Existence and Region of Critical Probabilities in Bootstrap Percolation on Inhomogeneous Periodic Trees

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Abstract. Bootstrap percolation is a growth model inspired by cellular automata. At the initial time $t = 0$, the bootstrap percolation process starts from an initial random configuration of active vertices on a given graph, and proceeds deterministically so that a node becomes active at time $t = 1, 2, \dots$ if sufficiently many of its neighbors are already active at the previous time $t - 1$. In the most basic model, all vertices have the same initial probability of being active in the initial configuration. One of the main questions is to determine the percolation threshold (if it exists) with the property that all nodes in the given graph become active asymptotically almost surely (a.a.s.) for the initial probability above this threshold, while this is not the case below the threshold. In this work, we study a scenario where the nodes do not all receive the same probabilities, but to keep the problem tractable, we impose conditions on the shape of the graph and the initial probabilities. Specifically, we consider infinite periodic trees, in which the degrees and initial probabilities of nodes on a path from the root node are periodic, with a given periodicity. Instead of the simple percolation threshold, we now obtain an entire region of possible probabilities for which all nodes in the tree become a.a.s. active. We show (i) that the unit cube, as the support of the initial probabilities, can be partitioned into two regions, denoted by W_0 and $\overline{W}_0$, such that the tree becomes (does not become) a.a.s. fully active for any initial probability vector that belongs to $\overline{W}_0$ (resp. W_0); (ii) for every node in the tree, we provide the probability that the node becomes eventually active, for any initial probability vector that belongs to W_0; (iii) further, we specify the boundary of W_0 and show how it can be numerically computed.

1 Introduction

In classical percolation theory, nodes of a graph become active according to certain probabilities to form a static configuration. *Bootstrap percolation* is a variant inspired by cellular automata that proceeds dynamically afterwards: starting

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* Supported by the National Research Foundation of South Africa under grant number 96236.

from an initial configuration (determined in the same way as in classical percolation), the process proceeds in discrete time-steps, where a node is active at time t if it is or sufficiently many of its neighbors are already active at time $t-1$. It may thus happen that all nodes become eventually active, which poses the natural question for the existence of a critical threshold probability (assuming that all nodes have the same initial probability of being active) such that all nodes (do not) become a.a.s. active if the initial probability is greater (resp. smaller) than the threshold. There is a large body of work on bootstrap percolation on different graph models: refgular or irregular, discrete or random, homogeneous or inhomogeneous, as well as in isotropic or anisotropic environment [1–13, 15–25].

In this work, we study a scenario where the nodes do not all receive the same probabilities, but to keep the problem tractable, we impose conditions on the shape of the graph and the probabilities and activation thresholds. Specifically, we consider infinite periodic trees, in which the degrees of nodes on a path from the root node are periodic, and also impose a periodicity condition on the probabilities. The existence of a percolation threshold for *periodic trees* has been established in [14] in the case where the initial probability of being active and the activation threshold are the same for all nodes.

In this work the initial probability and the activation threshold are functions of the node itself, which is the main generalization of the model analyzed in [14]. As motivation, consider a dynamical process (e.g. advertisement, rumor, or viral spread). It is usually the case that particles in the system become initially active (e.g. obtain the initial piece of information or become infected) with different probabilities, as well as that the activation threshold differs among particles (e.g. the level required to convince a customer to buy a new product or for one to become infected depends on the individual itself).

Hence in this work we consider the following object. A periodic tree corresponding to a sequence $d_0, d_1, \ldots, d_{\ell-1}$ is an infinite tree with a root node such that every vertex at distance $i \bmod \ell$ from the root has degree $d_i + 1$. In addition to the degrees, we specify activation thresholds $\theta_0, \theta_1, \ldots, \theta_{\ell-1}$ $(2 \leqslant \theta_i \leqslant d_i - 1)$ for the bootstrap percolation. This means that a node at distance $i \bmod \ell$ from the root will become active: either at the initial random phase $t = 0$, or once θ_i of its neighbors are active at the previous time step. Finally, and this is the main difference to all prior work, we allow the initial probabilities to be periodic as well (rather than fixed throughout the tree): at time 0, a node at distance $i \bmod \ell$ from the root becomes active with probability p_i. Note that the periods of d_i, θ_i and p_i do not a priori have to be equal, but we can assume so without loss of generality, since we can otherwise replace ℓ by the least common multiple of the periods.

In this work, we study bootstrap percolation on inhomogeneous periodic trees with different initial probabilities. Instead of the simple percolation threshold, as shown in [14], we now obtain an entire region of possible probabilities for which all nodes in the tree become a.a.s. active. We show (i) that the unit cube, as the support of the initial probabilities $p_0, p_1, \ldots, p_{\ell-1}$, can be partitioned into two regions, denoted by W_0 and $\overline{W}_0$, such that: the tree becomes (does not become) a.a.s. fully active for any initial probability vector that belongs to $\overline{W}_0$ (resp. W_0); (ii) for every node in the tree, we provide the probability that the node becomes eventually active, for any initial probability vector that belongs to W_0; (iii) further, we specify the boundary of W_0 and how it can be numerically computed. In fact we derive the explicit system of equations from which one can numerically compute the boundary of W_0.

2 Definitions and Preliminaries

Formally, bootstrap percolation is a cellular automaton defined on an underlying graph G with state space $\{0,1\}^{V(G)}$ whose initial configuration is chosen by a Bernoulli product measure. In other words, every node is in one of two different states 0 or 1, *inactive* or *active* respectively, and a node v is active with some initial probability p_v, independently of other nodes, within the initial configuration at $t = 0$. In this work the initial probability p_v is a function of the node itself. After drawing an initial configuration, a discrete time deterministic process updates the configuration according to a local rule: an inactive node v becomes active at time $t + 1$ if the number of its active neighbors at t is greater than or equal to some specified *activation parameter* θ_v, which is a function of the node v as well. Once an inactive node becomes active it remains active. A configuration that does not change at the next time step is a *stable* configuration. A configuration is *fully active* if all its nodes are active.

In this work we study the bootstrap percolation process on periodic trees defined as follows.

Definition 1. *(Periodic tree) Let $\ell, d_0, d_1, \ldots, d_{\ell-1} \in \mathbb{N}$. An ℓ-periodic tree $\mathbb{T}_{d_0,d_1,\ldots,d_{\ell-1}}$ is defined as follows. Consider a node v_0, called root. The nodes at distance $i \bmod \ell$ from v_0 have degree $d_i + 1$ for $i \in \mathbb{N}_0$. In particular, the root has degree $d_0 + 1$.*

An infinite d-regular tree is a special case: a 1-periodic tree where each node has degree $d + 1$.

We also need to define the following oriented tree.

Definition 2. *(Oriented periodic tree) Let $\ell, d_0, d_1, \ldots, d_{\ell-1} \in \mathbb{N}$. An oriented ℓ-periodic tree $\vec{\mathbb{T}}_{d_0,d_1,\ldots,d_{\ell-1}}$ is defined as follows. Consider a node v_0, called root. The nodes at distance $i \bmod \ell$ from v_0 have in-degree d_i and out-degree 1 for $i \in \mathbb{N}_0$, except for the root, which has out-degree 0.*

We note that an oriented ℓ-periodic tree is a periodic tree with all edges oriented towards the root, the exception being the root degree.

Definition 3. *For $\vec{p} = (p_0, p_1, \ldots, p_{\ell-1})$ define*

$$\vec{p}_{-i} := (p_0, p_1, \ldots, p_{i-1}, p_{i+1} \ldots, p_{\ell-1}) \tag{1}$$

to be a vector obtained by erasing the i-th coordinate from $\vec{p}$. Conversely, $\vec{p}$ is obtained by inserting p_i at the i-th coordinate in $\vec{p}_{-i}$, which we write as

$$\vec{p} \equiv (\vec{p}_{-i}|p_i) \ . \tag{2}$$

2.1 Notation.

Throughout this work, given $d_0, d_1, \ldots, d_{\ell-1}$, we will usually use $\mathbb{T}$ (resp. $\vec{\mathbb{T}}$) as a shorthand for $\mathbb{T}_{d_0,d_1,\ldots,d_{\ell-1}}$ (resp. $\vec{\mathbb{T}}_{d_0,d_1,\ldots,d_{\ell-1}}$).

Given a tree $\mathbb{T} = \mathbb{T}_{d_0,d_1,\ldots,d_{\ell-1}}$, partition the node set $V(\mathbb{T})$ into ℓ *classes* of nodes V_i, where V_i contains the nodes of $\mathbb{T}$ at distance $i \bmod \ell$ from the root with degree $d_i + 1$, activation threshold θ_i, and initial probability p_i, for $i \in \{0, 1, \ldots, \ell - 1\}$.

In the following, all indices will be considered modulo ℓ, e.g. $x_\ell = x_0$.

3 Bootstrap percolation

This section is devoted to bootstrap percolation on periodic inhomogeneous trees. The main result is given by Theorem 1 and Theorem 2, showing the regions of initial probabilities for which the tree $\vec{\mathbb{T}}$, respectively $\mathbb{T}$, become a.a.s. fully active. Finally, we show that these two regions are identical.

Functions of the form

$$\varphi_{d,p,\theta}(x) := p + (1-p) \sum_{k=\theta}^{d} \binom{d}{k} x^k (1-x)^{d-k}, \tag{3}$$

where $p \in [0,1]$, will play a key role, as they capture one time step in bootstrap percolation. Intuitively, the first term stands for the probability of a node to be initially active, the sum in the second term for the probability of becoming active because at least θ of its neighbors are. The following result appeared in different forms in [11, 17] and the proof is given in [14, Lemma 2.1].

Lemma 1. *Given $d, \theta \in \mathbb{N}$ such that $2 \leqslant \theta \leqslant d - 1$ and $p \in [0,1]$, there exists $p_c \in (0,1)$ such that for any $p > p_c$ we have $\varphi_{d,p,\theta}(x) > x$ for every $x \in (0,1)$, and 1 is the only solution of $\varphi_{d,p,\theta}(x) = x$ in $[0,1]$. For $p < p_c$, there are two solutions in $[0,1]$ other than 1, and for $p = p_c$ there is one other solution (of multiplicity 2).*

3.1 Bootstrap percolation on an oriented tree $\vec{\mathbb{T}}$.

Following the methodology of [17], we first show the existence of a threshold region for oriented trees, i.e. a region W_0 of probabilities p_i for which not all nodes become active asymptotically almost surely (a.a.s), while for a choice of probabilities in the complement $\overline{W}_0$, all nodes become active a.a.s. Later we show that the regions for oriented and unoriented periodic trees with the same parameters are actually the same.

The dynamics must be defined in a slightly different way for bootstrap percolation, though (in other words, we need to define an *oriented* version of bootstrap percolation): a node in class i becomes newly active if θ_i of its in-neighbors (neighbors for which the orientation of the associated edge is towards the node) are active in the previous step. For this slightly modified version, we obtain the following result:

Theorem 1. *Consider bootstrap percolation on an oriented tree* $\vec{\mathbb{T}}$ *with parameters* $(d_i, \theta_i, p_i)_{i=0}^{\ell-1}$*, where* $2 \leqslant \theta_i \leqslant d_i - 1$*. Let* W_0 *be the set of probability vectors* $\vec{p} \in [0,1]^\ell$ *such that there exists a solution in* $[0,1]^\ell$ *of the system*

$$x_i = p_i + (1-p_i) \sum_{k=\theta_i}^{d_i} \binom{d_i}{k} x_{i+1}^k (1-x_{i+1})^{d_i-k} \tag{4}$$

that is strictly less than $\vec{1}$*, i.e.* $x_i < 1$ *for every* $i \in \{0, 1, \ldots, \ell-1\}$*. Then (i) for every* $\vec{p} \in W_0$*,* $\vec{\mathbb{T}}$ *does not become fully active a.a.s.; (ii) for every* $\vec{p} \in \overline{W}_0 := [0,1]^\ell \setminus W_0$*,* $\vec{\mathbb{T}}$ *becomes fully active a.a.s. Moreover, there exist constants* $\delta, \sigma \in (0,1)$ *such that* $W_0 \supset [0,\delta]^\ell$ *and* $W_0 \subset [0,\sigma]^\ell$*.*

The dynamics of the bootstrap percolation process on $\vec{\mathbb{T}}$ are captured by knowing the states of every node $v \in V_i$, in every class V_i, at every time $t \in \mathbb{N}_0$. These states are denoted by $\vec{\zeta}_{i,t}(v) \in \{0,1\}$.

It is intuitive that the higher p_i, the higher the probability that a node in class i becomes eventually active. Also, if all p_i are equal to 0, the system is already in a state of equilibrium, where the state of every node in the tree is 0 (inactive). On the other hand, if all p_i are equal to 1, the system is in yet another equilibrium, where the state of every node in the tree is 1 (active).

3.2 Proof of Theorem 1.

The initial steps in the proof are analogous to those in [17] and [14]. However, the main difference is that we consider different degrees d_i, different activation thresholds θ_i and most importantly different initial probabilities p_i.

For every class V_i, choose any node $v \in V_i$. Conditioning upon whether this node v was active at time 0 or not (i.e., $\vec{\zeta}_{i,0}(v) = 0$ or $\vec{\zeta}_{i,0}(v) = 1$), the probability that the node v is active at time t is given by

$$\mathbb{P}\left(\vec{\zeta}_{i,t}(v) = 1\right) = \mathbb{P}\left(\vec{\zeta}_{i,0}(v) = 1\right) + \mathbb{P}\left(\vec{\zeta}_{i,0}(v) = 0\right) \mathbb{P}\left(\sum_{u \rightsquigarrow v} \vec{\zeta}_{i+1,t-1}(u) \geqslant \theta_i\right),$$

where the symbol "$\rightsquigarrow$" indicates that u is a neighbor of v in the oriented tree $\vec{\mathbb{T}}$ and the edge orientation is from u to v.

Given symmetry and the dynamical rules of the bootstrap percolation process, the $\vec{\zeta}_{i+1,t-1}(u)$ in the equation above are independent Bernoulli random variables with the same distribution; moreover, they are independent of $\vec{\zeta}_{i,0}(v)$. Introducing $\vec{z}_{i,t} := \mathbb{P}\left(\vec{\zeta}_{i,t}(v) = 1\right)$, we obtain the following system of recurrence equations:

$$\vec{z}_{i,t} = p_i + (1-p_i) \sum_{k=\theta_i}^{d_i} \binom{d_i}{k} \vec{z}_{i+1,t-1}^{\,k} \left(1 - \vec{z}_{i+1,t-1}\right)^{d_i-k} , \tag{5}$$

for $i = 0, 1, \ldots, \ell - 1$.

In order to simplify the notation, for given parameters $(d_i, \theta_i, p_i)_{i=0}^{\ell-1}$ that characterize a tree $\vec{\mathbb{T}}$ (as well as $\mathbb{T}$), we define the auxiliary functions $\phi_i(x; p_i) := \varphi_{d_i, p_i, \theta_i}(x)$ on $[0,1]$, i.e.

$$\phi_i(x; p_i) := p_i + (1-p_i) \sum_{k=\theta_i}^{d_i} \binom{d_i}{k} x^k (1-x)^{d_i-k} , \tag{6}$$

and the binomial tail

$$B_i(x) := \sum_{k=\theta_i}^{d_i} \binom{d_i}{k} x^k (1-x)^{d_i-k} . \tag{7}$$

Now, the recurrence system (5) can be rewritten as

$$\vec{z}_{i,t} = \phi_i(\vec{z}_{i+1,t-1}; p_i) , \tag{8}$$

for $i = 0, 1, \ldots, \ell-1$, and all $\vec{z}_{i,t}$ belong to $[0,1]$ since ϕ_i maps $[0,1]$ to $[0,1]$.

Claim. For every i, $\vec{z}_{i,t}$ is non-decreasing in t.

Proof. For $t = 0$, $\vec{z}_{i,0} = p_i$. From (5) $\vec{z}_{i,1} \geqslant p_i$, thus the claim holds for $t = 0$. Assume that for some t and every i, $\vec{z}_{i,t} \geqslant \vec{z}_{i,t-1}$. $B_i(x)$ is increasing in x. Thus

$$\vec{z}_{i,t+1} = \phi_i(\vec{z}_{i+1,t}; p_i) = p_i + (1-p_i) B_i(\vec{z}_{i+1,t}) \geqslant p_i + (1-p_i) B_i(\vec{z}_{i+1,t-1}) = \vec{z}_{i,t},$$

and the statement follows by mathematical induction.

So the $\vec{z}_{i,t}$ are non-decreasing in t and belong to $[0,1]$, thus by the monotone convergence theorem the limits $\vec{z}_{i,\infty} := \lim_{t\to\infty} \vec{z}_{i,t}$ exist, and they lie in $[0,1]$. By (8),

$$\vec{z}_{i,\infty} = \phi_i(\vec{z}_{i+1,\infty}; p_i) , \tag{9}$$

for all $i \in \{0, 1, \ldots \ell-1\}$. At this moment, we introduce the vector of the limiting values for $t \to \infty$:

$$\vec{z}_\infty := (\vec{z}_{0,\infty}, \vec{z}_{1,\infty}, \ldots, \vec{z}_{\ell-1,\infty}) \tag{10}$$

as well as the original ones at time $t = 0$:

$$\vec{p} := (p_0, p_1, \ldots, p_{\ell-1}) = (\vec{z}_{0,0}, \vec{z}_{1,0}, \ldots, \vec{z}_{\ell-1,0}) . \tag{11}$$

Applying (9) ℓ times, for every i, we obtain the equations of one variable

$$\vec{z}_{i,\infty} = \phi_i\left(\phi_{i+1}\left(\cdots\left(\phi_{i-1}\left(\vec{z}_{i,\infty}; p_{i-1}\right)\cdots\right); p_{i+1}\right); p_i\right) = F_i(\vec{z}_{i,\infty}) , \tag{12}$$

where we define

$$F_i := \phi_i \circ \phi_{i+1} \circ \cdots \circ \phi_{\ell-1} \circ \phi_0 \circ \cdots \circ \phi_{i-1} . \tag{13}$$

Notice that by (9) and (5), $\vec{z}_\infty \neq \vec{0}$ if and only if $\vec{p} \neq 0$. Next, we show that there exists a non-empty hypercube $[0,\delta]^\ell$ ($\delta > 0$) such that for all $\vec{p} \in [0,\delta]^\ell$, the limiting vector satisfies $\vec{0} < \vec{z}_\infty < \vec{1}$. In order to do so, define $d = \max_{i=0}^{\ell-1} d_i$ and $\theta = \min_{i=0}^{\ell-1} \theta_i \geqslant 2$ and introduce the function $\phi(x;p) : [0,1] \to [0,1]$ given by

$$\phi(x;p) := p + (1-p) \sum_{k=\theta}^{d} \binom{d}{k} x^k (1-x)^{d-k} \,. \tag{14}$$

In view of Lemma 1, there exists $p_c \in (0,1)$ such that 1 is the only solution in $[0,1]$ of $\phi(x;p) = x$ for all $p > p_c$. On the other hand there exist two solutions in $(0,1)$ for $p < p_c$, and one if $p = p_c$. From stochastic dominance it follows that

$$\mathbb{P}\left(\mathrm{Bin}\left(d_i, p_i\right) \geqslant \theta_i\right) \leqslant \mathbb{P}\left(\mathrm{Bin}\left(d, p_i\right) \geqslant \theta\right) \,, \tag{15}$$

so

$$\phi_i(x; p_i) \leqslant \phi(x; p_i) \tag{16}$$

for all x. Choose $0 < \delta < p_c$ and consider the following mapping with $\vec{Z}_0 = \delta$:

$$\vec{Z}_t = \phi(\vec{Z}_{t-1}; \delta). \tag{17}$$

The limit $\vec{Z}_\infty := \lim_{t\to\infty} \vec{Z}_t$ exists and $\vec{Z}_\infty < 1$ by the choice of δ, cf. [14]. Now choosing all $p_i \leqslant \delta$, it inductively follows from (17) and (16) that

$$\vec{z}_{i,t} = \phi_i(\vec{z}_{i+1,t-1}; p_i) \leqslant \phi(\vec{z}_{i+1,t-1}; \delta) \leqslant \phi(\vec{Z}_{t-1}; \delta) = \vec{Z}_t \,. \tag{18}$$

Hence $\vec{z}_{i,t} \leqslant \vec{Z}_t$ for every i and t, and $\vec{z}_{i,\infty} \leqslant \vec{Z}_\infty < 1$ for all i. This concludes the first part of the proof and shows that $[0,\delta]^\ell \subset W_0$.

By definition, for every $\vec{p} \notin W_0$ it follows that $\vec{z}_{i,\infty} = 1$ for some i, hence $\vec{z}_\infty = \vec{1}$, i.e. $\vec{\mathbb{T}}$ a.a.s. fully percolates for all $\vec{p} \in \overline{W}_0$. This proves statement (ii).

Finally, we want to show that W_0 is contained in some hypercube of volume σ^ℓ, where $\sigma > 0$. In order to do so, for every $\vec{p}_{-i} \in [0,1]^{\ell-1}$, define the critical value $h_c(\vec{p}_{-i})$ as the infimum of the probability p_i necessary such that $\vec{\mathbb{T}}$ fully percolates a.a.s.:

$$h_c(\vec{p}_{-i}) = \inf \Big\{ s : \vec{\mathbb{T}} \text{ a.a.s. fully percolates for probabilities } \vec{p} = (\vec{p}_{-i}|s) \Big\} \,. \tag{19}$$

Taking $p_i = 1$ will yield $\vec{z}_\infty = \vec{1}$, so the critical value $h_c(\vec{p}_{-i})$ is well defined. Next we want to show that h_c is not trivially identical to 1 on the entire domain $[0,1]^{\ell-1}$.

Lemma 2. *There exists a constant $\sigma \in (0,1)$ such that for every vector of initial probabilities $\vec{p} \in [0,1]^\ell$ and every coordinate $i \in \{0,1,\ldots,\ell-1\}$, the threshold function satisfies $h_c(\vec{p}_{-i}) \leqslant \sigma$.*

Proof. We have $\varphi_{d,p,\theta}(x) \geqslant \varphi_{d,p,d-1}(x) \geqslant \varphi_{d,0,d-1}(x)$ for every $x \in [0,1]$, so for all $i \in \{0,1,\ldots,\ell-1\}$,

$$\varphi_{d_i,p_i,\theta_i}(x) \geqslant \varphi_{d_i,p_i,d_i-1}(x) \geqslant \varphi_{d_i,0,d_i-1}(x) = d_i x^{d_i-1} - (d_i-1)x^{d_i} \,. \tag{20}$$

It is easy to show that $d_i x^{d_i-1} - (d_i-1)x^{d_i} = x$ has always one real solution in $(0,1)$, call it s_i, and that $d_i x^{d_i-1} - (d_i-1)x^{d_i} > x$ for $s_i < x < 1$. Without loss of generality, let s_0 be the maximum among all s_i. Choose any σ such that $s_0 < \sigma < 1$, and consider the recurrence system given by:

$$\begin{aligned} \vec{u}_{0,t} &= \varphi_{d_0,\sigma,d_0-1}(\vec{u}_{1,t-1})\,, \\ \vec{u}_{i,t} &= \varphi_{d_i,0,d_i-1}(\vec{u}_{i+1,t-1})\,, \end{aligned}$$

$i = 1, 2, \ldots, \ell-1$, with initial value $\vec{u}_0 = (\vec{u}_{0,0}, \vec{u}_{1,0}, \ldots, \vec{u}_{\ell-1,0}) = (\sigma, 0, \ldots, 0)$. The limit $\vec{u}_\infty := \lim_{t\to\infty}(\vec{u}_{0,t}, \vec{u}_{1,t}, \ldots, \vec{u}_{\ell-1,t})$ exists by the monotone convergence theorem. Moreover, $\vec{u}_{i,\infty} \geqslant \sigma$ for all i by the choice of σ, which in turn implies $\vec{u}_{i,\infty} = 1$ for all i (using the aforementioned fact that $\varphi_{d_i,p,d_i-1}(x) \geqslant d_i x^{d_i-1} - (d_i-1)x^{d_i} > x$ for $s_i < x < 1$). For every initial vector $\vec{p}$ for which $p_0 \geqslant \sigma$, it follows from (20) that $\vec{z}_{i,t} \geqslant \vec{u}_{i,t}$ for all i, t, hence $\vec{z}_\infty = \vec{1}$. Thus, for every $\vec{p}$, we have $h_c(\vec{p}_{-0}) \leqslant \sigma < 1$. In the same way, it follows that $h_c(\vec{p}_{-i}) \leqslant \sigma < 1$ for all i.

If $\vec{p} \notin [0,\sigma]^\ell$, then $p_i > \sigma \geqslant h_b(\vec{p}_{-i})$ for at least one i by Lemma 2. Hence $\vec{\mathbb{T}}$ a.a.s. fully percolates by definition of h_c. This means that $W_0 \subset [0,\sigma]^\ell$, which concludes the proof of Theorem 1.

3.3 Region of full percolation.

In the following lemma we provide better bounds on p_i for full percolation.

Lemma 3. *$\vec{\mathbb{T}}$ a.a.s. fully percolates for any initial vector of probabilities $\vec{p}$ such that for all $i \in \{0, 1, \ldots, \ell-1\}$, $p_i \in (1 - 1/\beta_i, 1]$, where*

$$\beta_i := d_i \binom{d_i-1}{\theta_i-1} \left(\frac{\theta_i-1}{d_i-1}\right)^{\theta_i-1} \left(\frac{d_i-\theta_i}{d_i-1}\right)^{d_i-\theta_i} . \tag{21}$$

We remark that $\beta_i \geqslant 1$, as will be shown below.

Proof. Consider again the function $F_i(z)$, given by (13). The first derivative of $F_i(z)$ is

$$F_i'(z) = \prod_{j=i}^{i-1 \bmod \ell} \phi_j'\left(\phi_{j+1}(\ldots\phi_{i-1}(z;p_{i-1})\ldots p_{j+1});p_j\right) . \tag{22}$$

For every $\phi_i(x;p_i)$, the first derivative is given by:

$$\phi_i'(x;p_i) = (1-p_i) d_i \binom{d_i-1}{\theta_i-1} x^{\theta_i-1}(1-x)^{d_i-\theta_i}, \tag{23}$$

and by differentiating again one finds that the maximum of $\phi_i'(x;p_i)$ is attained at $(\theta_i-1)/(d_i-1)$:

$$\max_{x\in[0,1]} \phi_i'(x;p_i) = \phi_i'\left(\frac{\theta_i-1}{d_i-1};p_i\right) = (1-p_i)\beta_i \tag{24}$$

by the definition of β_i. Note that β_i is the maximum of $B'_i(x)$, where B_i is given by (7). Since $B_i(0) = 0$ and $B_i(1) = 1$, it follows from the mean value theorem that $\beta_i \geqslant 1$.

For $p_i \in (0, 1)$, the first derivative given in (23) is strictly greater than 0, i.e. $\phi'_i(x; p_i) > 0$. For $p_i > 1 - 1/\beta_i$, the maximum of the first derivative is strictly less than 1, i.e. $\max_{0 \leqslant x \leqslant 1} \phi'_i(x; p_i) < 1$, see (24). Hence, for any vector of probabilities that satisfies $p_i > 1 - 1/\beta_i$ for all i, the convolution given in (22) yields

$$F'_i(z) - 1 < 0\,, \tag{25}$$

for all $i \in \{0, 1, \ldots, \ell - 1\}$. Thus the first derivative of the equation $F_i(z) - z$ is strictly negative on $[0, 1]$. Moreover $F(0) > 0$ and $F(1) - 1 = 0$, hence $z = 1$ is the only solution of $F_i(z) = z$ on $[0, 1]$. This implies that $\vec{z}_\infty = \vec{1}$ for any $\vec{p}$ that satisfies the condition of the lemma. Also note that if at least one p_i is 1, i.e. $\vec{z}_{i,0} = 1$, then $\vec{z}_{i,\infty} = 1$ for all i in view of (5), which concludes the proof.

3.4 Trajectory of $\vec{z}_t$.

In this section we analyze the trajectory of $\vec{z}_t$ over time $t = 0, 1, \ldots$. More precisely, we show necessary and sufficient conditions on $\vec{z}_t$ such that the initial vector $\vec{p}$ lies in W_0.

To start, consider again a function of the form $\phi(x; s)$ defined by (cf. (3))

$$\phi(x; s) := s + (1 - s) \sum_{k=\theta}^{d} \binom{d}{k} x^k (1 - x)^{d-k} \tag{26}$$

for certain parameters d and θ. Let $L(s) \leqslant R(s)$ be the real solutions of $\phi(x; s) = x$ in $(0, 1)$ if such solutions exist. We know that there exists some critical $s_c \in (0, 1)$, such that: (i) if $s < s_c$, there are two real solutions $L(s) < R(s)$ in $(0, 1)$; (ii) if $s = s_c$, there is one solution $L(s) = R(s)$ in $(0, 1)$; (iii) if $s > s_c$ there are no real solutions in $(0, 1)$, see Lemma 1. It is easy to show the following.

Lemma 4. *The limit of the sequence defined by the iteration $x_{t+1} := \phi(x_t; p)$ for $t = 0, 1, \ldots$ satisfies*

$$\lim_{t \to \infty} x_t = \begin{cases} L(p), \ x_0 \in [0, R(p)) \ \text{and} \ p \leqslant p_c, \\ R(p), \ x_0 = R(p) \ \text{and} \ p \leqslant p_c, \\ 1, \qquad \text{otherwise.} \end{cases}$$

We will write L_i and R_i for the functions of Lemma 4 associated with ϕ_i.

Lemma 5. *We have $\vec{z}_\infty < \vec{1}$ if and only if $\vec{z}_{i,t} \leqslant R_{i-1}(p_{i-1})$ for every i and every t.*

Proof. First, $\vec{z}_{i,t} = \phi_i(\vec{z}_{i+1,t-1}; p_i)$. Iterating this equation ℓ times it follows that

$$\vec{z}_{i,t+\ell} = \phi_i \left(\phi_{i+1} \left(\cdots \left(\phi_{i-1} \left(\vec{z}_{i,t}; p_{i-1} \right) \cdots \right); p_{i+1} \right); p_i \right)\,. \tag{27}$$

Assume that there exists some i such that $\vec{z}_{i,t} > R_{i-1}(p_{i-1})$. Let $b_0 := \vec{z}_{i,t}$ and define $b_k := \phi_{i-1}\left(b_{k-1}; p_{i-1}\right)$ for $k \geqslant 1$. The composition

$$\phi_i \circ \phi_{i+1} \circ \cdots \phi_{\ell-1} \circ \phi_0 \circ \cdots \circ \phi_{i-2}$$

is increasing, as the convolution of increasing functions, hence from (27) we obtain $\vec{z}_{i,t+\ell\cdot k} \geqslant b_k$. From Lemma 4 it follows that $\lim_{k\to\infty} b_k = 1$, so $\vec{z}_{i,\infty} = 1$ and consequently $\vec{z}_{j,\infty} = 1$ for all j. Conversely, if $\vec{z}_{i,t} \leqslant R_{i-1}(p_{i-1})$ for every i and every t, then $\vec{z}_{i,\infty} \leqslant R_{i-1}(p_{i-1}) < 1$ for all i and thus $\vec{z}_\infty < \vec{1}$.

Lemma 6. *Let a be the index for which $R_j(p_j)$ is maximal, and set $R_{\max} := R_a(p_a)$. We have $\vec{z}_\infty < \vec{1}$ if and only if $\vec{z}_{i,t} \leqslant R_{\max}$ for every i and t.*

Proof. First, let us recall the following two facts: (1) $\vec{z}_{a,\infty} < 1$ if and only if $\vec{z}_\infty < \vec{1}$; (2) $\vec{z}_{a,\infty} = 1$ if and only if $\vec{z}_\infty = \vec{1}$.

To prove sufficiency, assume that there exist i and t such that $\vec{z}_{i,t} > R_{\max}$. Then by Lemma 4 it follows that $\vec{z}_{i,\infty} = 1$ for every i, thus $\vec{z}_\infty = \vec{1}$.

To prove necessity, let $\vec{z}_{i,t} \leqslant R_{\max}$ for all i and t. It follows immediately that $\vec{z}_{i,\infty} \leqslant R_{\max} < 1$ for all i, completing the proof.

3.5 Bootstrap percolation on an unoriented tree $\mathbb{T}$.

To determine the critical region for bootstrap percolation on $\mathbb{T}$, we use the result of Section 3.1 on oriented trees, as in [14]. Let z_t be the probability that the root is active at time t, and define the limiting probability $z_\infty := \lim_{t\to\infty} z_t$.

Theorem 2. *The probability z_∞ is given by*

$$z_\infty = p_0 + (1-p_0) \sum_{k=\theta_0}^{d_0+1} \binom{d_0+1}{k} \vec{z}_{1,\infty}^{\,k} (1-\vec{z}_{1,\infty})^{d_0+1-k} . \tag{28}$$

Proof. As before, p_0 simply stands for the probability that the root is initially active, so we focus on the case that it is initially inactive, which happens with probability $1-p_0$. In this case, it can become active if at least θ_0 of its d_0+1 neighbors become active in the process. For the root activity, it is immaterial whether or not a node can contribute to activating neighboring nodes that are further away from the root, so we can consider the d_0+1 root branches as oriented trees (oriented towards the root) on which oriented bootstrap percolation is performed. Thus we know that $\vec{z}_{1,\infty}$ is the limiting probability for a root neighbor to become active (if the root is not initially), which proves the desired formula.

Theorem 3. *The percolation regions on oriented tree $\vec{\mathbb{T}}$ and unoriented tree $\mathbb{T}$ are the same and equal to $\overline{W}_0$.*

Proof. If the unoriented tree fully percolates a.a.s., then in particular $z_\infty = 1$. Note that $z_\infty = 1$ by Theorem 2 if and only if $\vec{z}_{1,\infty} = 1$ (the case $p_0 = 1$ is trivial). However, if $\vec{z}_{1,\infty} = 1$, then also $\vec{z}_{i,\infty} = 1$ for all i, which means that even the oriented tree percolates a.a.s. The converse is clear as well.

4 Numerical estimation of $\boldsymbol{W_0}$

For a given tree $\vec{\mathbb{T}}$ (as well as $\mathbb{T}$ by Theorem 2), the region W_0 is determined in Theorem 1. That is, W_0 is the set of the initial probabilities $(p_0, p_1, \ldots, p_{\ell-1}) \in [0,1]^\ell$ such that there exists a solution $(x_0, x_1, \ldots, x_{\ell-1}) \in [0,1)^\ell$ (note: all $x_i < 1$) of the system given by $x_i = \phi_i(x_{i+1}; p_i)$, where $i = 0, 1, \ldots, \ell-1$. At the same time, this gives a criterion how one can decide whether $(p_0, p_1, \ldots, p_{\ell-1}) \in [0,1]^\ell$ belongs to W_0.

In Figure 1, we demonstrate results of this method. We present the boundary that separates W_0 and $\overline{W}_0$ in dimension 2 for a few different values of degrees d_0, d_1 and activation thresholds θ_0, θ_1. Specifically, $(d_0, d_1; \theta_0, \theta_1)$ takes the values $(7,8;5,3), (7,8;3,3), (7,8;4,4)$ in the upper and $(10,4;5,3), (4,10;5,3), (4,10;3,3)$ in the lower diagram. One can observe monotonicity of the boundary in (θ_0, θ_1). Looking at these diagrams, it is also tempting to conjecture that W_0 is always convex.

5 Conclusion

We examined and showed the existence of the region of critical probabilities in bootstrap percolation on infinite inhomogeneous periodic trees. The main difference to prior work is that we allow the initial probabilities and activation thresholds to be periodic, rather than fixed throughout the entire tree. We characterized the entire region of possible probabilities for which all nodes in the tree become a.a.s. active, as well as provided the probability that a node becomes eventually active, for any initial probability vector that does not belong to this region. Finally, the region is specified through a set of equations whose solution gives the boundary of the region. We demonstrated how one can numerically find this boundary and provided a few numerical examples in dimension two.

References

1. Aizenman, M., and Lebowitz, J. L. Metastability effects in bootstrap percolation. *Journal of Physics A: Mathematical and General 21*, 19 (1988), 3801–3813.
2. Amini, H. Bootstrap percolation and diffusion in random graphs with given vertex degrees. *Electronic Journal of Combinatorics 17*, #R25 (2010).
3. Amini, H., and Fountoulakis, N. What I tell you three times is true: Bootstrap percolation in small worlds. In *Internet and Network Economics - 8th International Workshop, WINE 2012, Liverpool, UK, December 10-12, 2012. Proceedings* (2012), pp. 462–474.
4. Amini, H., and Fountoulakis, N. Bootstrap percolation in power-law random graphs. *Journal of Statistical Physics 155*, 1 (2014), 72–92.
5. Amini, H., Fountoulakis, N., and Panagiotou, K. Bootstrap percolation in inhomogeneous random graphs. arXiv:1402.2815.
6. Balogh, J., Bollobás, B., Duminil-copin, H., and Morris, R. The sharp threshold for bootstrap percolation in all dimensions. *Trans. Amer. Math. Soc. 364*, 5 (2012), 2667–2701.

7. BALOGH, J., BOLLOBÁS, B., AND MORRIS, R. Majority bootstrap percolation on the hypercube. *Combinatorics, Probability and Computing 18*, 1-2 (2009), 17–51.
8. BALOGH, J., BOLLOBÁS, B., AND MORRIS, R. Bootstrap percolation in high dimensions. *Combinatorics, Probability and Computing 19*, 5-6 (2010), 643–692.
9. BALOGH, J., PERES, Y., AND PETE, G. Bootstrap percolation on infinite trees and non-amenable groups. *Combinatorics, Probability & Computing 15*, 5 (2006), 715–730.
10. BALOGH, J., AND PITTEL, B. Bootstrap percolation on the random regular graph. *Random Structures & Algorithms 30*, 1-2 (2007), 257–286.
11. BISKUP, M., AND SCHONMANN, R. H. Metastable behavior for bootstrap percolation on regular trees. *Journal of Statistical Physics 136*, 4 (2009), 667–676.
12. BOLLOBÁS, B., GUNDERSON, K., HOLMGREN, C., JANSON, S., AND PRZYKUCKI, M. Bootstrap percolation on Galton-Watson trees. *Electron. J. Probab. 19* (2014), no. 13, 1–27.
13. BRADONJIĆ, M., AND SANIEE, I. Bootstrap percolation on random geometric graphs. *Probability in the Engineering and Informational Sciences 28*, 2 (2014), 169–181.
14. BRADONJIĆ, M., AND SANIEE, I. Bootstrap percolation on periodic trees. In *Proceedings of the Twelfth Workshop on Analytic Algorithmics and Combinatorics, ANALCO 2015, San Diego, CA, USA, January 4, 2015* (2015), pp. 89–96.
15. CHALUPA, J., LEATH, P. L., AND REICH, G. R. Bootstrap percolation on a Bethe lattice. *J. Phys. C 12*, L31 (1979).
16. DUMINIL-COPIN, H., AND VAN ENTER, A. C. D. Sharp metastability threshold for an anisotropic bootstrap percolation model. *Ann. Probab. 41*, 3A (05 2013), 1218–1242.
17. FONTES, L., AND SCHONMANN, R. Bootstrap percolation on homogeneous trees has 2 phase transitions. *Journal of Statistical Physics 132*, 5 (2008), 839–861.
18. GRAVNER, J., HOLROYD, A. E., AND MORRIS, R. A sharper threshold for bootstrap percolation in two dimensions. *Probability Theory and Related Fields 153*, 1-2 (2012), 1–23.
19. HOLROYD, A. E. Sharp metastability threshold for two-dimensional bootstrap percolation. *Probability Theory and Related Fields 125* (2003), 195–224.
20. JANSON, S., LUCZAK, T., TUROVA, T., AND VALLIER, T. Bootstrap percolation on the random graph $G_{n,p}$. *Ann. of Appl. Probab 22*, 5 (2012), 1989–2047.
21. SCHONMANN, R. Critical points of two-dimensional bootstrap percolation-like cellular automata. *Journal of Statistical Physics 58*, 5-6 (1990), 1239–1244.
22. SCHONMANN, R. H. On the behavior of some cellular automata related to bootstrap percolation. *The Annals of Probability 20*, 1 (1992), 174–193.
23. VAN ENTER, A., ADLER, J., AND DUARTE, J. Finite-size effects for some bootstrap percolation models. *Journal of Statistical Physics 60*, 3-4 (1990), 323–332.
24. VAN ENTER, A., AND FEY, A. Metastability thresholds for anisotropic bootstrap percolation in three dimensions. *Journal of Statistical Physics 147*, 1 (2012), 97–112.
25. VAN ENTER, A., AND HULSHOF, T. Finite-size effects for anisotropic bootstrap percolation: Logarithmic corrections. *Journal of Statistical Physics 128*, 6 (2007), 1383–1389.

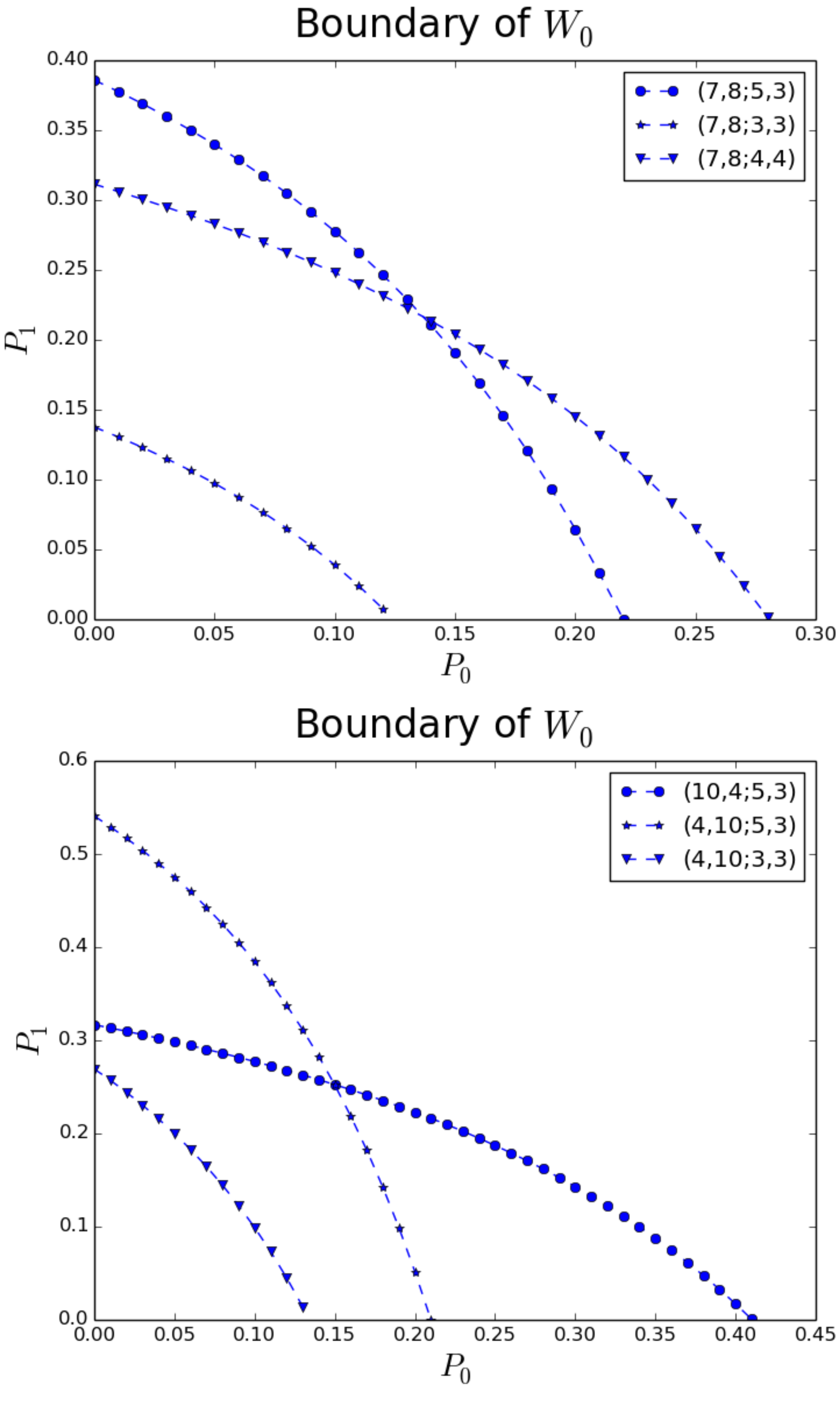

Fig. 1. Numerical evaluation of the boundary of the critical region W_0 in dimension two for different values of degrees and activation thresholds. Concretely, $(d_0, d_1; \theta_0, \theta_1)$ takes the values $(7, 8; 5, 3), (7, 8; 3, 3), (7, 8; 4, 4)$ in the upper and $(10, 4; 5, 3), (4, 10; 5, 3), (4, 10; 3, 3)$ in the lower diagram.