Analytic Combinatorics – Part V: Saddle point method

Stephan Wagner

Stellenbosch University

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For functions that are entire (e.g. $A(x) = e^{e^x - 1}$) or have essential singularities (e.g. $A(x) = e^{x/(1-x)}$), the saddle point method is often the method of choice.

It can be considered a complex version of Laplace’s method. The main steps are:

- In Cauchy’s integral formula

$$
\frac{1}{2\pi i} \oint_{C} \frac{A(x)}{x^{n+1}} \, dx,
$$

identify the so-called “saddle point” of the integrand $\frac{A(x)}{x^{n+1}}$.

- Choose a suitable contour $C$ (typically a circle) passing (at least approximately) through the saddle point.

- Split the integral into a “central” part and the “tails”.

The function $A(x) = e^{x/(1-x)}$ is the exponential generating function for partitions of \{1, 2, \ldots, n\} into ordered lists.

It has an essential singularity at 1, so singularity analysis does not apply.

Substitute $x = Re^{iu}$ (for $R$ to be determined), and rewrite the coefficient-extracting integral as

$$\frac{a_n}{n!} = [x^n]A(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left(\frac{Re^{iu}}{1 - Re^{iu}} - n \log R - i nu\right) du.$$
Examples

The radius $R$ needs to be chosen in such a way that the derivative

$$\frac{\partial}{\partial u} \left( \frac{Re^{iu}}{1 - Re^{iu}} - n \log R - i nu \right) \bigg|_{u=0}$$

vanishes, at least approximately. A suitable choice is $R = 1 - 1/\sqrt{n}$.

The integral is now split into a central part ($u$ close to 0), where it is approximated by a Gaussian integral, and the rest. This procedure yields

$$\frac{a_n}{n!} \sim \frac{1}{2\sqrt{\pi}} n^{-3/4} e^{2\sqrt{n}-1/2},$$

or

$$a_n \sim \frac{1}{\sqrt{2}} n^{n-1/4} e^{-n+2\sqrt{n}-1/2}.$$
The function \( B(x) = e^{ex} - 1 \) is the exponential generating function for the Bell numbers, which count set partitions (partitions of \( \{1, 2, \ldots, n\} \) into unordered lists):

\[
B(x) = e^{ex} - 1 = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.
\]

This function is entire (no singularities), so again singularity analysis cannot apply.

Setting \( x = Re^{iu} \), we obtain a saddle point equation that cannot be solved explicitly (or only by means of special functions, specifically the Lambert W-function): \( Re^R = n \).
However, if we let $R$ be this implicitly defined quantity, then the remaining steps of the saddle point method can be carried out again, and we obtain

$$\frac{B_n}{n!} \sim \frac{1}{\sqrt{2\pi n(1 + R)}} e^{nR - n \log n + n/R - 1}.$$

Since $R = \log n - \log \log n + O\left(\frac{\log \log n}{\log n}\right)$, this yields

$$\log B_n = n \log n - n \log \log n - n + O\left(\frac{n \log \log n}{\log n}\right).$$
Examples

An example where identifying the saddle point and evaluating the derivatives is more complicated is the generating function of integer partitions:

\[ P(x) = \prod_{j=1}^{\infty} (1 - x^j)^{-1}. \]

Here, the saddle point method can be used to show that the number \( p_n \) of partitions of \( n \) (ways to write \( n \) as an unordered sum of positive integers) satisfies

\[ p_n \sim \frac{e^{\pi \sqrt{2n/3}}}{4\sqrt{3n}}. \]

There are even more precise formulas, as well as generalisations to different types of partitions.
For more information, consult the “bible” of analytic combinatorics: