Determinant identities for Laplace matrices

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Abstract

We show that every minor of an $n \times n$ Laplace matrix, i.e. a symmetric matrix whose row- and column sums are 0, can be written in terms of those $\binom{n}{2}$ minors that are obtained by deleting two rows and the corresponding columns. The proof is based on a classical determinant identity due to Sylvester. Furthermore, we show how our result can be applied in the context of electrical networks and spanning tree enumeration.

Key words: determinant identity, minors, electrical networks, spanning trees

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1. Introduction

Identities between various minors of a matrix have a long tradition that dates back at least to the eighteenth century; the book of Muir [1] provides an excellent treatise on the theory of determinants. In combinatorics, determinants are frequently used to solve enumeration problems, in particular in the context of graph-theoretical problems: it is well-known that every minor of the Laplace matrix of a graph can be interpreted as the number of certain spanning forests of the graph, see for example [2, 3, 4]. In particular, the determinant of a matrix that is obtained by deleting any single row and any column of the Laplace matrix is, except possibly for the sign, the number of spanning trees of the corresponding graph—Kirchhoff's celebrated Matrix-Tree Theorem [5]. Kirchhoff's motivation was the study of electrical networks: an edge-weighted graph can be regarded as an electrical network, where the weights are the conductances of the respective edges. The effective conductance between two specific vertices v, w can be written as the quotient of the (weighted) number of spanning trees and the (weighted) number of so-called thickets, i.e. spanning forests with exactly two components and the property that each of the components contains precisely one of the vertices v, w [6]. By the aforementioned properties of the Laplace matrix, this can be rewritten as the quotient of two minors of the Laplace matrix.

Noticing that an electrical network on n vertices is uniquely determined by $\binom{n}{2}$ conductances, a natural question is: is it possible to reconstruct them from the $\binom{n}{2}$ effective conductances? While the step from conductances to effective conductances only involves the computation of certain determinants, the reverse step is not quite as obvious: it is known that the effective conductances determine the network uniquely, see for example [7], but a priori, determining all conductances amounts to solving a nonlinear system of equations in $\binom{n}{2}$ unknowns. To the best of our knowledge, nobody has ever treated the question whether an explicit formula for the conductances of an electrical network in terms of the effective conductances exists.

In this paper, we will show that such a formula indeed exists and that it can be obtained from a determinant identity for Laplace matrices. This identity is actually more general: it relates any minor of a Laplace matrix to the specific minors that are obtained by deleting two rows and the corresponding columns. The proof of our identity is based on a classical result of Sylvester.

Our second motivation is the problem of enumerating spanning trees in graphs with a high degree of symmetry. In a recent paper by the authors [8], the following theorem was given as a byproduct:

Theorem. *Let G be a connected (multi-)graph, and let $\Theta \subseteq V$ be a subset of θ distinguished vertices. Suppose that G is strongly symmetric with respect to Θ, i.e. the restriction of the automorphism group of G to Θ is either the entire symmetric group or the alternating group. If $r(A)$ denotes the number of all rooted spanning forests of G whose roots are the elements of A and $\tau(G)$ is the number of spanning trees of G, then we have*

$$r(A) = m\rho^{m-1}\theta^{1-m}\tau(G)$$

for all sets $A \subseteq \Theta$ of cardinality m. Here, ρ is the so-called resistance scaling factor of G with respect to Θ (for a precise definition, see Section 4).

We will show that this is also a corollary of our determinant identity and that it even holds in the somewhat more general case that the automorphism group acts 2-homogeneously on the set Θ.

In the last section, we will describe how our determinant identity can be exploited to provide a very general method for the enumeration of spanning trees; roughly stated, if any part of a graph is replaced by an electrically equivalent graph, the number of spanning trees only changes by a factor that is independent of the rest of the graph. This allows us to determine the number of spanning trees in a graph by the same methods that are used to simplify electrical networks. The described technique proves to be most useful if the graphs under consideration are highly symmetric; in particular, it can be applied to the enumeration of spanning trees in self-similar graphs such as the Sierpiński graphs, a problem which has recently gained attention in physics [9].

2. Main result

Let L be a square matrix. Given a set $A = \{a_1, \dots, a_m\}$ of row indices and a set $B = \{b_1, \dots, b_m\}$ of column indices we write L_B^A for the submatrix of L, where rows in A and columns in B are deleted, and write $D_B^A = \det L_B^A$ for the associated minor. For convenience, we write D_{kl}^{ij} instead of $D_{\{k,l\}}^{\{i,j\}}$. We will make use of the following identity for minors of a matrix that is due to Sylvester, see [1, 10] and the references therein.

Theorem 1. *Let $A = \{a_1 < a_2 < \cdots < a_m\}$ and $B = \{b_1 < b_2 < \cdots < b_m\}$ be sets of row and column indices of the matrix L, respectively. Then, for any k and l,*

$$D_B^A \big(D_{b_l}^{a_k}\big)^{m-2} = (-1)^{k+l} \sum_{\substack{\pi \in S_m \\ \pi(l)=k}} \operatorname{sgn} \pi \prod_{\substack{1 \le i \le m \\ i \ne l}} D_{b_i b_l}^{a_{\pi i} a_k}. \tag{1}$$

In the following we always assume that the matrix L is symmetric and that it has zero row/column sum. Then L is a (weighted) Laplace matrix of a graph G with edge weights $c(e)$, $e \in EG$. We note that all graphs under consideration are allowed to have parallel edges and loops. By the matrix-tree theorem the cofactors $(-1)^{a+b} D_b^a$ are all equal and count the number of (weighted) spanning trees in G, as mentioned in the introduction. We denote their common value by $\tau = \tau(G) = \tau(G, c)$. More generally, the absolute value of D_B^A counts (weighted) spanning forests each of which components contains exactly one vertex from A and one from B, see [2, 3, 4]. Whenever edge weights are given, the number of spanning trees and similar objects is always counted with respect to these weights.

Using the symmetry condition and the zero row sum condition we express the left hand side $D_B^A \tau^{m-2}$ of Equation (1) in terms of minors of the form D_{rs}^{rs}. In order to state the following theorem, define $\mathcal{G}(A, B)$ to be the family of graphs Λ which satisfy the following properties:

- The vertex set $V\Lambda$ is $A \cup B$.
- The edge set $E\Lambda$ has size $m - 1$.
- The set of components consists of paths (including isolated vertices) and cycles (excluding loops, but allowing 2-cycles).
- The vertices of cyclic components are contained in $A \cap B$.
- Path components of length 1 and more have one end-vertex in A and the other in B. All internal vertices are contained in $A \cap B$.

As a consequence a graph Λ in $\mathcal{G}(A, B)$ has exactly $|A \setminus B| + 1$ path components and there are unique vertices $a \in A$ and $b \in B$ ($a = b$ is allowed) so that $\Lambda + ab$ has constant degree 1 on the symmetric difference $A \triangle B$ and constant degree 2 on $A \cap B$.

Theorem 2. *Let A and B be sets of row and column indices of the matrix L with $|A| = |B| = m$. Then*

$$D_B^A \tau^{m-2} = \sum_{\Lambda \in \mathcal{G}(A,B)} \alpha(\Lambda) \prod_{rs \in E\Lambda} D_{rs}^{rs}.$$

The coefficients $\alpha(\Lambda)$ are of the form $\pm\left(\frac{1}{2}\right)^{\nu}$, where the sign and the non-negative integer ν can be computed in terms of Λ, see Lemma 6 and its proof.

For the proof of this theorem, we need a sequence of lemmas. Note first that by symmetry $D_Y^X = D_X^Y$ for any index sets X and Y. For convenience we set $D_{kl}^{ii} = D_{ii}^{kl} = 0$ for arbitrary (possibly equal) i, k, l. The following lemma expresses all minors D_X^Y with $|X| = |Y| = 2$ in terms of minors of the form D_{rs}^{rs}.

Lemma 3. *If $i \le j$ and $k \le l$, then*

$$D_{kl}^{ij} = \tfrac{1}{2}(-1)^{i+j+k+l}\big(D_{il}^{il} + D_{jk}^{jk} - D_{ik}^{ik} - D_{jl}^{jl}\big). \tag{2}$$

Proof. If $i = j$ and/or $k = l$ then we get 0 on both sides. If $i = k$ and $j = l$, then the statement is also trivial. For certain fixed indices r and s, denote by $v_1, v_2, \ldots$ the columns of L^{rs} (rows r and s are deleted). If now $i < j < k$, then we get

$$\begin{aligned} 0 &= \det(v_i + v_j + v_k, v_1, v_2, \ldots) \\ &= \det(v_i, v_1, v_2, \ldots) + \det(v_j, v_1, v_2, \ldots) + \det(v_k, v_1, v_2, \ldots) \\ &= (-1)^{i-1} D_{jk}^{rs} + (-1)^{j-2} D_{ik}^{rs} + (-1)^{k-3} D_{ij}^{rs} \end{aligned}$$

by the zero row sum property, where the columns v_i, v_j, v_k are omitted in the sequence $v_1, v_2 \ldots$ inside determinants. Denote the right hand side of the last equation by $\mathrm{RHS}(r, s)$; then by symmetry

$$\begin{aligned} 0 &= (-1)^{k-3}\,\mathrm{RHS}(i, j) + (-1)^{j-2}\,\mathrm{RHS}(i, k) - (-1)^{i-1}\,\mathrm{RHS}(j, k) \\ &= 2(-1)^{j+k+1} D_{ik}^{ij} + D_{ij}^{ij} + D_{ik}^{ik} - D_{jk}^{jk}. \end{aligned}$$

Solving this for D_{ik}^{ij} yields

$$D_{ik}^{ij} = \tfrac{1}{2}(-1)^{j+k}\big(D_{ij}^{ij} + D_{ik}^{ik} - D_{jk}^{jk}\big).$$

By similar calculations we get

$$D^{ij}_{jk} = \tfrac{1}{2}(-1)^{i+k}\bigl(D^{ik}_{ik} - D^{ij}_{ij} - D^{jk}_{jk}\bigr),$$
$$D^{ik}_{jk} = \tfrac{1}{2}(-1)^{i+j}\bigl(D^{ik}_{ik} + D^{jk}_{jk} - D^{ij}_{ij}\bigr).$$

Note that the three identities above match the statement of the lemma since $D^{ii}_{ii} = 0$, etc. If $i < j < k < l$, then

$$\begin{aligned} 0 = \mathrm{RHS}(k,l) &= (-1)^{i-1}D^{kl}_{jk} + (-1)^{j-2}D^{kl}_{ik} + (-1)^{k-3}D^{kl}_{ij} \\ &= \tfrac{1}{2}(-1)^{i+j+l-1}\bigl(D^{jl}_{jl} - D^{jk}_{jk} - D^{kl}_{kl}\bigr) \\ &\quad + \tfrac{1}{2}(-1)^{i+j+l-2}\bigl(D^{il}_{il} - D^{ik}_{ik} - D^{kl}_{kl}\bigr) + (-1)^{k-3}D^{kl}_{ij} \\ &= \tfrac{1}{2}(-1)^{i+j+l}\bigl(D^{il}_{il} + D^{jk}_{jk} - D^{ik}_{ik} - D^{jl}_{jl}\bigr) + (-1)^{k-3}D^{kl}_{ij} \end{aligned}$$

and therefore

$$D^{kl}_{ij} = \tfrac{1}{2}(-1)^{i+j+k+l}\bigl(D^{il}_{il} + D^{jk}_{jk} - D^{ik}_{ik} - D^{jl}_{jl}\bigr).$$

Similarly, considering the equations $0 = \mathrm{RHS}(j,l)$ and $0 = \mathrm{RHS}(i,l)$ yields the identity for D^{ik}_{jl} and D^{il}_{jk}. □

Now we substitute (2) into Sylvester's identity (1) for $k = l = m$ and obtain

$$\begin{aligned} D^A_B\,\tau^{m-2} = (-1)^{\Sigma A + \Sigma B}\bigl(-\tfrac{1}{2}\bigr)^{m-1} \times \\ \times \sum_{\pi\in S_{m-1}} \operatorname{sgn}\pi \prod_{1\le i<m} \Bigl(D^{a_{\pi i}b_i}_{a_{\pi i}b_i} + D^{a_m b_m}_{a_m b_m} - D^{a_{\pi i}b_m}_{a_{\pi i}b_m} - D^{a_m b_i}_{a_m b_i}\Bigr) \end{aligned} \tag{3}$$

after some simplification, where $\Sigma A = a_1 + \cdots + a_m$ and $\Sigma B = b_1 + \cdots + b_m$. When the products are expanded, a fair amount of cancellation occurs. In a first step we temporarily consider the minors D^{rs}_{rs} as a set of indeterminates which do not satisfy $D^{rs}_{rs} = D^{sr}_{sr}$ or $D^{rr}_{rr} = 0$. Hence, whenever we come across a minor D^{rs}_{rs} in the expanded right hand side of (3), we can conclude that $r \in A$ and $s \in B$. It turns out that all cancellation already takes place in this first step. In a second step, we collect terms involving $D^{rs}_{rs} = D^{sr}_{sr}$ for $r, s \in A \cap B$.

First of all, let us expand the product

$$\prod_{1\le i<m} \Bigl(D^{a_{\pi i}b_i}_{a_{\pi i}b_i} + D^{a_m b_m}_{a_m b_m} - D^{a_{\pi i}b_m}_{a_{\pi i}b_m} - D^{a_m b_i}_{a_m b_i}\Bigr) \tag{4}$$

for some $\pi \in S_{m-1}$. Then, for each $1 \leq i < m$, we heave four choices. We collect those indices i for which the first summand is chosen in a set M_1, collect those indices i for which the second summand is chosen in a set M_2, and so on. Then every term that we get after expansion of (4) can be written as

$$\Pi(M,\pi) = \prod_{i\in M_1} D^{a_{\pi i}b_i}_{a_{\pi i}b_i} \prod_{i\in M_2} D^{a_m b_m}_{a_m b_m} \prod_{i\in M_3} D^{a_{\pi i}b_m}_{a_{\pi i}b_m} \prod_{i\in M_4} D^{a_m b_i}_{a_m b_i}$$

for $M = (M_1, M_2, M_3, M_4)$. Therefore the product (4) is equal to

$$\sum_M (-1)^{|M_3|+|M_4|} \Pi(M,\pi),$$

where the sum is taken over all tuples $M = (M_1, M_2, M_3, M_4)$ with the property that $M_1 \uplus M_2 \uplus M_3 \uplus M_4 = \{1, \ldots, m-1\}$. We replace the product by this sum in (3) to obtain

$$D^A_B\, \tau^{m-2} = (-1)^{\Sigma A + \Sigma B} \left(-\tfrac{1}{2}\right)^{m-1} \sum_M (-1)^{|M_3|+|M_4|} \sum_{\pi\in S_{m-1}} \operatorname{sgn}\pi\ \Pi(M,\pi) \quad (5)$$

after changing the order of summation.

Lemma 4. *Let $M = (M_1, M_2, M_3, M_4)$ be a tuple of index sets as before. If $|M_2| + |M_4| \geq 2$, then*

$$\sum_{\pi\in S_{m-1}} \operatorname{sgn}\pi\ \Pi(M,\pi) = 0.$$

Proof. If $|M_2| + |M_4| \geq 2$, then there exist two distinct elements $k, l \in M_2 \cup M_4$. Write $\tau = (k, l)$ for the transposition of k and l. Note that $a_{\pi k}$ and $a_{\pi l}$ do not occur as indices of minors in $\Pi(M,\pi)$ for any $\pi \in S_{m-1}$, since the summand that is chosen from the kth factor of the product (4) is either $D^{a_m b_k}_{a_m b_k}$ or $D^{a_m b_m}_{a_m b_m}$ in this case; the same holds analogously for l. Therefore we may freely interchange them without changing the monomials: $\Pi(M,\pi) = \Pi(M,\pi\tau)$ for all $\pi \in S_{m-1}$. We decompose S_{m-1} into the disjoint sets A_{m-1} and $A_{m-1}\tau$ and obtain

$$\begin{aligned}\sum_{\pi\in S_{m-1}} \operatorname{sgn}\pi\ \Pi(M,\pi) &= \sum_{\pi\in A_{m-1}} \big(\operatorname{sgn}\pi\ \Pi(M,\pi) + \operatorname{sgn}\pi\tau\ \Pi(M,\pi\tau)\big) \\ &= \sum_{\pi\in A_{m-1}} \Pi(M,\pi)\big(\operatorname{sgn}\pi + \operatorname{sgn}\pi\tau\big) = 0. \qquad \square\end{aligned}$$

Performing all cancellations on the right hand side of (5), we obtain

$$D_B^A \tau^{m-2} = (-1)^{\Sigma A + \Sigma B} \left(\tfrac{1}{2}\right)^{m-1} \sum_K \pm \prod_{(a,b)\in K} D_{ab}^{ab}, \tag{6}$$

where the sum is taken over certain sets $K \subseteq A \times B$ of size $m-1$. By definition, every K in this sum covers a row index $a \in A$, $a \neq a_m$, at most once. The lemma above shows that a_m is also covered at most once by a set K involved in the sum. Since the rôles of rows and columns are interchangeable, the same must be true for all column indices in B: K covers every column index in B at most once. Therefore, the sum in Equation (6) is taken over all partial matchings $K \subseteq A \times B$ of size $m-1$. In other words, after cancellation the sum in Equation (5) runs over all $M = (M_1, M_2, M_3, M_4)$ which satisfy $M_1 \uplus M_2 \uplus M_3 \uplus M_4 = \{1, \ldots, m-1\}$ and $|M_2| + |M_3| < 2$ as well as $|M_2| + |M_4| < 2$.

Lemma 5. *We have*

$$D_B^A \tau^{m-2} = (-1)^{\Sigma A + \Sigma B} \left(-\tfrac{1}{2}\right)^{m-1} \sum_{\sigma \in S_m} \operatorname{sgn} \sigma \sum_{k=1}^{m} \prod_{\substack{i=1 \\ i \neq k}}^{m} D_{a_{\sigma i} b_i}^{a_{\sigma i} b_i}. \tag{7}$$

Proof. We claim that the right hand side of (7) is equal to the right hand side of (5), which will prove the statement. Given a pair (σ, k) with $\sigma \in S_m$ and $1 \leq k \leq m$ we associate a permutation $\pi \in S_{m-1}$ and a tuple $M = (M_1, M_2, M_3, M_4)$, so that

$$\operatorname{sgn} \sigma \prod_{\substack{i=1 \\ i \neq k}}^{m} D_{a_{\sigma i} b_i}^{a_{\sigma i} b_i} = (-1)^{|M_3| + |M_4|} \operatorname{sgn} \pi \, \Pi(M, \pi) \tag{8}$$

holds. First note that the indices $a_{\sigma k}$ and b_k do not occur on the left hand side of the equation above. The rough idea is that the left hand side was generated by choosing the second, third or fourth summand in the expansion of Equation (3) when $i = k$ and/or $\pi i = \sigma k$. To make this precise we have to distinguish several cases:

Case 1: $k = m$ and $\sigma m = m$. This corresponds to the case that the first summand $D_{a_{\pi i} b_i}^{a_{\pi i} b_i}$ is always chosen in the expansion. Accordingly, we set $\pi = \sigma$ regarding π as a permutation in S_{m-1} and set $M_2 = M_3 = M_4 = \varnothing$.

Case 2: $k = m$ and $\sigma m \neq m$. This amounts to the case that the fourth summand $D^{a_m b_i}_{a_m b_i}$ is chosen when $i = \sigma^{-1} m$ and the first one in all other cases. Hence we set $\pi = (\sigma m, m) \circ \sigma$ and $M_4 = \{\sigma^{-1} m\}$, $M_2 = M_3 = \varnothing$.

Case 3: $k \neq m$ and $\sigma k = m$. In this case the third summand $D^{a_{\pi i} b_m}_{a_{\pi i} b_m}$ is chosen when $i = k$ and the first one otherwise. Thus we set $\pi = \sigma \circ (k, m)$ and $M_3 = \{k\}$, $M_2 = M_4 = \varnothing$.

Case 4: $k \neq m$ and $\sigma k \neq m$ and $\sigma m = m$. This corresponds to the case that the second summand $D^{a_m b_m}_{a_m b_m}$ is chosen when $i = k$ and the first in all other cases. Therefore we set $\pi = \sigma$ and $M_2 = \{k\}$, $M_3 = M_4 = \varnothing$.

Case 5: $k \neq m$ and $\sigma k \neq m$ and $\sigma m \neq m$. In this final case, the third summand is chosen when $i = k$, the fourth summand when $i = \sigma^{-1} m$, and the first in all remaining cases. Consequently we set $\pi = (\sigma k, m) \circ \sigma \circ (k, m)$ and $M_3 = \{k\}$, $M_4 = \{\sigma^{-1} m\}$, $M_2 = \varnothing$.

In all cases M_1 is defined to be $\{1, \ldots, m-1\} \setminus (M_2 \cup M_3 \cup M_4)$. It is now easy to see that Equation (8) holds. Furthermore, the map $(\sigma, k) \mapsto (\pi, M)$ is a one-to-one correspondence between $S_m \times \{1, \ldots, m\}$ and S_{m-1} times the set of tuples $M = (M_1, M_2, M_3, M_4)$ satisfying $M_1 \uplus M_2 \uplus M_3 \uplus M_4 = \{1, \ldots, m-1\}$, $|M_2| + |M_3| < 2$, and $|M_2| + |M_4| < 2$. This proves the claim. $\square$

In a second step of simplifying the right hand side of Equation (3), we collect terms on the right hand side of (7). If $A \cap B \neq \varnothing$, then any minor D^{rs}_{rs} with $r, s \in A \cap B$ also occurs in the form D^{sr}_{sr}. Now we regard them as equal again and also use the convention that $D^{rr}_{rr} = 0$. Given $\sigma \in S_m$ and $1 \leq k \leq m$ consider the monomial

$$\prod_{\substack{i=1 \\ i \neq k}}^{m} D^{a_{\sigma i} b_i}_{a_{\sigma i} b_i} = \prod_{(a,b) \in K}^{m} D^{ab}_{ab}$$

where $K = \{(a_{\sigma i}, b_i) : 1 \leq i \leq m, i \neq k\}$. If K contains an element (r, r) for some $r \in A \cap B$, then the monomial above is 0, since $D^{rr}_{rr} = 0$. Otherwise, regarding the elements of K as unordered pairs, K is the edge (multi-)set of a graph in $\mathcal{G}(A, B)$. Therefore

$$D^A_B \, \tau^{m-2} = \sum_{\Lambda \in \mathcal{G}(A,B)} \alpha(\Lambda) \prod_{rs \in E\Lambda}^{m} D^{rs}_{rs}$$

for suitable coefficients $\alpha(\Lambda)$. Recall that all cyclic components of a graph in $\mathcal{G}(A,B)$ are contained in $A \cap B$.

Lemma 6. *Let $\Lambda \in \mathcal{G}(A,B)$, then*

$$\alpha(\Lambda) = \pm\left(\tfrac{1}{2}\right)^{m-1} \prod_{C \in \mathcal{C}\Lambda} \beta(C),$$

where $\mathcal{C}\Lambda$ is the set of all components of Λ and $\beta(C)$ is given as follows: $\beta(C) = 1$ if C is a single vertex, a 2-cycle, or a path of length $\ell \geq 0$ with a vertex in $A \triangle B$, whereas $\beta(C) = 2$ if C is a cycle of length $\ell \geq 3$, or a path of length $\ell \geq 1$ in $A \cap B$. Additionally, the sign of $\alpha(\Lambda)$ can be computed in terms of Λ.

Proof. We may assume that $a_i = b_i$ for $i = 1, \ldots, |A \cap B|$. Otherwise reorder rows and columns appropriately. Note that this yields a global factor ± 1. For $\sigma \in S_m$ define a directed graph X_σ as follows: the vertex set of X_σ is $A \cup B$ and the edges are $(a_{\sigma i}, b_i)$ for $1 \leq i \leq m$. Obviously, X_σ has constant out-degree 1 on A and constant in-degree 1 on B, and $\sigma \mapsto X_\sigma$ is one-to-one.

Let $\Lambda \in \mathcal{G}(A,B)$. There are unique indices $a \in A$ and $b \in B$ such that $\Lambda + ab$ has constant degree 1 on $A \triangle B$ and constant degree 2 on $A \cap B$. We determine the number of permutations $\sigma \in S_m$ with

$$\prod_{\substack{i=1\\ i \neq k}}^{m} D^{a_{\sigma i} b_i}_{a_{\sigma i} b_i} = \prod_{rs \in E\Lambda}^{m} D^{rs}_{rs} \tag{9}$$

and show that they all have the same sign. Assume that $\sigma \in S_m$ satisfies (9). Then X_σ is an orientation of $\Lambda + ab$. Since a path component in $\Lambda + ab$ has one end-vertex in $A \setminus B$ and the other in $B \setminus A$, there is only one allowed orientation of the component. Thus σ is uniquely determined by $\Lambda + ab$ on indices i, so that b_i is contained in a path component of $\Lambda + ab$. If C is a cyclic component of $\Lambda + ab$, then a cyclic orientation of C is a component of X_σ too. As a cyclic component of X_σ defines a cycle of σ of the same length, the cyclic structure and thus the sign of σ are determined by $\Lambda + ab$. The number of cyclic orientations of cyclic components in $\Lambda + ab$ explains the value of $\beta(C)$ (there are two possible orientations for a cycle unless it is a 2-cycle or a loop), with one exception: if C is a 2-cycle of $\Lambda + ab$ so that ab is an edge of C, then we have two choices for the edge ab, which yields a factor 2 in this case, although there is only one cyclic orientation. □

This finishes the proof of Theorem 2.

3. A special case

In this section we study the case $A = B$, since we can write down a very explicit formula for the coefficients $\alpha(\Lambda)$ in terms of components. The following result was conjectured by the authors, see [11].

Corollary 7. *The number D_A^A of rooted spanning forests with root set A of size $m \geq 2$ satisfies*

$$D_A^A\, \tau^{m-2} = \sum_{\Lambda \in \mathcal{G}(A,A)} \alpha(\Lambda) \prod_{rs \in E\Lambda} D_{rs}^{rs}.$$

The coefficient $\alpha(\Lambda)$ is given by

$$\alpha(\Lambda) = (-1)^{|\mathcal{C}\Lambda|-1} \left(\tfrac{1}{2}\right)^{m-1} \prod_{C \in \mathcal{C}\Lambda} \beta(C),$$

where $\mathcal{C}\Lambda$ is the set of all components of Λ and $\beta(C)$ is given in Lemma 6.

Proof. Except for the sign of $\alpha(\Lambda)$ the statement follows from Theorem 2 and Lemma 6. It remains to determine the sign. Let Λ be a graph in $\mathcal{G}(A, A)$. Note that Λ has exactly one path component; hence there are unique elements $a, b \in A$ such that $\Lambda + ab$ is 2-regular. If $\sigma \in S_m$ satisfies (9), then (as shown in the proof of Lemma 6) the cycle structure of σ is completely determined by the cyclic components of $\Lambda + ab$. Since Λ and $\Lambda + ab$ have the same number of components, we get $\operatorname{sgn} \sigma = (-1)^{m+|\mathcal{C}\Lambda|}$, which proves the statement. □

4. Electrical networks

Let G be a graph with loops and parallel edges and let $c : EG \to [0, \infty)$ define weights (conductances) on the edges. The Laplace matrix $L = L(G)$ is defined by its entries

$$L_{x,y} = -\sum_{\substack{e \in EG \\ e \text{ connects } x,y}} c(e) \qquad \text{and} \qquad L_{x,x} = -\sum_{\substack{z \in VG \\ z \neq x}} L_{x,z}$$

where x, y are vertices in VG, $x \neq y$. We say that two edge-weighted graphs (networks) G and H are *electrically equivalent* with respect to $\Theta \subseteq VG \cap VH$, if they cannot be distinguished by applying voltages to Θ and measuring the resulting currents on Θ. By Kirchhoff's current law this means that the rows corresponding to Θ of $L_G H_\Theta^{VG}$ and $L_H H_\Theta^{VH}$ are equal, where H_Θ^{VG} is the matrix associated to harmonic extension, see for instance [7, 12]. If $u, v \in VG$ are vertices in G and H is the complete graph with vertex set $\{u, v\}$, then there exists a conductance $c_{\text{eff}}(u, v)$ on the single edge of H, so that G and H equipped with $c_{\text{eff}}(u, v)$ are equivalent with respect to $\{u, v\}$. The number $c_{\text{eff}}(u, v)$ is called *effective conductance* and the number $r_{\text{eff}}(u, v) = c_{\text{eff}}(u, v)^{-1}$ is called *effective resistance* of u and v. By Kirchhoff's famous result connecting currents and spanning trees (see for example [6]), the effective resistance is given by

$$r_{\text{eff}}(u, v) = \tau^{-1} D_{uv}^{uv}, \tag{10}$$

where D_{uv}^{uv} counts rooted spanning forests with root set $\{u, v\}$ (so-called thickets, see [6]), and τ is the number of spanning trees in G. Given all effective resistances on a simple graph (no loops or parallel edges), one may ask whether it is possible to reconstruct the edge weights. Indeed, this is possible, as it is shown in [7, Section 2.1] using an inductive argument. As a consequence of Theorem 2 we can give an explicit solution to this inverse problem. Without loss of generality we may assume that our network forms a complete graph, since non-existent edges can be regarded as edges of weight 0.

Corollary 8. *Let G be a complete graph with three or more vertices and let $c : EG \to [0, \infty)$ define conductances, so that $\tau \neq 0$. If all effective resistances are known, then it is possible to reconstruct the original conductances on G; assume that $VG = \{1, \ldots, n\}$; then the edge weight $c(e)$ of the edge $e = kl$ ($k, l \in VG$, $k \neq l$) can be computed as follows: Set $A = VG \setminus \{k\}$ and $B = VG \setminus \{l\}$, define edge weights $\tilde{c}(e)$ by*

$$\tilde{c}(e) = \sum_{\Lambda \in \mathcal{G}(A,B)} \alpha(\Lambda) \prod_{rs \in E\Lambda} r_{\text{eff}}(r, s),$$

and write $\tilde{\tau}$ to denote the number of spanning trees in G with respect to the weights $\tilde{c}$. Then

$$c(e) = \tilde{\tau}^{-1/(n-2)}\, \tilde{c}(e).$$

Proof. Since $c(e) = D_B^A$ and $D_{rs}^{rs} = \tau r_{\text{eff}}(r, s)$, Theorem 2 implies

$$c(e) = \tau \sum_{\Lambda \in \mathcal{G}(A,B)} \alpha(\Lambda) \prod_{rs \in E\Lambda} r_{\text{eff}}(r, s) = \tau \tilde{c}(e).$$

By the matrix-tree theorem it follows that $\tau = \tilde{\tau}\,\tau^{n-1}$, which yields the statement. $\square$

Remark 1. We note that in the situation of the corollary above, the sign of $\alpha(\Lambda)$ for $\Lambda \in \mathcal{G}(A,B)$ is given by $(-1)^{|\mathcal{C}\Lambda|+\varepsilon}$, where

$$\varepsilon = \begin{cases} 1 & \text{if } k \text{ and } l \text{ are connected by a path in } \Lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2. If $\tau = 0$ in the situation of the previous corollary, then we restrict ourselves to components induced by edges of positive weight. Note that these components can be determined from the effective resistances as well.

In combinatorics unit conductances are of great interest because of the well-known relation between electrical networks and the number of spanning trees. Let G be a graph and c_G be unit conductances on the edges of G. We say that G has *resistance scaling factor* $\rho = \rho_\Theta$ with respect to $\Theta \subseteq VG$, if (G, c_G) is electrically equivalent to $(H, \rho^{-1} c_H)$, where H is a complete graph with vertex set $VH = \Theta$ and c_H are unit conductances on H. Note that the effective resistance of vertices u and v in a graph with unit conductances is exactly the resistance scaling factor with respect to $\{u,v\}$. The following result was proved by the authors in [8] under stronger assumptions, whereas the form here seems to be best possible.

Corollary 9. *Let G be a connected graph and let $\Theta \subseteq VG$ be a subset of θ distinguished vertices. Suppose that the restriction of the automorphism group of G to Θ is 2-homogeneous, i.e. for all $u,v,w,x \in \Theta$ with $u \neq v$ and $w \neq x$ there is an automorphism φ with $\varphi(\Theta) = \Theta$ and $\varphi(\{u,v\}) = \{w,x\}$. Then we have*

$$D_A^A = m\rho^{m-1}\theta^{1-m}\tau$$

for all sets $A \subseteq \Theta$ of cardinality m, where ρ is the resistance scaling factor of G with respect to Θ.

Proof. Let H be a complete graph with vertex set Θ and unit resistances. By assumption, we have $r_{\text{eff}}^G(r,s) = \rho\, r_{\text{eff}}^H(r,s)$ for $r,s \in \Theta$. Then, using the identity (10) and Theorem 2, we get

$$\frac{D_A^A(G)}{\tau(G)} = \sum_{\Lambda \in \mathcal{G}(A,B)} \alpha(\Lambda) \prod_{rs \in E\Lambda} r_{\text{eff}}^G(r,s) = \rho^{m-1} \cdot \frac{D_A^A(H)}{\tau(H)}.$$

It is well known that $\tau(H) = \theta^{\theta-2}$, and $D_A^A(H) = m\,\theta^{\theta-m-1}$. Putting everything together yields the statement. $\square$

5. Counting spanning trees

In this section, we show how our determinant identity can be applied to the enumeration of spanning trees. Specifically, we prove that if a subgraph of a graph G is replaced by an electrically equivalent graph, the number of spanning trees only changes by a factor that does not depend on G. This allows us to employ techniques from the theory of electrical networks—such as the Wye-Delta transform—to determine the number of spanning trees of a graph. This is particularly useful when one is working with graphs with a high degree of symmetry; several examples are given at the end of this section. Formally, the main result of this section reads as follows:

Theorem 10. *Suppose that X is a (possibly edge-weighted) graph that is decomposed into two graphs G and H in the following way: $EX = EG \uplus EH$ (i.e. the edge set of X is partitioned into the edge sets of G and H) and $VX = VG \cup VH$, where $VG \cap VH = M$. Furthermore, we assume that $\tau(X) \neq 0$ and $\tau(H) \neq 0$. Now suppose that H' is another graph with the property that $EG \cap EH' = \varnothing$ and $VG \cap VH' = M$, and suppose that H and H' are electrically equivalent with respect to M. Finally, set $X' = G \cup H'$. Then, the following formula holds:*

$$\frac{\tau(X')}{\tau(X)} = \frac{\tau(H')}{\tau(H)}.$$

Proof. Any spanning tree of X induces spanning forests on G and H; these spanning forests must have the additional property that any of their components contains a vertex of M. For a fixed spanning forest F on G with this property, let $\sigma_F(H)$ be the number of spanning forests F' on H with the property that $F \cup F'$ is a spanning tree on X. Then

$$\tau(X) = \sum_F \sigma_F(H),$$

where the sum is taken over all possible forests F. We will show that $\sigma_F(H)$ is proportional to $\tau(H)$, given the effective resistances of H with respect to the vertex set M that G and H have in common.

The connected components of F induce certain connections on M; If we contract the vertices that are connected by F to single vertices, we obtain a new graph H_F; this contraction may result in additional parallel edges. It is easy to see that spanning forests F' in H with the aforementioned property correspond exactly to spanning trees in the contracted graph H_F, and so we have $\sigma_F(H) = \tau(H_F)$. The effect of the contraction on the Laplace matrix is also quite simple: the rows respectively columns of contracted vertices are added to form a single row respectively column. Because of the multilinearity of the determinant, the determinant of the new Laplace matrix (i.e. the Laplace matrix of H_F), reduced by one row and one column (so that it gives exactly $\tau(H_F)$, can be written as the sum of minors of the original Laplace matrix of H, where only rows and columns corresponding to vertices in M are removed. By Theorem 2, each of these minors can be written as $\tau(H) \cdot P(\mathbf{r}_{\text{eff}}(H))$, where P is a polynomial and $\mathbf{r}_{\text{eff}}(H)$ is the vector of all effective resistances of H with respect to M. Hence, there exists a polynomial Σ_F such that

$$\sigma_F(H) = \tau(H_F) = \tau(H) \cdot \Sigma_F(\mathbf{r}_{\text{eff}}(H)).$$

Since H and H' were assumed to be electrically equivalent with respect to M, we obtain

$$\sigma_F(H') = \tau(H') \cdot \Sigma_F(\mathbf{r}_{\text{eff}}(H')) = \tau(H') \cdot \Sigma_F(\mathbf{r}_{\text{eff}}(H)) = \frac{\tau(H')}{\tau(H)} \cdot \sigma_F(H).$$

Summing over all possible forests F finally yields the desired result. □

In the following, we list the effect of some simple transformations on the number of spanning trees:

1. Parallel edges: If two parallel edges with conductances a and b are merged into a single edge with conductance $a+b$, the (weighted) number of spanning trees remains the same.
2. Serial edges: If two serial edges with conductances a and b are merged into a single edge with conductance $\frac{ab}{a+b}$, the weighted number of spanning trees changes as follows:
$$\tau(X') = \frac{1}{a+b} \cdot \tau(X).$$
3. Wye-Delta transform: if a star with conductances a, b, c (see Figure 1) is changed into an electrically equivalent triangle with conductances $x = \frac{bc}{a+b+c}$, $y = \frac{ac}{a+b+c}$ and $z = \frac{ab}{a+b+c}$, the weighted number of spanning trees changes as follows:
$$\tau(X') = \frac{1}{a+b+c} \cdot \tau(X).$$

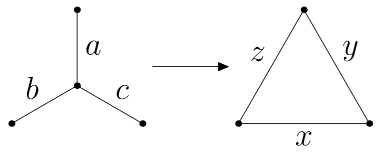

Figure 1: Wye-Delta transform.

4. Delta-Wye transform: if a triangle with conductances a, b, c (see Figure 2) is changed into an electrically equivalent star with conductances $x = \frac{ab+bc+ca}{a}$, $y = \frac{ab+bc+ca}{b}$ and $z = \frac{ab+bc+ca}{c}$, the weighted number of spanning trees changes as follows:

$$\tau(X') = \frac{(ab+bc+ca)^2}{abc} \cdot \tau(X).$$

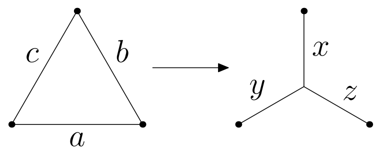

Figure 2: Delta-Wye transform.

Let us apply these simple transforms to determine the number of spanning trees of a small graph.

Example 1. Consider the graph that is shown in Figure 3; a few applications of the aforementioned transformations suffice to determine the correct number of spanning trees. It is clear that the weighted number of spanning trees

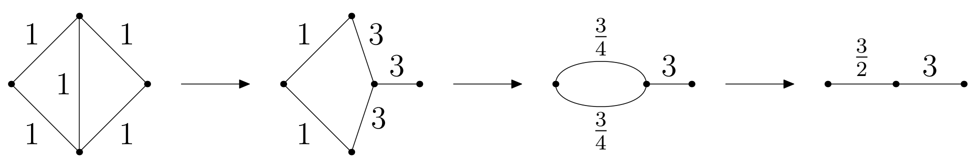

Figure 3: A simple example.

in the final graph is $\frac{3}{2} \cdot 3 = \frac{9}{2}$. The factors that we obtain from the three transformations are $\frac{1}{9}$, 4^2 and 1, which shows that the original graph has

$$\frac{1}{9} \cdot 4^2 \cdot 1 \cdot \frac{9}{2} = 8$$

spanning trees.

Admittedly, the exhibited method is unnecessarily complicated in this example, and ad-hoc reasoning would be much faster, but the technique of replacing parts of a graph by electrically equivalent graphs becomes powerful when one is working with symmetric graphs; to this end, we extend our list of operations a little further: a star $K_{1,n}$ is electrically equivalent to a complete graph K_n with conductances $\frac{1}{n}$, which yields the following:

5. if a star $K_{1,n}$ with conductances a is changed into an electrically equivalent complete graph K_n with conductances $\frac{a}{n}$, the weighted number of spanning trees changes as follows:

$$\tau(X') = \frac{(a/n)^{n-1}\tau(K_n)}{a^n} \cdot \tau(X) = \frac{1}{an} \cdot \tau(X).$$

The factor $(a/n)^{n-1}$ arises from the fact that every spanning tree of the complete graph K_n has exactly $n-1$ edges, whose associated conductances are all $\frac{a}{n}$ in this case. Note that this operation is essentially a generalization of the Wye-Delta transform (in the case that all conductances are the same). Of course there is also an analogous reverse operation. The well-known formula for the number of spanning trees in a complete bipartite graph follows immediately as an example:

Example 2. Consider the complete bipartite graph $K_{n,m}$ ($n \geq 2$); it can be seen as the union of m stars with n edges each. Replace each of these stars by an electrically equivalent complete graph with n vertices and conductances $\frac{1}{n}$. The resulting graph is a complete graph with n vertices and conductances $\frac{m}{n}$; now we obtain from Theorem 10 that $\tau(K_{n,m})$ is given by

$$\tau(K_{n,m}) = n^m \cdot \left(\frac{m}{n}\right)^{n-1} \tau(K_n) = m^{n-1}n^{m-n+1}n^{n-2} = m^{n-1}n^{m-1}.$$

It is actually even possible to deduce Cayley's formula for the number of spanning trees in a complete graph in this vein without circular reasoning:

Example 3. Consider the complete graph K_n ($n \geq 3$); replace the star that is formed by all edges going out from a certain vertex by a complete graph with conductances $\frac{1}{n-1}$. The resulting graph is a complete graph with conductances $1 + \frac{1}{n-1} = \frac{n}{n-1}$; hence its weighted number of spanning trees is $\left(\frac{n}{n-1}\right)^{n-2}\tau(K_{n-1})$. Now Theorem 10 yields

$$\tau(K_n) = \frac{1}{(n-1)^{-(n-2)}\tau(K_{n-1})} \cdot \left(\frac{n}{n-1}\right)^{n-2} \tau(K_{n-1}) = n^{n-2}.$$

Note that the precise value of $\tau(K_{n-1})$ was not actually used, since it cancels in our calculation.

In the following example, we show how the number of spanning trees of the Petersen graph can be determined by hand in three simple steps without having to compute a single determinant:

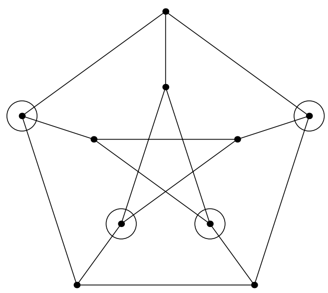

Figure 4: Petersen graph.

Example 4. In the Petersen graph (Figure 4), replace four stars by triangles (the centers are indicated in the figure) to obtain a complete graph with six vertices; all edges have conductance $\frac{1}{3}$ (indicated by dashed lines in Figure 5), except for three remaining edges whose conductances are still equal to 1. We

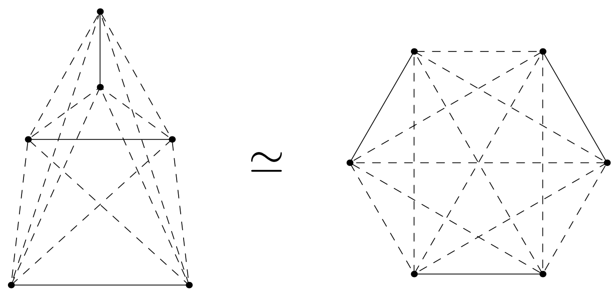

Figure 5: First step in the reduction of the Petersen graph.

regard each of them as two parallel edges with conductances $\frac{1}{3}$ and $\frac{2}{3}$ and replace the complete graph that is formed by all edges with conductance $\frac{1}{3}$ by a star with conductances equal to two. The resulting graph consists of three triangles joined at a common vertex (Figure 6); the last step is to determine the number of its spanning trees; it would be possible to reduce further, but it is easy enough to determine the number directly: a spanning

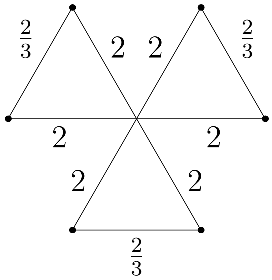

Figure 6: Second step in the reduction of the Petersen graph.

tree in this graph must consist of spanning trees in each of the three triangles, which shows that the weighted number of spanning trees is $(2^2 + 2 \cdot 2 \cdot \frac{2}{3})^3 = \frac{8000}{27}$. The factors that we obtain from the two transformations are 3^4 and $\frac{1}{12}$ respectively, which shows that the number of spanning trees of the Petersen graph is

$$3^4 \cdot \frac{1}{12} \cdot \frac{8000}{27} = 2000.$$

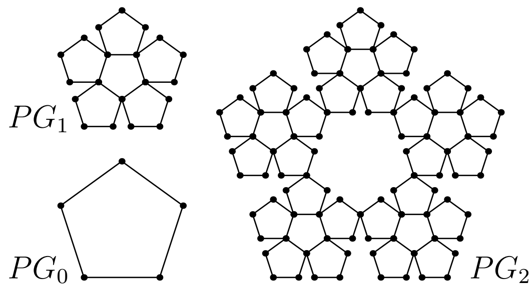

Figure 7: The Pentagasket: a pentagonal analogue of the Sierpiński gasket.

Example 5. Finally we would like to exhibit the type of problem where our transformation theorem proves to be most useful: self-similar graphs such as the *Pentagasket* that is shown in Figure 7: it has been shown [13] that the level-n Pentagasket PG_n is electrically equivalent to a pentagon (in graph-theoretic terms, a complete graph K_5) whose outer edges have conductance a_n and whose diagonal edges have conductance b_n; (a_n, b_n) are given as iterates of the following map:

$$R(a,b) = \left(\frac{5(8a+7b)(a^2+3ab+b^2)}{176a^2+228ab+71b^2}, \frac{5(4a+b)(a^2+3ab+b^2)}{176a^2+228ab+71b^2} \right). \tag{11}$$

The initial values are $(a_0, b_0) = (1, 0)$. Since PG_{n+1} is made up of five copies of PG_n, we may replace each of these parts by an electrically equivalent pentagon with conductances a_n and b_n. The weighted number of spanning trees of the resulting graph (denoted by Y_n) is easily determined explicitly by means of a computer (since it only consists of 20 vertices). The same applies to the weighted pentagon (denoted by Z_n), so that we obtain the following formula that is a direct consequence of Theorem 10:

$$\begin{aligned} \tau(PG_{n+1}) &= \frac{\tau(Y_n)}{\tau(Z_n)^5} \cdot \tau(PG_n)^5 \\ &= \frac{6250(2a_n + 3b_n)(a_n^2 + 3a_nb_n + b_n^2)^9}{(5(a_n^2 + 3a_nb_n + b_n^2)^2)^5} \cdot \tau(PG_n)^5 \qquad (12) \\ &= \frac{2(2a_n + 3b_n)}{a_n^2 + 3a_nb_n + b_n^2} \cdot \tau(PG_n)^5. \end{aligned}$$

Set $q_n = \frac{2(2a_n+3b_n)}{a_n^2+3a_nb_n+b_n^2}$; it is not difficult to check that q_n satisfies the recurrence

$$q_n = \tfrac{9}{5}q_{n-1} + \tfrac{4}{5}q_{n-2},$$

with initial values $q_0 = 4$ and $q_1 = \frac{56}{5}$. Thus

$$q_n = \left(2 + \tfrac{38}{\sqrt{161}}\right)\rho^n + \left(2 - \tfrac{38}{\sqrt{161}}\right)\bar{\rho}^n,$$

where

$$\rho = \tfrac{1}{10}\left(9 + \sqrt{161}\right) \qquad \text{and} \qquad \bar{\rho} = \tfrac{1}{10}\left(9 - \sqrt{161}\right)$$

are the roots of the characteristic equation. Now iteration yields

$$\tau(PG_n) = \tau(PG_0)^{5^n} \cdot \prod_{k=0}^{n-1} q_k^{5^{n-k-1}} = 5^{5^n} \cdot \prod_{k=0}^{n-1} q_k^{5^{n-k-1}}.$$

It is also possible to deduce the asymptotic behavior from this formula: take logarithms to obtain

$$\log\tau(PG_n) = 5^n \log 5 + 5^n \sum_{k=0}^{\infty} 5^{-k-1} \log q_k - \sum_{k=n}^{\infty} 5^{n-k-1} \log q_k.$$

The infinite sum converges, since

$$\log q_k = k\,\log\rho + c + O(\varepsilon^k),$$

where $c = \log\bigl(2 + \frac{38}{\sqrt{161}}\bigr)$ and $\varepsilon = |\bar{\rho}/\rho| < 1$. Furthermore,

$$\sum_{k=n}^{\infty} 5^{n-k-1} \log q_k = \tfrac{1}{4} n \log \rho + \tfrac{1}{16}(\log \rho + 4c) + O(\varepsilon^n).$$

Finally, we obtain

$$\begin{aligned}\tau(PG_n) &= \exp\bigl(-\tfrac{1}{16}(\log \rho + 4c) - \tfrac{1}{4} n \log \rho\bigr) \cdot C^{5^n} \left(1 + O(\varepsilon^n)\right) \\ &= A \cdot \rho^{-n/4} \cdot C^{5^n} \left(1 + O(\varepsilon^n)\right),\end{aligned}$$

where the numerical values of A and C are given by

$$A \doteq 0.637317153240 \qquad \text{and} \qquad B \doteq 7.514181930576.$$

Remark 3. The method that was shown in this example does not only apply to the specific example of the Pentagasket; it can be used to any sequence $X_0, X_1, \ldots$ of self-similar graphs that is defined in a similar way; we refer to [7, 14, 15] for precise definitions. Roughly speaking, we start with $X_0 = K_\theta$ and say that all θ vertices are "boundary" vertices. Now, given X_n and θ boundary vertices of X_n, we construct X_{n+1} as the union of s copies of X_n, where some boundary vertices are glued together by a prescribed rule. Additionally, θ boundary vertices of X_{n+1} are chosen according to a prescribed rule, too. The boundary vertices are ordered by the rule, so that we may speak about the first, second, etc. boundary vertex of X_n.

Given some conductances c_0 on X_0, the graphs $X_1, X_2, \ldots$ inherit weights in a natural way from X_0 (every edge in X_n is a copied version of a unique edge in X_0). Especially, we write $S(c_0)$ to denote the conductances on X_1 inherited from (X_0, c_0). On the other hand, given conductances c_1 on X_1 there are unique weights $\bar{c}_1$ on the complete graph $\bar{X}_1$ whose vertices are the boundary vertices of X_1, so that the networks (X_1, c_1) and $(\bar{X}_1, \bar{c}_1)$ are electrically equivalent with respect to the boundary vertices. $(\bar{X}_1, \bar{c}_1)$ is often called the trace of X_1 with respect to the boundary vertices. The so-called renormalization map R is the composition of copying conductances from X_0 to X_1, taking the trace from X_1 to $\bar{X}_1$, and identifying $\bar{X}_1$ with X_0 using the ordering of boundary vertices. Consequently, R maps conductances of X_0 into itself. If we write $T(c_1)$ for the conductances which emerge by taking the trace, we have $R = T \circ S$ up to identification. In the case of the Pentagasket the map R is given by Equation (11).

Let c_0 be some conductances on X_0 and denote by c_n the conductances on X_n inherited from X_0. Then it is easy to see that (X_n, c_n) is electrically equivalent to $(X_0, R^n(c_0))$ with respect to the boundary vertices, where R^n denotes the n-fold iterate of R. The method employed above can be generalized as follows: Fix some initial conductances c_0. The graph X_{n+1} is an amalgamation of s copies of X_n. If we replace each copy by the electrically equivalent network $(X_0, R^n(c_0))$, we get $(X_1, S(R^n(c_0)))$, where the conductances $S(R^n(c_0))$ on X_1 are inherited from $(X_0, R^n(c_0))$. Using Theorem 2 we obtain

$$\tau(X_{n+1}) = \frac{\tau(X_1, S(R^n(c_0)))}{\tau(X_0, R^n(c_0))^s} \cdot \tau(X_n)^s,$$

which is the general form of Equation (12). Therefore the counting problem is closely related to the dynamical behavior of the renormalization map R. Whenever there are a factor ρ and conductances c_∞ on X_0 solving the non-linear eigenvalue problem $c_\infty = \rho R(c_\infty)$, so that $\rho^n R^n(c_0)$ converges to c_∞, then

$$\tau(X_n) \sim A \cdot \rho^{-n/(s-1)} \cdot C^{s^n},$$

by the reasoning of the example above, where A, C are constants. The number ρ solving the non-linear eigenvalue problem is called *resistance scaling factor*, see [14]. The dynamical behavior of R was studied in [15] (see also the references therein). In the case where the sequence $X_0, X_1, \ldots$ satisfies a strong symmetry condition, a closed formula for the number of spanning trees was shown before in [8].

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