Exact and asymptotic enumeration of perfect matchings in self-similar graphs

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Abstract

We consider self-similar graphs following a specific construction scheme: in each step, several copies of the level-n graph X_n are amalgamated to form X_{n+1}. Examples include finite Sierpiński graphs or Viček graphs. For the former, the problem of counting perfect matchings has recently been considered in a physical context by Chang and Chen, and we aim to find more general results. If the number of

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amalgamation vertices is small or if other conditions are satisfied, it is possible to determine explicit counting formulæ for this problem, while generally it is not even easy to obtain asymptotic information. We also consider the statistics “number of matching edges pointing in a given direction” for Sierpiński graphs and show that it asymptotically follows a normal distribution. This is also shown in more generality in the case that only two vertices of X_n are used for amalgamation in each step.

Key words: Perfect matchings, self-similar graphs, exact enumeration, asymptotics

1 Introduction

The enumeration of perfect matchings belongs to the classical counting problems in graph theory. In view of its applications to the dimer problem in statistical physics, the enumeration of perfect matchings is particularly well-studied for square and hexagonal lattices—this line of investigation has been started by Kasteleyn's fundamental work (see [5]), and there is a vast variety of subsequent papers on the enumeration of perfect matchings and the equivalent problem of counting domino and lozenge tilings; [7] provides a good survey of this topic. Some other papers deal with perfect matchings in trees, cacti and other families of graphs, see for instance [3,6].

For basically the same class of graphs, it was shown in [9] that remarkable explicit formulæ can be given for the problem of counting spanning trees. It turns out that there are explicit formulæ for the number of perfect matchings

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in some special cases as well, and these cases will thus be of particular interest in this paper. Section 4 deals with these special cases.

We will also consider the statistics "number of edges in a fixed direction" for random perfect matchings, which is particularly natural in the aforementioned special case of the Sierpiński gasket. A normal limit law for this quantity with explicit mean and variance will be proved in Section 5.

2 Construction

In order to define the graph sequences we are going to investigate, the following essential ingredients are needed (cf. the construction in [8,9]):

- An edgeless graph G with $\theta \geq 2$ distinguished vertices given by $\eta : \Theta \to VG$ ($\Theta = \{1, \ldots, \theta\}$).
- $s \geq 2$ substitutions, defined by injective maps $\sigma_i : \Theta \to VG$ for $i \in S = \{1, \ldots, s\}$ such that $VG = \bigcup_{i=1}^{s} \sigma_i(\Theta)$.

Now, for any (multi-)graph X and any injective map $\varphi : \Theta \to VX$, a new multigraph Y together with an injective map $\psi : \Theta \to VY$ is constructed as follows:

For each $i \in S$ let Z_i be an isomorphic copy of the (multi-)graph X, so that the vertex sets $VZ_1, \ldots, VZ_s$, and VG are mutually disjoint. The isomorphism between X and Z_i is denoted by $\zeta_i : VX \to VZ_i$. Let Z be the disjoint union of G and $Z_1, \ldots, Z_s$ and define the relation $\sim$ on VZ as the reflexive, symmetric, and transitive hull of

$$\bigcup_{i=0}^{s}\Big\{(\sigma_i(j),\zeta_i(\varphi(j)))\,:\,j\in\Theta\Big\}\subseteq VZ\times VZ.$$

Then the multigraph Y is defined by its vertex set $VY = VZ/{\sim}$ and edge (multi-)set

$$EY=\Big\{\{[v],[w]\}\,:\,\{v,w\}\in EZ\Big\},$$

where $[v]$ denotes the equivalence class of a vertex v. The map $\psi:\Theta\to VY$ is defined by $\psi(i)=[\eta(i)]\in VY$. We call $\varphi(\Theta)$ (and $\psi(\Theta)$) the *distinguished vertices* (or boundary vertices) of X (and of Y, respectively).

If the pair (Y,ψ) is constructed as above from (X,φ), write $(Y,\psi)=\mathsf{Copy}(X,\varphi)$. Since we fix G, η, and $\{\sigma_i\,:\,i\in S\}$, the dependence on these items is suppressed. Note that Y is the amalgamation of s isomorphic copies of X (thus we need the additional condition that $VG=\bigcup_{i=1}^{s}\sigma_i(\Theta)$, which means that there are no isolated vertices): for $i\in S$ define $\bar{Z}_i$ by

$$V\bar{Z}_i=\Big\{[v]\,:\,v\in VZ_i\Big\}\quad\text{and}\quad E\bar{Z}_i=\Big\{\{[v],[w]\}\,:\,\{v,w\}\in EZ_i\Big\}.$$

Then $\bar{Z}_i$ is isomorphic to X and the isomorphism is given by

$$\bar{\zeta}_i\,:\,VX\to V\bar{Z}_i,\quad v\mapsto[\zeta_i(v)].$$

The subgraph $\bar{Z}_i$ is called the *i-th part* of Y. On the i-th part of Y distinguished vertices are given by

$$\Theta\to V\bar{Z}_i,\qquad j\mapsto\bar{\zeta}_i(\varphi(j))=[\sigma_i(j)].$$

In the following, we will be interested in sequences of graphs obtained by iterating this construction, i.e. X_0 is some initial graph with distinguished vertices given by a map φ_0, and $(X_n,\varphi_n)=\mathsf{Copy}(X_{n-1},\varphi_{n-1})$. We will also need some symmetry condition in the following sections: it will be assumed that the graphs X_n are *strongly symmetric* with respect to the boundary $\varphi_n(\Theta)$, i.e. the automorphism group of X_n acts like the alternating group or the symmetric group on $\varphi_n(\Theta)$. If this condition is satisfied, then we have the following simple yet important property:

Lemma 1. For any two subsets $K_1, K_2 \subseteq \Theta$ with $|K_1| = |K_2|$ and any non-negative integer n, there is an automorphism π of X_n such that $\pi(\varphi_n(K_1)) = \pi(\varphi_n(K_2))$.

2.1 Examples

In this subsection we present some examples of self-similar graphs illustrating the construction. Note that all examples satisfy the symmetry condition.

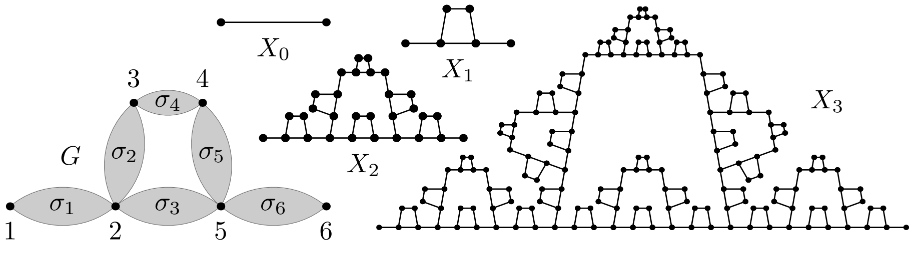

Fig. 1. An example of a sequence of finite self-similar graphs.

2.1.1 An example with two distinguished vertices

Let $\theta = 2$ and $s = 6$ and define G by $VG = \{1,2,3,4,5,6\}$. Furthermore, define the maps η and σ_j by the following table:

i	$\eta(i)$	$\sigma_1(i)$	$\sigma_2(i)$	$\sigma_3(i)$	$\sigma_4(i)$	$\sigma_5(i)$	$\sigma_6(i)$
1	1	1	2	2	3	4	5
2	6	2	3	5	4	5	6

With these definitions we build a sequence of finite self-similar graphs X_n by setting $X_0 = K_2$ and $(X_n, \varphi_n) = \mathsf{Copy}(X_{n-1}, \varphi_{n-1})$ (Figure 1).

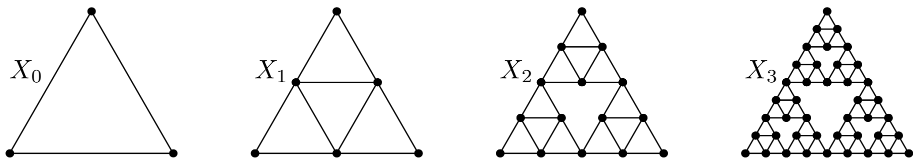

Fig. 2. The first few finite 2-dimensional Sierpiński graphs.

2.1.2 Finite Sierpiński graphs

Fix some $d \in \mathbb{N}$ and let $s = \theta = d + 1$. Define the edgeless graph G by

$$VG = \left\{ \boldsymbol{x} \in \mathbb{N}_0^{d+1} \,:\, x_1 + x_2 + \ldots + x_{d+1} = 2 \right\}$$

and the map $\eta : \Theta \to VG$ by $\eta(i) = 2\boldsymbol{e}_i$, where $\boldsymbol{e}_i$ is the i-th canonical basis vector of $\mathbb{R}^{d+1}$. In addition, set $\sigma_i(j) = \boldsymbol{e}_i + \boldsymbol{e}_j \in VG$ for $i \in S$ and $j \in \Theta$ (note that $\Theta = S = \{1, \ldots, d+1\}$). It is easy to see that $|VG| = \frac{1}{2}(d+2)(d+1)$. The usual finite d-dimensional Sierpiński graphs are then obtained by setting $X_0 = K_{d+1}$ and iterating $(X_n, \varphi_n) = \mathsf{Copy}(X_{n-1}, \varphi_{n-1})$ for $n \in \mathbb{N}$. See Figure 2 for the case $d = 2$.

2.1.3 Finite Viček graphs

Fix some integer $\theta \geq 2$ and set $s = \theta + 1$. Recall that $\Theta = \{1, 2, \ldots, \theta\}$ and define $VG = \Theta \times \Theta$. Then define the maps η and σ_i by $\eta(i) = (i, 1)$ and

$$\sigma_i(j) = \begin{cases} (i, j) & \text{if } i \in \Theta, \\ (j, 2) & \text{if } i = s = \theta + 1 \end{cases}$$

With this data the finite Viček graphs are defined as follows: the initial graph X_0 is the complete graph K_θ, and X_n is defined by $(X_n, \varphi_n) = \mathsf{Copy}(X_{n-1}, \varphi_{n-1})$, as always. Figure 3 shows the first few Viček graphs for $\theta = 4$.

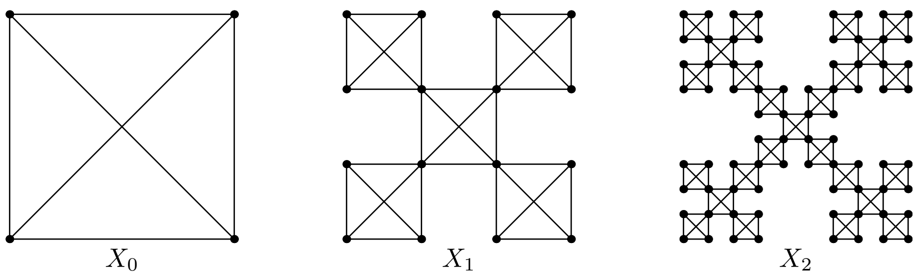

Fig. 3. The first few finite Viček graphs.

3 Perfect Matchings

A *matching* is a set of disjoint edges of a graph, a *perfect matching* is a matching which covers all vertices of a graph. Let a graph X with θ distinguished vertices (defined by an injective map $\varphi : \Theta \to VX$, as in the previous section) be given such that X is strongly symmetric with respect to $\varphi(\Theta)$. We denote the set of matchings by $\mathcal{M}(X)$ and define $\mathcal{M}_K(X)$ to be the set of all perfect matchings of $X \setminus \varphi(K)$ for any set $K \subseteq \Theta$. Then, in view of strong symmetry, the size of $\mathcal{M}_K(X)$ only depends on the cardinality of K, and we may define $m_k(X) = |\mathcal{M}_K(X)|$ for any set K of cardinality $|K| = k$. Note that

$m_k(X) = 0$ if $k \not\equiv |VX| \bmod 2$.

Now, if $Y = \mathsf{Copy}(X)$, we want to express the values of $m_k(Y)$ in terms of the $m_j(X)$. To this end, note that every matching M in $\mathcal{M}_K(Y)$ induces matchings on all parts $\bar{Z}_i$ of Y. For each i, let M_i be the restriction of M to the i-part $\bar{Z}_i$: $M_i = M \cap E\bar{Z}_i$. The matching M_i has to cover all vertices of $\bar{Z}_i$ except possibly some of the distinguished vertices of $\bar{Z}_i$. Hence $\bar{\zeta}_i^{-1}(M_i)$ belongs to $\mathcal{M}_{L_i}(X)$ for some set L_i. Moreover, for each $v \in VG \setminus \eta(K)$, there is exactly one $i = \rho(v)$ such that the vertex $[v] \in VY$ is covered by an edge in the part $\bar{Z}_i$.

Conversely, let a set K be given. Define a map $\rho : VG\setminus\eta(K) \to S$ such that $v \in \sigma_{\rho(v)}(\Theta)$ for all v, and choose a perfect matching M_i in $X \setminus \varphi(\Theta \setminus \sigma_i^{-1}(\rho^{-1}(i)))$ for each $i \in S$ (i.e. in the preimage of $\bar{Z}_i$, reduced by all vertices which are not covered within $\bar{Z}_i$), if possible. Then $\bigcup_{i=1}^{s} \bar{\zeta}_i(M_i)$ is a matching in $\mathcal{M}_K(Y)$. So we have established a bijective correspondence between $\mathcal{M}_K(Y)$ and all possible tuples $(\rho, M_1, \ldots, M_s)$. Here, M_i is a matching in $\mathcal{M}_{L_i}$ for some set L_i of cardinality $\theta - |\rho^{-1}(i)|$. Hence, the formula

$$m_k(Y) = \sum_{\rho} \prod_{i=1}^{s} m_{\theta - |\rho^{-1}(i)|}(X) \tag{1}$$

holds, where the sum is over all possible functions ρ which satisfy the above condition, and K is an arbitrary set of size k. The following simple lemma is an immediate consequence:

Lemma 2. For every $0 \le k \le \theta$, there exist nonnegative integer coefficients $a(k, \boldsymbol{\nu})$ such that

$$m_k(Y) = \sum_{\boldsymbol{\nu}} a(k, \boldsymbol{\nu}) \prod_{j=0}^{\theta} m_j(X)^{\nu_j},$$

where the sum is over all $(\theta+1)$-tuples $\boldsymbol{\nu} = (\nu_0, \ldots, \nu_\theta)$ of nonnegative integers such that

$$\sum_{j=0}^{\theta} \nu_j = s \qquad \text{and} \qquad \sum_{j=0}^{\theta} j\nu_j = s\theta - |VG| + k.$$

Proof. We only have to check that in Equation (1), the identity

$$\sum_{i=1}^{s} (\theta - |\rho^{-1}(i)|) = s\theta - |VG| + k$$

holds. Then, the lemma follows easily from (1). However, this is equivalent to

$$|\mathrm{dom}(\rho)| = \sum_{i=1}^{s} |\rho^{-1}(i)| = |VG| - k,$$

which is obviously true. ■

In the following, some examples for the resulting recurrences are provided and analyzed. The special cases $\theta = 2$ and $\theta = 3$ have particularly nice properties yielding explicit formulæ, and so we are going to deal with them first.

Note that the number of vertices in X_n satisfies a first-order linear recurrence, namely

$$|VX_n| = s|VX_{n-1}| + |VG| - s\theta.$$

Hence there are three possibilities, depending on s and $\delta = s\theta - |VG|$:

- $|VX_n|$ is even for all $n > 0$, so that $m_k(X_n)$ can only be positive if k is even. This happens if s and δ are both even or if s is odd and $\delta, |VX_0|$ are even.
- $|VX_n|$ is odd for all $n > 0$, so that $m_k(X_n)$ can only be positive if k is odd. This happens if s is even and δ odd or if $s, |VX_0|$ are odd and δ even.
- $|VX_n|$ is alternately odd and even, and $m_k(X_n)$ behaves accordingly. This happens if s, δ are both odd.

4 The special cases of two or three distinguished vertices

In the cases $\theta = 2$ and $\theta = 3$ it is possible to derive exact formulæ for the quantities $m_k(X_n)$, as will be exhibited in this section. Specifically, we have the following theorem:

Theorem 3. Assume that $|VX_n|$ is always even or always odd for $n > 0$ and $\theta = 2$ or $\theta = 3$. Then there are constants C_k, γ, τ, and β, so that

$$m_k(X_n) = C_k \, \gamma^{(\tau - k/2)n} \, \beta^{s^n}$$

holds for all $n > 0$ and all k.

Now assume that $|VX_n|$ is alternately odd and even for $n > 0$. If $\theta = 2$, then $m_k(X_n)$ is eventually 0 for all k. If $\theta = 3$, then $m_k(X_n)$ is given by the formula above for every other n depending on parity.

The proof of this result is provided in the following two subsections. Note that $\gamma > 0$ can be arbitrarily close to 0 as well as arbitrarily close to ∞, see Example 4.3.3.

4.1 Two distinguished vertices

Since we are mostly interested in counting perfect matchings, we deal with the case when $|VX_n|$ is always even first. Then Lemma 2 shows that there are coefficients a, b such that

$$m_0(X_n) = a \, m_0(X_{n-1})^{\nu} \, m_2(X_{n-1})^{s-\nu},$$

$$m_2(X_n) = b \, m_0(X_{n-1})^{\nu-1} \, m_2(X_{n-1})^{s-\nu+1},$$

where $\nu = \frac{1}{2}|VG|$. Note that no symmetry condition at all is necessary to obtain this recursive relation. It also follows that a is precisely the number of perfect matchings in $\mathsf{Copy}(K_2, \varphi)$ (φ being the trivial map from $\{1, 2\}$ to the vertices of K_2) and that b is the number of perfect matchings in $\mathsf{Copy}(K_2, \varphi) \setminus \psi(\{1, 2\})$. Dividing the two equations yields

$$\frac{m_0(X_n)}{m_2(X_n)} = \frac{a}{b} \cdot \frac{m_0(X_{n-1})}{m_2(X_{n-1})},$$

which shows that

$$m_2(X_n) = Q \left(\tfrac{b}{a}\right)^n m_0(X_n),$$

where $Q = \frac{m_2(X_0)}{m_0(X_0)}$. We use this in the formula for $m_0(X_n)$ to obtain

$$m_0(X_n) = a\, Q^{s-\nu} \left(\tfrac{b}{a}\right)^{(s-\nu)(n-1)} m_0(X_{n-1})^s$$

with the explicit solution

$$m_0(X_n) = C_0\, \gamma^{\tau n}\, \beta^{s^n}, \qquad m_2(X_n) = C_2\, \gamma^{(\tau-1)n}\, \beta^{s^n},$$

where the constants C_0,C_2,γ,τ, and β are given as follows:

$$\gamma = \tfrac{a}{b}, \quad \tau = \tfrac{s-\nu}{s-1}, \quad C_0 = (a^{-1}\gamma^{\tau})^{1/(s-1)}\, Q^{-\tau},$$
$$C_2 = C_0\, Q, \quad \beta = C_0^{-1}\, m_0(X_0).$$

The case when $|VX_n|$ is always odd is less interesting. We already know that $m_0(X_n) = m_2(X_n) = 0$ for all $n > 0$. In view of Lemma 2, $m_1(X_n) = 0$ holds as well for almost all n unless $|VG| = s + 1$ (otherwise, there is no solution to $\nu_1 = s$ and $\nu_1 = 2s - |VG| + 1$). Then, however, there exists a constant a so that $m_1(X_n) = a\, m_1(X_{n-1})^s$ with the simple solution

$$m_1(X_n) = a^{\frac{1}{1-s}} \cdot \left(m_1(X_0)\, a^{\frac{1}{s-1}}\right)^{s^n}.$$

Finally, we consider the case when the parity of $|VX_n|$ is alternating. In this case, the quantities $m_0(X_n)$, $m_1(X_n)$, and $m_2(X_n)$ are equal to 0 for almost all n, which can be shown by similar arguments: first, note that either $m_0(X_n)$ or $m_2(X_n)$ has to be 0 for almost all n by Lemma 2 (there cannot exist solutions to the systems $\nu_1 = s$, $\nu_1 = 2s - |VG|$ and $\nu_1 = s$, $\nu_1 = 2s - |VG| + 2$ simultaneously). Thus, suppose for instance that $m_2(X_n) = 0$ for almost all n and that $|VG| = s$ (so that there exists ν_1 with $\nu_1 = s = 2s - |VG|$). But then, $m_1(X_n) = 0$ for almost all n, as there is no solution of the system $\nu_0 = s$, $0 = 2s - |VG| + 1 = s + 1$. The second case is treated similarly.

4.2 *Three distinguished vertices*

First, let us consider the case when $|VX_n|$ is always even again. Then, we have

$$m_0(X_n) = a\, m_0(X_{n-1})^{\nu}\, m_2(X_{n-1})^{s-\nu},$$
$$m_2(X_n) = b\, m_0(X_{n-1})^{\nu-1}\, m_2(X_{n-1})^{s-\nu+1}$$

for some integer coefficients a, b, where $\nu = \frac{1}{2}(|VG| - s)$. This system of recurrences is basically the same as in the case $\theta = 2$.

The case when $|VX_n|$ is always odd is also completely analogous. We obtain a system

$$m_1(X_n) = a\, m_1(X_{n-1})^{\nu}\, m_3(X_{n-1})^{s-\nu},$$
$$m_3(X_n) = b\, m_1(X_{n-1})^{\nu-1}\, m_3(X_{n-1})^{s-\nu+1},$$

where $\nu = \frac{1}{2}(|VG| - 1)$. The solution follows again along the same lines.

Finally, let us consider the case when the parity of $|VX_n|$ is alternating. Then we obtain the system

$$m_0(X_n) = a_0\, m_1(X_{n-1})^{\nu}\, m_3(X_{n-1})^{s-\nu},$$
$$m_1(X_n) = a_1\, m_0(X_{n-1})^{\kappa}\, m_2(X_{n-1})^{s-\kappa},$$
$$m_2(X_n) = a_2\, m_1(X_{n-1})^{\nu-1}\, m_3(X_{n-1})^{s-\nu+1},$$
$$m_3(X_n) = a_3\, m_0(X_{n-1})^{\kappa-1}\, m_2(X_{n-1})^{s-\kappa+1}$$

for certain integers a_0, a_1, a_2, a_3, where $\nu = \frac{1}{2}|VG|$ and $\kappa = \frac{1}{2}(|VG| - s - 1)$. We iterate this system once to obtain

$$m_0(X_n) = c_0\, m_0(X_{n-2})^{\lambda}\, m_2(X_{n-2})^{s^2-\lambda},$$
$$m_2(X_n) = c_2\, m_0(X_{n-2})^{\lambda-1}\, m_2(X_{n-2})^{s^2-\lambda+1}$$

for integer coefficients c_0, c_2 and $\lambda = \nu + (\kappa - 1)s = \frac{1}{2}\Big((s+1)|VG| - s^2 - 3s\Big)$. Again, this system can be solved as in Section 4.1.

4.3 Examples

Let us now apply Theorem 3 to the examples of Section 2.1.

4.3.1 An example with two distinguished vertices

See Section 2.1.1 for the construction of this example. We have $\theta = 2$, $s = 6$, and $|VG| = 6$. Since $\delta = s\theta - |VG| = 6$ and $|VX_0| = 2$, the number $|VX_n|$ is always even. It is easy to see that the following system of recurrence equations holds:

$$m_0(X_n) = m_0(X_{n-1})^3 m_2(X_{n-1})^3,$$
$$m_2(X_n) = 2m_0(X_{n-1})^2 m_2(X_{n-1})^4.$$

Now the results of Section 4.1 imply that

$$m_0(X_n) = 2^{3(6^n-5n-1)/25} \qquad \text{and} \qquad m_2(X_n) = 2^{(3\cdot 6^n+10n-3)/25}.$$

Notice that the quantity γ equals $\frac{1}{2}$ in this case.

4.3.2 Two-dimensional Sierpiński graphs

For the construction see Section 2.1.2. Here, we have $s = \theta = 3$, $|VG| = 6$ and thus $\delta = s\theta - |VG| = 3$. Hence, the parity of $|VX_n|$ is alternating. The following system of recurrences holds:

$$\begin{aligned}
m_0(X_n) &= 2m_1(X_{n-1})^3,\\
m_1(X_n) &= 2m_0(X_{n-1})\, m_2(X_{n-1})^2,\\
m_2(X_n) &= 2m_1(X_{n-1})^2\, m_3(X_{n-1}),\\
m_3(X_n) &= 2m_2(X_{n-1})^3,
\end{aligned}$$

which reduces to

$$\begin{aligned}
m_0(X_n) &= 16\, m_0(X_{n-2})^3\, m_2(X_{n-2})^6,\\
m_2(X_n) &= 16\, m_0(X_{n-2})^2\, m_2(X_{n-2})^7.
\end{aligned}$$

Since $m_1(X_0) = m_3(X_0) = 1$, we have $m_0(X_n) = m_2(X_n)$ and $m_1(X_n) = m_3(X_n)$ for all n. Therefore, $m_0(X_n)$ is given by the closed formula

$$m_0(X_n) = m_2(X_n) = 2^{\frac{3^n-1}{2}}$$

for all odd values of n. Note that $\gamma = 1$.

This result was also found by Chang and Chen in [1], where two-dimensional Sierpiński graphs with a larger number b of subdivisions are considered as well (see Figure 4 for the case $b = 3$; the above case of ordinary Sierpiński graphs

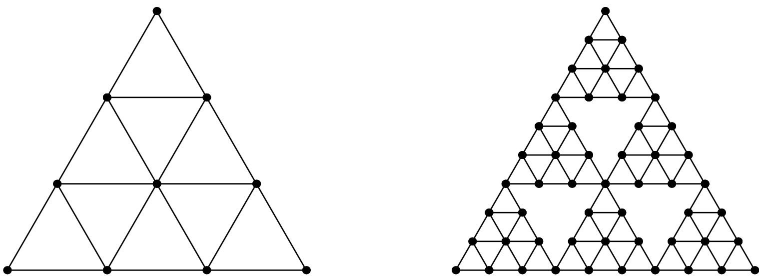

Fig. 4. Sierpinski graph of level 1 and 2 with three subdivisions.

corresponds to $b = 2$). It turns out that $\gamma = 1$ for arbitrary b. To this end, we show by a simple bijection that $m_0(X_n) = m_2(X_n)$ and $m_1(X_n) = m_3(X_n)$ for all n, regardless of the number of subdivisions b.

Lemma 4. Consider the sequence X_n of two-dimensional Sierpiński graphs with arbitrary number b of subdivisions. Then,

$$m_0(X_n) = m_2(X_n) \qquad \text{and} \qquad m_1(X_n) = m_3(X_n)$$

for all n.

Proof. We construct a bijection between matchings covering the left and right corners and those not covering these two corners. Given a matching of the first kind, consider all edges between vertices of the first (bottom) and second row. Each of these edges is replaced by an edge connecting the same second-row vertex with its other first-row neighbor. The horizontal matching edges in the first row are moved accordingly (it is not difficult to see that this is possible). The result is a matching of the second kind, and the process is also reversible. See Figure 5 for an example. ∎

It follows immediately that $\gamma = 1$ for an arbitrary number of subdivisions. Furthermore, one has

$$m_1(X_n) = m_3(X_n) = m_1(X_1)^{(s^n-1)/(s-1)}$$

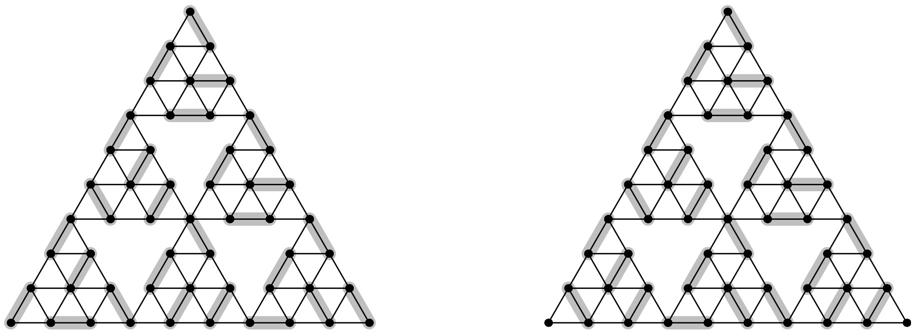

Fig. 5. The bijection that proves Lemma 4.

for $b \equiv 0 \bmod 4$ or $b \equiv 1 \bmod 4$ (so that $|VX_n|$ is odd for all $n \geq 1$), where $s = \binom{b+1}{2}$. For $b \equiv 2 \bmod 4$ or $b \equiv 3 \bmod 4$, the formula is slightly more complicated, but essentially the same.

Hence, the problem is reduced to that of counting perfect matchings in triangular grids (see [7] in this regard). The asymptotic growth constants

$$\alpha_b = \lim_{n\to\infty} \frac{\log m_k(X_n)}{|VX_n|}$$

(where k is chosen appropriately such that $m_k(X_n)$ is nonzero) can now be determined explicitly for small b, see Table 1. For $b \leq 5$, these were given in the aforementioned paper of Chang and Chen [1].

4.3.3 Examples for small and large γ

We construct two families of self-similar graphs depending on a parameter $\mu \in \mathbb{N}$. Since $\theta = 2$ in both cases the methods of Section 4.1 apply, where γ is given by $\gamma = \mu$ in the first case and $\gamma = \mu^{-1}$ in the second case. For the first family let $s = 3\mu$ and $|VG| = 2\mu+2$ and for the second one let $s = 3\mu+2$ and $|VG| = 2\mu+4$. For both families the initial graph X_0 is K_2. The constructions are indicated in Figure 6.

$$\alpha_2 = \tfrac{1}{3}\log 2 = 0.2310490602$$

$$\alpha_3 = \tfrac{1}{7}\log 6 = 0.2559656385$$

$$\alpha_4 = \tfrac{1}{12}\log 28 = 0.2776837092$$

$$\alpha_5 = \tfrac{1}{18}\log 200 = 0.2943509648$$

$$\alpha_6 = \tfrac{1}{550}\log\left(1386 \cdot 2196^{21}\right) = 0.3069389564$$

$$\alpha_7 = \tfrac{1}{924}\log\left(16814 \cdot 37004^{27}\right) = 0.3178972533$$

$$\alpha_8 = \tfrac{1}{42}\log 957304 = 0.3279018162$$

$$\alpha_9 = \tfrac{1}{52}\log 38016960 = 0.3356450564$$

$$\alpha_{10} = \tfrac{1}{3528}\log\left(220240306 \cdot 2317631400^{55}\right) = 0.3416156081$$

$$\alpha_{11} = \tfrac{1}{4950}\log\left(10032960146 \cdot 216893681800^{65}\right) = 0.3474147262$$

$$\alpha_{12} = \tfrac{1}{88}\log 31159166587056 = 0.3530696544$$

Table 1

The values α_b for small b.

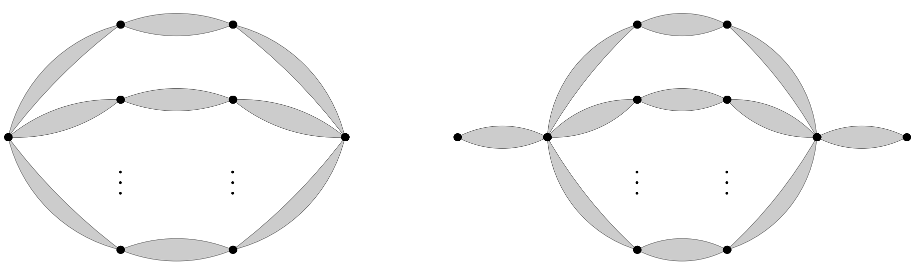

Fig. 6. Construction schemes for two families of self-similar graphs.

5 Statistics

Once it is possible to count perfect matchings, it is natural to consider certain shape statistics. Let us exhibit this for a particular example first. Consider the two-dimensional Sierpiński graph again, as in Section 4.3.2. An edge included in a perfect matching can point in three different directions: up, down or horizontal. We are interested in the distribution of the number of edges in a certain direction (by symmetry, the distribution is the same for all three directions) in a random perfect matching of the level-n Sierpiński graph. In Figure 7 below, there are 7 "up" edges, 9 "down" edges, and 5 horizontal edges in the indicated perfect matching.

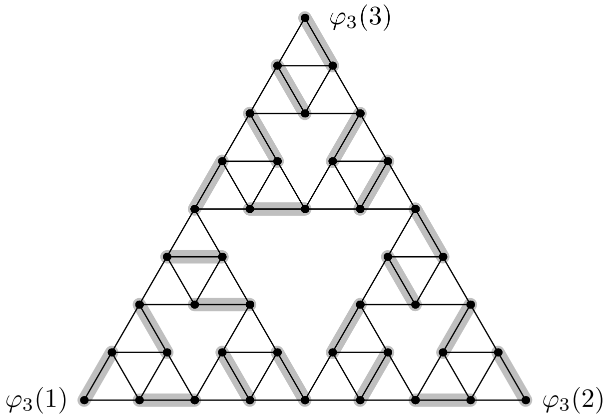

Fig. 7. An example of a perfect matching in a Sierpiński graph of level 3.

In order to analyze this parameter, we slightly modify our definitions: we consider univariate polynomials now, where the coefficient of x^k gives the number of perfect matchings with exactly k horizontal edges. Furthermore, we need more different variables, since the symmetry is not as strong any longer. For a subset K of $\{1,2,3\}$, we let $m_K(X_n) = m_K(X_n, x)$ be the polynomial that corresponds to perfect matchings of $X_n \setminus \varphi_n(K)$. Note that it is still true that $m_K(X_n) = 0$ if $|K| \equiv n \bmod 2$. Furthermore, we have $m_{\{1\}}(X_n) = m_{\{2\}}(X_n)$ and $m_{\{1,3\}}(X_n) = m_{\{2,3\}}(X_n)$ by symmetry. Finally, we obtain a system of recurrences given in Table 2. The initial values are given by

$$m_\emptyset(X_0) = 0, \qquad m_{\{1\}}(X_0) = 1, \qquad m_{\{2\}}(X_0) = 1, \qquad m_{\{3\}}(X_0) = x,$$
$$m_{\{1,2\}}(X_0) = 0, \quad m_{\{1,3\}}(X_0) = 0, \quad m_{\{2,3\}}(X_0) = 0, \quad m_{\{1,2,3\}}(X_0) = 1.$$

Straightforward induction shows that $m_{\{3\}}(X_n) = x m_{\{1,2,3\}}(X_n)$ and $m_\emptyset(X_n) =$

$$
\begin{aligned}
m_{\emptyset}(X_n) &= 2m_{\{1\}}(X_{n-1})m_{\{2\}}(X_{n-1})m_{\{3\}}(X_{n-1}) \\
&= 2m_{\{1\}}(X_{n-1})^2 m_{\{3\}}(X_{n-1}), \\
m_{\{1\}}(X_n) &= m_{\emptyset}(X_{n-1})\big(m_{\{1,2\}}(X_{n-1})^2 + m_{\{1,3\}}(X_{n-1})^2\big), \\
m_{\{3\}}(X_n) &= m_{\emptyset}(X_{n-1})\big(m_{\{1,3\}}(X_{n-1})^2 + m_{\{2,3\}}(X_{n-1})^2\big) \\
&= 2m_{\emptyset}(X_{n-1})m_{\{1,3\}}(X_{n-1})^2, \\
m_{\{1,2\}}(X_n) &= m_{\{1,2,3\}}(X_{n-1})\big(m_{\{1\}}(X_{n-1})^2 + m_{\{2\}}(X_{n-1})^2\big) \\
&= 2m_{\{1,2,3\}}(X_{n-1})m_{\{1\}}(X_{n-1})^2, \\
m_{\{1,3\}}(X_n) &= m_{\{1,2,3\}}(X_{n-1})\big(m_{\{1\}}(X_{n-1})^2 + m_{\{3\}}(X_{n-1})^2\big), \\
m_{\{1,2,3\}}(X_n) &= 2m_{\{1,2\}}(X_{n-1})m_{\{1,3\}}(X_{n-1})m_{\{2,3\}}(X_{n-1}) \\
&= 2m_{\{1,2\}}(X_{n-1})m_{\{1,3\}}(X_{n-1})^2,
\end{aligned}
$$

Table 2

Recurrences for matching polynomials

$xm_{\{1,2\}}(X_n)$ (this can also be seen from the bijection used in the proof of Lemma 4), which allows us to simplify a little further:

$$
\begin{aligned}
m_{\emptyset}(X_n) &= 2m_{\{1\}}(X_{n-1})^2 m_{\{3\}}(X_{n-1}), \\
m_{\{1\}}(X_n) &= m_{\emptyset}(X_{n-1})\Big(m_{\{1,3\}}(X_{n-1})^2 + x^{-2}m_{\emptyset}(X_{n-1})^2\Big), \\
m_{\{3\}}(X_n) &= 2m_{\emptyset}(X_{n-1})m_{\{1,3\}}(X_{n-1})^2, \\
m_{\{1,3\}}(X_n) &= x^{-1}m_{\{3\}}(X_{n-1})\Big(m_{\{1\}}(X_{n-1})^2 + m_{\{3\}}(X_{n-1})^2\Big).
\end{aligned}
$$

Let us now consider the case when n is odd (so that a perfect matching exists), the other case being analogous. Then, it is sufficient to consider $m_{\emptyset}(X_n)$ and $m_{\{1,3\}}(X_n)$. Setting

$$
\begin{aligned}
a_r &= a_r(x) = m_{\emptyset}(X_{2r+1}), \\
b_r &= b_r(x) = m_{\{1,3\}}(X_{2r+1}),
\end{aligned}
$$

and iterating the above recurrences yields

$$a_r = 4a_{r-1}^3 b_{r-1}^2 \Big(x^{-2} a_{r-1}^2 + b_{r-1}^2\Big)^2,$$
$$b_r = 2x^{-1} a_{r-1} b_{r-1}^2 \Big(4a_{r-1}^2 b_{r-1}^4 + a_{r-1}^2 \Big(x^{-2} a_{r-1}^2 + b_{r-1}^2\Big)^2\Big),$$

with initial values $m_\emptyset(X_1) = 2x$ and $m_{\{1,3\}}(X_1) = 1 + x^2$. Now define the quotient q_r by

$$q_r = q_r(x) = \frac{x b_r}{a_r}.$$

From the above equations, it follows that

$$a_{r+1} = a_r^9 \cdot 4x^{-6} q_r^2 (1 + q_r^2)^2 \tag{2}$$

and $q_{r+1} = f(q_r)$, where f is the rational function

$$f(t) = \frac{1}{2} + \frac{2t^4}{(1 + t^2)^2}.$$

The initial values are $a_0 = 2x$ and $q_0 = \frac{1}{2}(1 + x^2)$. Note that $\frac{1}{2} \le f(t) \le \frac{5}{2}$ for all $t \in (0, \infty)$; furthermore, it is not difficult to show that $|f(1 + u) - 1| \le 2(r + 1)^{-1/2}$ if $|u| \le 2r^{-1/2}$, and so straightforward induction shows that $|q_r - 1| \le 2r^{-1/2}$ for all r, implying that q_r tends to 1, uniformly in x. Taking logarithms in (2) yields

$$\log a_{r+1} = 9 \log a_r + \log 16 - 6 \log x + \log \frac{q_r^2 (1 + q_r^2)^2}{4}.$$

Set $\varepsilon_r = \varepsilon_r(x) = \log \frac{q_r^2(1+q_r^2)^2}{4}$ and note that $\varepsilon_r = O(r^{-1/2})$. Hence,

$$\begin{aligned} \log a_r &= 9^r \log a_0 + \sum_{j=0}^{r-1} 9^{r-j-1} (\log 16 - 6 \log x + \varepsilon_j) \\ &= 9^r \log a_0 + \frac{9^r - 1}{8} (\log 16 - 6 \log x) \\ &\qquad + 9^r \sum_{j=0}^{\infty} 9^{-j-1} \varepsilon_j - \sum_{j=r}^{\infty} 9^{r-j-1} \varepsilon_j \\ &= \frac{6 \log x - \log 16}{8} + 9^r G(x) + O(r^{-1/2}), \end{aligned}$$

where $G(x)$ is given by

$$G(x) = \log a_0 - \frac{6\log x - \log 16}{8} + \sum_{j=0}^{\infty} 9^{-j-1}\varepsilon_j(x).$$

From this we obtain

$$a_r = m_\emptyset(X_{2r+1}) = 2^{-1/2}\, x^{3/4}\, e^{9^r G(x)}\,(1 + O(r^{-1/2})) \tag{3}$$

uniformly for $x > 0$. Another simple induction shows that $q_r(1) = q_r'(1) = q_r''(1) = 1$ for all r. Hence, differentiating the explicit formula for $\log a_r$ yields

$$\frac{a_r'(1)}{a_r(1)} = 9^r - 6\sum_{j=0}^{r-1} 9^{r-1-j} + 4\sum_{j=0}^{r-1} 9^{r-1-j} = \frac{3^{2r+1}+1}{4}$$

and

$$\frac{a_r''(1)}{a_r(1)} - \left(\frac{a_r'(1)}{a_r(1)}\right)^2 = -9^r + 6\sum_{j=0}^{r-1} 9^{r-1-j} + 2\sum_{j=0}^{r-1} 9^{r-1-j} = -1,$$

which implies that the mean of the number of horizontal edges is exactly

$$\frac{a_r'(1)}{a_r(1)} = \frac{3^{2r+1}+1}{4}$$

(one third of the total number of edges in a perfect matching, as it was to be expected), while the variance is

$$\frac{a_r''(1)}{a_r(1)} + \frac{a_r'(1)}{a_r(1)} - \left(\frac{a_r'(1)}{a_r(1)}\right)^2 = \frac{3^{2r+1}-3}{4}.$$

In the same way, one finds $G'(1) = \frac{3}{4}$ and $G''(1) = 0$. Finally, let H_r denote the number of horizontal edges in a random perfect matching of X_{2r+1}, and consider the normalized random variable

$$N_r = \frac{H_r - \mu_r}{\sigma_r}, \quad \text{where} \quad \mu_r = \frac{3^{2r+1}+1}{4} \quad \text{and} \quad \sigma_r^2 = \frac{3^{2r+1}-3}{4}.$$

Its moment generating function is given by

$$\mathbb{E}(e^{tN_r}) = e^{-\mu_r t/\sigma_r}\,\mathbb{E}(e^{tH_r/\sigma_r}) = e^{-\mu_r t/\sigma_r}\,\frac{a_r\left(e^{t/\sigma_r}\right)}{a_r(1)}.$$

Making use of the asymptotic formula (3), we obtain

$$\begin{aligned}
\mathbb{E}\left(e^{tN_r}\right) &= \exp\biggl(-\frac{\mu_r t}{\sigma_r} + \frac{3t}{4\sigma_r} + 9^r\Bigl(G(e^{t/\sigma_r}) - G(1)\Bigr)\biggr)(1 + O(r^{-1/2})) \\
&= \exp\biggl(-\frac{\mu_r t}{\sigma_r} + \frac{3t}{4\sigma_r} + 9^r\biggl(G'(1)\frac{t}{\sigma_r} + G'(1)\frac{t^2}{2\sigma_r^2} + G''(1)\frac{t^2}{2\sigma_r^2}\biggr)\biggr) \\
&\qquad\qquad \times \left(1 + O\left(r^{-1/2} + \frac{9^r t^3}{\sigma_r^3}\right)\right) \\
&= \exp\left(\frac{t^2}{2} + O(r^{-1/2})\right)
\end{aligned}$$

uniformly in t on any compact subset of $(-\infty, \infty)$. Therefore, by Curtiss' Theorem [2], the normalized random variable tends weakly to a normal distribution. Summing up, we have the following theorem:

Theorem 5. The random variable "number of horizontal edges in a random perfect matching of X_n", where n is odd, is asymptotically normal, with mean $\frac{3^n+1}{4}$ and variance $\frac{3^n-3}{4}$.

Generally, if a sequence of graphs X_n is constructed as described in this paper, any edge in X_n can be "traced back" to an edge in X_0, and one can consider the number of edges in a random perfect matching that can be traced back to one specific edge in X_0. For $\theta = 2$, i.e. two distinguished vertices, it follows quite immediately that the limit distribution is either normal (as in the example above) or degenerate, which can be seen as follows. Note that no symmetry condition at all was necessary, so we can consider polynomials $m_0(X_n, x)$ and $m_2(X_n, x)$ instead of the ordinary counting sequences $m_0(X_n)$ and $m_2(X_n)$. The solution is still the same—the polynomial $m_0(X_n, x)$ can be explicitly written as

$$m_0(X_n, x) = C_0(x)\,\gamma^{\tau n}\,\beta(x)^{s^n},$$

where $C_0(x)$ and $\beta(x)$ are given by

$$C_0(x) = (a^{-1}\gamma^\tau)^{1/(s-1)}\, Q(x)^{-\tau},$$
$$\beta(x) = C_0^{-1}\, m_0(X_0, x),$$
$$Q(x) = \frac{m_2(X_0, x)}{m_0(X_0, x)}$$

with $a, b, s, \nu, \gamma, \tau$ as in Section 4.1. The normalized polynomial $m_0(X_n, x)/m_0(X_n, 1)$ is thus given by

$$\frac{m_0(X_n, x)}{m_0(X_n, 1)} = \left(\frac{Q(x)}{Q(1)}\right)^{-\tau} \left(\frac{Q(x)^\tau m_0(X_0, x)}{Q(1)^\tau m_0(X_0, 1)}\right)^{s^n},$$

and now there are several ways to show asymptotic normality (unless the distribution is degenerate), for instance Hwang's quasi-power theorem [4].

Generally, for $\theta \geq 3$, it can be expected that the distribution is still asymptotically normal or degenerate, but this seems to be difficult to prove, considering that mere counting of perfect matchings becomes more intricate for $\theta > 3$ (see the following section).

6 The general case

In this section, we consider the case of arbitrary θ. First, we use the example of higher-dimensional Sierpiński graphs to exhibit the problems arising in the general case. Then, we consider Vičeck graphs, for which it is still possible to obtain explicit formulæ. This is further generalized and discussed in Section 6.2.

6.1 Examples

6.1.1 Higher-dimensional Sierpiński graphs

For the construction see Section 2.1.2. Let us consider the three-dimensional case: $d = 3$. Then $s = \theta = 4$, $|VG| = 10$, and $\delta = 6$. Since $|VX_0| = 4$, the number $|VX_n|$ is always even. A short calculation yields the following recurrences:

$$\begin{aligned}
m_0(X_{n+1}) &= 8m_0(X_n)m_2(X_n)^3,\\
m_2(X_{n+1}) &= 4m_0(X_n)m_2(X_n)^2m_4(X_n) + 4m_2(X_n)^4,\\
m_4(X_{n+1}) &= 8m_2(X_n)^3m_4(X_n).
\end{aligned}$$

The initial values are given by $(m_0(X_0), m_2(X_0), m_4(X_0)) = (3, 1, 1)$.

It is obvious from the recurrences that

$$\frac{m_0(X_n)}{m_4(X_n)} = 3$$

for all n. Furthermore, if we set

$$q_n = \frac{m_2(X_n)}{m_4(X_n)}, \qquad \text{then} \qquad q_{n+1} = \frac{q_n^2 + 3}{2q_n},$$

and so q_n converges to $\sqrt{3}$ at a doubly exponential rate, i.e. $q_n = \sqrt{3} + O(C^{2^n})$ for some $0 < C < 1$. The same follows for the quotient

$$\frac{m_0(X_n)}{m_2(X_n)} = \frac{3}{q_n},$$

and so we have

$$m_0(X_{n+1}) = \frac{8}{3\sqrt{3}} m_0(X_n)^4 \Big(1 + O(C^{2^n})\Big).$$

Using the same techniques as in the previous section, we obtain

$$m_0(X_n) \sim \alpha \cdot \beta^{4^n},$$

where $\alpha = \frac{\sqrt{3}}{2}$ and $\beta = 2.3582688182$. β can also be expressed explicitly as

$$\beta = 2 \cdot \prod_{j=0}^{\infty} q_j^{3 \cdot 4^{-j-1}}.$$

This constant, without the precise asymptotic behavior, was also determined in [1].

Due to the fact that the polynomials in the recurrences are no longer monomials, there is no explicit formula any more. The asymptotic behavior can be obtained for Sierpiński graphs of higher dimension by essentially the same ideas (compare again [1]), but the technical details become increasingly tedious, and it is not quite clear how a general result for higher dimensions might be found.

6.1.2 Viček graphs

See Section 2.1.3 for definitions. Here we have $s = \theta + 1$, $|VG| = \theta^2$, $\delta = \theta$, $|VX_0| = \theta$. If θ is even, then $|VX_n|$ is always even, too. So let us restrict to this case. It is then easy to check that

$$m_k(X_n) = m_0(X_{n-1})^{\theta-k} m_2(X_{n-1})^k m_{\theta-k}(X_{n-1})$$

for even k. We assume that $\theta \geq 6$, the other cases being degenerate and thus easier. It is sufficient to consider the quantities $m_0(X_n)$, $m_2(X_n)$, $m_{\theta-2}(X_n)$, and $m_\theta(X_n)$:

$$\begin{aligned} m_0(X_n) &= m_0(X_{n-1})^\theta m_\theta(X_{n-1}), \\ m_2(X_n) &= m_0(X_{n-1})^{\theta-2} m_2(X_{n-1})^2 m_{\theta-2}(X_{n-1}), \\ m_{\theta-2}(X_n) &= m_0(X_{n-1})^2 m_2(X_{n-1})^{\theta-1}, \\ m_\theta(X_n) &= m_0(X_{n-1}) m_2(X_{n-1})^\theta, \end{aligned}$$

Using basic linear algebra it is easy to derive closed formulæ from these recurrences. Since the formulæ are rather long, we will not state them here. However, by taking logarithms we obtain $\boldsymbol{x}_n = \boldsymbol{A}\, \boldsymbol{x}_{n-1}$, where

$$\boldsymbol{A} = \begin{pmatrix} \theta & 0 & 0 & 1 \\ \theta-2 & 2 & 1 & 0 \\ 2 & \theta-1 & 0 & 0 \\ 1 & \theta & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{x}_n = \begin{pmatrix} \log m_0(X_n) \\ \log m_2(X_n) \\ \log m_{\theta-2}(X_n) \\ \log m_\theta(X_n) \end{pmatrix}.$$

The eigenvalues of $\boldsymbol{A}$ are $s = \theta+1, 1, 1, -1$ (taking algebraic multiplicity into account), where the eigenvalue 1 has geometric multiplicity 1.

6.2 A special case

For simplicity we restrict to the case when VX_n is always even for $n > 0$. As in the cases $\theta = 2$ and $\theta = 3$, there are also examples of self-similar graphs with $\theta \geq 4$ (such as the Viček graphs discussed above), where the recurrences for $m_k(X_n)$ have the special form

$$m_{2k}(X_n) = b_k \prod_i m_{2i}(X_{n-1})^{a_{k,i}},$$

which leads to exact formulæ for the quantities $m_{2k}(X_n)$. To this end, set $x_{k,n} = \log m_{2k}(X_n)$ and $\boldsymbol{x}_n = (x_{0,n}, x_{1,n}, \dots)$; then

$$\boldsymbol{x}_n = \boldsymbol{A}\boldsymbol{x}_{n-1} + \boldsymbol{c},$$

where $\boldsymbol{A} = (a_{k,i})_{k,i}$ and $\boldsymbol{c} = (\log b_k)_k$. The recurrence equation above can be solved easily by means of linear algebra:

Proposition 1. *For even k the quantity $\log m_k(X_n)$ is given by the solution of a linear recurrence equation. Moreover, s and 1 are eigenvalues of the matrix $\boldsymbol{A}$.*

Proof. The first part is plain. Using the homogeneity of the recurrences $\boldsymbol{A}\mathbf{1} = s\mathbf{1}$ follows ($\mathbf{1} = (1, 1, 1, \dots)$). The second restriction on the exponents of the monomials in the system (see Lemma 2) implies that $\boldsymbol{A}\boldsymbol{f} = \delta\mathbf{1} + \boldsymbol{f}$, where $\boldsymbol{f} = (0, 2, 4, \dots)$. Together with $\boldsymbol{A}\mathbf{1} = s\mathbf{1}$ we obtain

$$\boldsymbol{A}\Big(\boldsymbol{f} - \tfrac{\delta}{s-1}\mathbf{1}\Big) = \Big(\boldsymbol{f} - \tfrac{\delta}{s-1}\mathbf{1}\Big). \qquad \blacksquare$$

Of course a similar result holds when the parity of $|VX_n|$ is odd or alternating.

6.3 Final Remark

As demonstrated, there so no hope for closed formulæ in the general case. However, the examples suggest that $\log m_0(X_n)$ is always asymptotically equal to the solution of a linear recurrence. Furthermore, it is likely that such a solution contains only powers of the form 1^n, $(-1)^n$ and s^n. Note that this was the case in all examples so far. Moreover, we have verified this conjecture for a subclass where the structure of the self-similar graphs is "tree-like", as for the Viček graphs.

References

[1] S.-C. Chang, L.-C. Chen, Dimer coverings on the sierpinski gasket with possible vacancies on the outmost vertices, J. Stat. Phys. 131 (4) (2008) 631–650, arXiv:0711.0573v1.

[2] J. H. Curtiss, A note on the theory of moment generating functions, Ann. Math. Statistics 13 (1942) 430–433.

[3] E. J. Farrell, Matching in cacti, J. Math. Sci. (Calcutta) 8 (2) (1997) 85–96.

[4] H.-K. Hwang, On convergence rates in the central limit theorems for combinatorial structures, European J. Combin. 19 (3) (1998) 329–343.

[5] P. W. Kasteleyn, Graph theory and crystal physics, in: Graph Theory and Theoretical Physics, Academic Press, London, 1967, pp. 43–110.

[6] J. W. Moon, The number of trees with a 1-factor, Discrete Math. 63 (1) (1987) 27–37.

[7] J. Propp, Enumeration of matchings: problems and progress, in: New perspectives in algebraic combinatorics (Berkeley, CA, 1996–97), vol. 38 of Math. Sci. Res. Inst. Publ., Cambridge Univ. Press, Cambridge, 1999, pp. 255–291.

[8] E. Teufl, S. Wagner, Enumeration problems for classes of self-similar graphs, J. Combin. Theory Ser. A 114 (7) (2007) 1254–1277.

[9] E. Teufl, S. Wagner, The number of spanning trees in self-similar graphs, preprint (2008).