

A NOTE ON THE NUMBER OF DOMINATING SETS OF A GRAPH

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ABSTRACT. In a recent article by Bród and Skupień, sharp upper and lower bounds for the number of dominating sets in a tree were determined. In the present paper we show that the lower bound even holds for graphs without isolated vertices and characterise the cases of equality. Further results on this graph parameter are a Turán-type theorem for the number of dominating sets of a graph with given domination number, and a Nordhaus-Gaddum inequality.

1. INTRODUCTION

While various parameters related to domination in graphs have been studied extensively (see the book [4] for a comprehensive treatment of different aspects of domination), this does not yet seem to be the case for the number of dominating sets of a graph (henceforth denoted by $\partial(G)$ in accordance with [1]). Only recently, it was shown that the number of dominating sets of a graph is always odd (with a particularly elegant proof given by Lex Schrijver, see [2]), which is quite a remarkable property.

The most natural problem in connection with graph parameters is usually to determine good upper and lower bounds under certain assumptions. It is trivial that one has $1 \leq \partial(G) \leq 2^n - 1$ for any graph G on n vertices, with equality for the empty and complete graph respectively. Upper and lower bounds for trees were determined in another interesting recent paper by Bród and Skupień [1], following analogous investigations on other graph parameters such as the number of independent sets [6, 7], the number of maximal independent sets [8, 9] or the number of maximal matchings [3]. While the maximum of $2^{n-1} + 1$ (among all trees with n vertices) is attained for the star (and by the path for $n = 4$ and $n = 5$), the lower bound is more complicated, and it is attained for a rich class of trees. The result of Bród and Skupień can be stated as follows (the actual formulation in [1] is slightly different yet equivalent):

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Theorem 1 (Bród and Skupień [1]). *Let T be a tree on $n > 1$ vertices. Then the following inequality holds:*

$$\partial(T) \geq \begin{cases} 5^{n/3} & n \equiv 0 \pmod{3}, \\ 9 \cdot 5^{(n-4)/3} & n \equiv 1 \pmod{3}, \\ 3 \cdot 5^{(n-2)/3} & n \equiv 2 \pmod{3}. \end{cases}$$

Equality holds

- *for $n \equiv 0 \pmod{3}$ if and only if every internal (non-leaf) vertex is adjacent to exactly two leaves,*
- *for $n \equiv 1 \pmod{3}$ if and only if every internal vertex is adjacent to exactly two leaves, with either one exceptional internal vertex that is adjacent to three leaves, or two exceptional internal vertices that are adjacent to one leaf each,*
- *for $n \equiv 2 \pmod{3}$ if and only if every internal vertex is adjacent to exactly two leaves, with one exceptional vertex that is adjacent to only one leaf.*

One aim of this note is to show that there is actually nothing particularly special about trees in this context: the lower bound remains correct and sharp if arbitrary connected graphs or even more generally graphs without isolated vertices are considered (see Section 2). Furthermore, we present a Turán-type result on graphs with given domination number as well as a Nordhaus-Gaddum inequality for the number of dominating sets of a graph (Section 3).

2. THE LOWER BOUND

The number of trees that actually attain the lower bound is quite considerable (as also remarked in [1]): if n is divisible by 3, then this number is just the number of trees with $n/3$ vertices. However, there are even more graphs for which the lower bound is attained. It is obvious that a spanning subgraph H of a graph G can have at most as many dominating sets as G (since any dominating set in H is automatically a dominating set in G). It turns out that equality can hold even if $H \neq G$ —it is interesting to compare this to the situation of independent sets: if H is a spanning subgraph of G and $H \neq G$, then H must always have strictly more independent sets than G . The following theorem characterises all connected graphs that attain the lower bound for the number of dominating sets:

Theorem 2. *Let G be a connected graph on $n > 1$ vertices. Then the following inequality holds:*

$$\partial(G) \geq \begin{cases} 5^{n/3} & n \equiv 0 \pmod{3}, \\ 9 \cdot 5^{(n-4)/3} & n \equiv 1 \pmod{3}, \\ 3 \cdot 5^{(n-2)/3} & n \equiv 2 \pmod{3}. \end{cases}$$

Equality holds

- *for $n \equiv 0 \pmod{3}$ if and only if every vertex of degree > 1 is adjacent to exactly two vertices of degree 1,*

- for $n \equiv 1 \pmod{3}$ if and only if every vertex of degree > 1 is adjacent to exactly two vertices of degree 1, with either one exceptional vertex that is adjacent to three such vertices, or two exceptional vertices that are adjacent to one vertex of degree 1 each,
- for $n \equiv 2 \pmod{3}$ if and only if every vertex of degree > 1 is adjacent to exactly two vertices of degree 1, with one exceptional vertex that is adjacent to only one such vertex.

Proof. Since any spanning tree T of G has at most as many dominating sets as G , the bounds follow trivially from Theorem 1. Equality can only hold if T is one of the trees described in the statement of this proposition. It remains to show that all leaves of T are still vertices of degree 1 in G if equality holds. Suppose to the contrary that v is a leaf in T whose neighbour in T is w , and that v is connected to $w' \neq w$ in G . If $n > 2$ (the case $n = 2$ being trivial), then the degree of w must be greater than 1. Therefore, the set that consists of all vertices of G except for v and w is a dominating set in G (v is dominated by w' , w is dominated by a neighbour other than v), but it is not a dominating set in T (since v is not dominated). Therefore, one has $\partial(G) > \partial(T)$, so that equality cannot hold in the theorem. ■

Interestingly enough, the theorem even remains true if disconnected graphs are allowed, as long as there are no isolated vertices. It is well known that any graph without isolated vertices has a dominating set of at most $\frac{n}{2}$ vertices, and any superset of such a dominating set is still a dominating set, so one can expect at least $2^{n/2}$ dominating sets in a graph without isolated vertices, but this trivial bound is quite far from the actual bound of (essentially) $5^{n/3}$:

Corollary 3. *Theorem 2 remains true if G is restricted to graphs without isolated vertices.*

Proof. By induction on the number r of components. The case $r = 1$ is already given by Theorem 2. For the induction step, note first that $\partial(G) = \prod_{j=1}^k \partial(G_j)$ if G_1, G_2, \dots, G_k are the connected components of a graph G . Now let H_1 be one of the components of G and $H_2 = G \setminus H_1$. Then we can apply the induction hypothesis to both H_1 and H_2 . Let n_1 and n_2 be the number of vertices of H_1 and H_2 respectively. Then, depending on the remainders of n_1 and n_2 modulo 3 (denoted h_1 and h_2 respectively), we have to distinguish 9 cases summarised in the following table: the lower bound obtained from the induction hypothesis is given in each case.

	$h_2 = 0$	$h_2 = 1$	$h_2 = 2$
$h_1 = 0$	$5^{n/3}$	$9 \cdot 5^{(n-4)/3}$	$3 \cdot 5^{(n-2)/3}$
$h_1 = 1$	$9 \cdot 5^{(n-4)/3}$	$81 \cdot 5^{(n-8)/3} > 3 \cdot 5^{(n-2)/3}$	$27 \cdot 5^{(n-6)/3} > 5^{n/3}$
$h_1 = 2$	$3 \cdot 5^{(n-2)/3}$	$27 \cdot 5^{(n-6)/3} > 5^{n/3}$	$9 \cdot 5^{(n-4)/3}$

In each case, we obtain the desired lower bound, which completes the induction. A closer look at the induction step also shows that equality can only hold if almost all components

have a number of vertices that is divisible by 3, with the exception of either at most one component whose cardinality is $\equiv 1 \pmod 3$ or at most two components whose cardinality is $\equiv 2 \pmod 3$ (but not both). Together with Theorem 2, applied to each of the components, this shows that the characterisation of the cases of equality is still the same as in Theorem 2. ■

3. A TURÁN-TYPE THEOREM AND A NORDHAUS-GADDUM INEQUALITY

As mentioned earlier, the fact that any graph without isolated vertices contains a dominating set of size at most $\frac{n}{2}$ guarantees the existence of at least $2^{n/2}$ dominating sets. Generally, the existence of a “small” dominating set of size k implies that the number of dominating sets is at least 2^{n-k} (all supersets). It soon becomes clear, however, that this is far from best possible. The following theorem addresses the question how many dominating sets a graph (without isolated vertices, to avoid trivialities) must have at least if the domination number is k . The analogous problem for the number of cliques was treated in a paper of Hedman [5], where the extremal graphs were found to be Turán graphs. The construction of the extremal graphs in our case is also somewhat reminiscent of the Turán graphs:

Theorem 4. *Suppose that the domination number of a graph G of order n without isolated vertices is k ; write $n = ak + b$, where a and b are integers with $0 \leq b < k$. Then the following lower bound for $\partial(G)$ holds:*

$$\partial(G) \geq (2^{a-1} + 1)^{k-b} (2^a + 1)^b,$$

with equality if and only if G contains a dominating set K of size k with the property that all vertices outside of K have degree 1 and that every vertex of K is adjacent to either $a - 1$ or a of these vertices.

Proof. Suppose that G has the smallest possible number of dominating sets among all graphs with n vertices and domination number k . Let K be a dominating set of size k , which exists by assumption. First we show that K can be chosen in such a way that every vertex outside of K has degree 1. If K contains a vertex v of degree 1, consider two possibilities:

- If the unique neighbour of v also has degree 1, then the two vertices form a component of G , and they can henceforth be ignored.
- Otherwise, replace v by its neighbour in K ; clearly, the resulting set is still dominating.

From now on, we may assume that K does not contain vertices of degree 1. Now let v be a vertex outside of K . Since K is dominating, v must have a neighbour w in K . Assume that v has other neighbours, and consider the graph G' that results if all edges incident to v , except for vw , are removed. Clearly $\partial(G') \leq \partial(G)$, and the inequality is even strict: the set that comprises all vertices of G except for v and w is dominating in G , but not in G'

(here it is essential that the degree of w is not 1). Since we obtain a contradiction, we can conclude that all vertices outside of K have degree 1.

If we want to construct a dominating set, we have the following possibilities for each of the vertices in K :

- We can include the vertex and any subset of its neighbours that are not in K (there is at least one such vertex, since otherwise K would not be a minimum dominating set; here it is important that there are no isolated vertices), or
- not include the vertex and include all of its neighbours in K .

For a vertex v in K , let $e(v)$ be the number of its neighbours outside of K . The above argument shows that one has $2^{e(v)} + 1$ choices, independently for each of the elements of K , and thus

$$\partial(G) = \prod_{v \in K} (2^{e(v)} + 1).$$

It remains to show that $e(v)$ is either $a - 1$ or a for each of the elements in K . To this end, it suffices to prove that there cannot be two vertices v and w in K such that $e(v) - e(w) \geq 2$.

Suppose that this was the case. Move one of the neighbours of v whose degree is 1 to w ; then $e(v)$ decreases by 1, while $e(w)$ increases by 1. Everything else remains unchanged. Note, however, that

$$(2^{e(v)-1} + 1)(2^{e(w)+1} + 1) - (2^{e(v)} + 1)(2^{e(w)} + 1) = 2^{e(w)} - 2^{e(v)-1} < 0,$$

which shows that the resulting graph has a smaller number of dominating sets. This completes the proof. \blacksquare

Remark. If graphs with isolated vertices are taken into consideration as well, then the lower bound decreases to $2^{n-k} + 1$, which is attained for a star of order $n - k + 1$ together with $k - 1$ isolated vertices.

Inequalities that relate a graph and its complement are usually referred to as *Nordhaus-Gaddum inequalities*, see for instance Chapter 9 of [4] for such inequalities in the context of domination. The following theorem provides such an inequality for the number of dominating sets:

Theorem 5. *For any graph G on n vertices and its complement \overline{G} , the inequality*

$$\partial(G) + \partial(\overline{G}) \geq 2^n$$

holds, and this inequality is sharp.

Proof. Note that equality holds for the complete graph (and many other graphs as well, e.g. the star). To prove the inequality, consider a set S of vertices that is not a dominating set of G . Then there exists a vertex v that is not dominated by S —in other words, v does not have any neighbours in S . But this implies that v is connected to all vertices of S in the complement graph \overline{G} , so that the set complement \overline{S} of S is a dominating set of \overline{G} .

So we can conclude that \overline{G} has at least as many dominating sets as there are non-dominating sets in G ; the inequality

$$\partial(G) + \partial(\overline{G}) \geq \partial(G) + (2^n - \partial(G)) = 2^n$$

follows immediately. ■

It seems to be much more difficult to determine the maximum of $\partial(G) + \partial(\overline{G})$ as G ranges over all possible graphs on n vertices; let us leave this as an open problem:

Problem 1. *Determine the smallest upper bound for $\partial(G) + \partial(\overline{G})$ in terms of the number of vertices of G .*

4. CONCLUSION

Compared to similar graph parameters such as the number of independent sets, the extremal graphs (in particular those of Theorem 2) show a rather unexpected structure. The fact that there is a very large class of graphs for which the lower bound is attained (for many graph parameters, the extremal graphs are unique, or at least there is a bounded number of extremal graphs) also shows that the parameter ∂ is relatively unaffected by other properties of the graph, such as diameter, girth, or cyclomatic number.

Many natural related problems remain for further study; instead of dominating sets, one might want to consider total dominating sets, minimal dominating sets (minimal with respect to set inclusion) or minimum cardinality dominating sets and try to determine upper and lower bounds.

REFERENCES

- [1] D. Bród and Z. Skupień. Trees with extremal numbers of dominating sets. *Australas. J. Combin.*, 35:273–290, 2006.
- [2] A. E. Brouwer. The number of dominating sets of a finite graph is odd. Preprint, <http://www.win.tue.nl/~aeb/preprints/domin2.pdf>, 2009.
- [3] J. Górska and Z. Skupień. Trees with maximum number of maximal matchings. *Discrete Math.*, 307(11–12):1367–1377, 2007.
- [4] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater. *Fundamentals of domination in graphs*, volume 208 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 1998.
- [5] B. Hedman. Another extremal problem for Turán graphs. *Discrete Math.*, 65(2):173–176, 1987.
- [6] S. B. Lin and C. Lin. Trees and forests with large and small independent indices. *Chinese J. Math.*, 23(3):199–210, 1995.
- [7] H. Prodinger and R. F. Tichy. Fibonacci numbers of graphs. *Fibonacci Quart.*, 20(1):16–21, 1982.
- [8] B. E. Sagan. A note on independent sets in trees. *SIAM J. Discrete Math.*, 1(1):105–108, 1988.
- [9] H. S. Wilf. The number of maximal independent sets in a tree. *SIAM J. Algebraic Discrete Methods*, 7(1):125–130, 1986.

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