1$, $r^k_j = 1$ and $q^k_j = l$, we remove q^k_j and either r^k_{j-1} (this occurs if $x^k_{j-1} < x^k_{j+1}$) or r^k_j (otherwise), and add q^k_j to q^k_{j-1} in the former case and to q^k_{j+1} in the latter.
(4) Before the last step of the algorithm, one has $r^k = (-1, 1)$ or $r^k = (1, -1)$.

It turns out that knowing all sequences r^k is enough to characterise the entire set of pulses, which is a consequence of the following simple lemma:

Lemma 3. *If r^{k-1}, q^{k-1} and r^k are known, then q^k is uniquely determined.*

Proof. Knowing r^{k-1} and q^{k-1}, one can determine which bumps/pits will be removed in the following (kth) step. Looking at r_k, one can determine for each of these bumps/pits which of the two possibilities in part (3a) (or (3b)) of the construction applies. This suffices to construct q^k as well. ∎

It follows inductively that knowledge of all sequences r^k is enough to reconstruct all sequences q^k and thus the positions of all pulses. Hence it is sufficient to count all possible sequences r^k (or, equivalently, all pairs of sequences r^k, q^k) that can arise. We call a pair of sequences (r^k, q^k) *admissible* if it can be obtained from the above steps, that is,

(1) $r^0, r^1, \ldots$ are nonconstant sequences with elements ± 1, and the length of r^0 is $n-1$.
(2) $q^0 = (1, 1, \ldots, 1)$, and the length of q^0 is n.
(3) (a) If $k = 2l - 1$ is odd, then for each j with $r^k_{j-1} = 1$, $r^k_j = -1$ and $q^k_j = l$, q^k_j and either r^k_{j-1} or r^k_j are removed; q^k_j is added to q^k_{j-1} in the former case and to q^k_{j+1} in the latter.
(b) If $k = 2l$ is even, then for each j with $r^k_{j-1} = -1$, $r^k_j = 1$ and $q^k_j = l$, q^k_j and either r^k_{j-1} or r^k_j are removed; q^k_j is added to q^k_{j-1} in the former case and to q^k_{j+1} in the latter.
(4) The last element of the sequence $r^0, r^1, \ldots$ is of the form $r^K = (-1, 1)$ or $r^K = (1, -1)$.

A priori, it is not clear that every admissible sequence corresponds to a possible choice of x^0 and thus to a valid pulse basis. The following lemma shows, however, that this is the case:

Lemma 4. *If the sequences r^k, q^k are admissible, then there is a corresponding sequence x^0 so that one obtains r^k, q^k by applying the roadmaker's algorithm to x^0.*

Proof. Suppose that r^K, q^K are the last elements of our admissible sequence. We construct $x^K, x^{K-1}, \ldots, x^0$ iteratively to prove the lemma. It suffices to construct the associated sequences $v^K, v^{K-1}, \ldots, v^0$ that contain the elements without multiplicity, the multiplicities being given by $q^K, q^{K-1}, \dots, q^0$. If $r^K = (-1, 1)$, we set $v^K = (0, -1, 0)$, and analogously $v^K = (0, 1, 0)$ if $r^K = (1, -1)$. It is obvious that $r_j^K = \operatorname{sgn}(v_{j+1}^K - v_j^K)$, as it should be.

Now suppose that v^{k+1} has already been constructed; let us consider the case of even $k = 2l - 2$, the other case being analogous. Knowing r^k and q^k, one can determine all positions j for which $r_{j-1}^k = 1$, $r_j^k = -1$ and $q_j^k = l$ (bumps that are removed in step k). If J is the set of these positions, choose $|J|$ distinct real numbers that are larger than any of the elements of v^{k+1}. These are now inserted in v^{k+1} in such a way that they become the elements v_j^k, $j \in J$. Let us exhibit this idea for a simple example; if r^k and q^k are given by

k	q^k	r^k
3	$(5, 2, 2)$	$(-1, 1)$
2	$(3, 2, 2, 2)$	$(1, -1, 1)$
1	$(2, 1, 2, 1, 1, 2)$	$(-1, 1, -1, 1, 1)$
0	$(1, 1, 1, 1, 1, 1, 1, 1, 1)$	$(1, -1, 1, 1, -1, 1, 1, -1)$

then v^k evolves as follows:

k	v^k
3	$(0, -1, 0)$
2	$(0, \mathbf{1}, -1, 0)$
1	$(0, -\mathbf{2}, 1, -\mathbf{3}, -1, 0)$
0	$(0, \mathbf{2}, -2, 1, \mathbf{3}, -3, -1, \mathbf{4}, 0)$

It is clear that the condition $v_j^k = \operatorname{sgn}(r_{j+1}^k - r_j^k)$ remains true. At the end of this process, one obtains a sequence $x^0 = v^0$ that satisfies the required condition. ∎

Now we are finally ready to solve our main problem, the enumeration of cones (equivalently, pulse bases), which is now reduced to counting admissible sequences.

3. The combinatorial result

Theorems 1 and 2 are simple consequences of the following lemma:

Lemma 5. *If the elements $r^0, r^1, \dots, r^k$ and $q^0, q^1, \dots, q^k$ of an admissible sequence are given, then there are $\binom{\ell(r^k)-2}{h(r^k)-1}$ possible ways to complete the sequence, where $\ell(r^k)$ is the length of r^k and $h(r^k)$ the number of ones in r^k.*

Proof. By backwards induction. At the end of the procedure, one must have $\ell(r^k) = 2$ and $h(r^k) = 1$, and there is only one possibility, in accordance with the lemma. Now assume that the algorithm is not yet at its end. The only part of the definition of an admissible sequence that involves a choice is (3). Suppose that the number of bumps or peaks to be removed in step $k+1$ is m (this number can be determined from r^k and q^k). For each of these bumps/peaks, we choose independently whether a 1 or a -1 should be removed. Therefore, there are $\binom{m}{j}$ possibilities to remove precisely j of the ones. The number of possibilities to complete the admissible sequence after that step is $\binom{\ell(r^k)-m-2}{h(r^k)-j-1}$ by the induction hypothesis. Note that this potentially includes the cases $h(r^k) - j = 0$ and

$h(r^k) - j = \ell(r^k) - m$, which are both forbidden in the definition of an admissible sequence (r^{k+1} would be constant in these two cases); however, the binomial coefficient evaluates to zero in both cases. Therefore, the total number of ways to complete the admissible sequence is

$$\sum_{j=0}^{h(r^k)} \binom{m}{j} \binom{\ell(r^k) - m - 2}{h(r^k) - j - 1} = \binom{\ell(r^k) - 2}{h(r^k) - 1}$$

by the Vandermonde identity [1], which completes the induction. ■

Since admissible sequences are equivalent to pulse bases and thus in turn cones, Theorem 1 now follows from the observation that there are $\binom{n-1}{h}$ possible initial values for r^0 (the length of r^0 must be $n-1$, and the number of ascents is exactly the number of ones in r^0). Finally, Theorem 2 is obtained by summing over all possible values of h, making use of the Vandermonde identity once again:

$$\begin{aligned} \sum_{h=1}^{n-2} \binom{n-1}{h} \binom{n-3}{h-1} &= \sum_{h=1}^{n-2} \binom{n-1}{h} \binom{n-3}{n-h-2} \\ &= \binom{2(n-2)}{n-2}, \end{aligned}$$

which completes the proof of our main result.

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