A GENERAL ASYMPTOTIC SCHEME FOR MOMENTS OF PARTITION STATISTICS

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Abstract. In order to compute means, variances and higher moments of various partition statistics, one often has to study generating functions of the form $P(x)F(x)$, where $P(x)$ is the generating function for the number of partitions. In this paper, we show how asymptotic expansions can be obtained in a quasi-automatic way from expansions of $F(x)$ around $x = 1$. Numerous examples from the literature, as well as some new statistics are treated via this methodology. In addition, we show how to compute further terms in the asymptotic expansions of previously studied partition statistics.

1. Introduction

In the analysis of partition statistics, one often has to study generating functions of the form $P(x)F(x)$, where

$$P(x) = \prod_{j=1}^{\infty}(1 - x^j)^{-1}$$

is the generating function for the number of partitions. In this paper, we want to develop a general asymptotic scheme that allows one to derive an asymptotic formula for the n-th coefficient of $P(x)F(x)$ from the behaviour of $F(x)$ as $x \to 1$. It is well known that $p(n) = [x^n]P(x)$ essentially behaves like $\frac{1}{4\sqrt{3}n}\exp\left(\pi\sqrt{2n/3}\right)$, which is made much more precise by Rademacher's celebrated formula

$$p(n) = \frac{1}{\pi\sqrt{2}}\sum_{k=1}^{\infty}A_k(n)\sqrt{k}\frac{d}{dn}\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{n-1/24}}, \tag{1}$$

a sum formula that is both exact and asymptotic (in the sense that the asymptotic order of the summands is decreasing). See Apostol's book [1] for an excellent exposition. This result depends heavily on a functional equation for $P(x)$, a special case of which is given by

$$P(e^{-t}) = \sqrt{\frac{t}{2\pi}}\exp\left(\frac{\pi^2}{6t} - \frac{t}{24}\right)P(e^{-4\pi^2/t}). \tag{2}$$

We will make frequent use of this functional equation throughout the paper. Let us now consider a few examples that lead to generating functions of the form $P(x)F(x)$.

The length of a partition. The length (number of parts) of a partition has bivariate generating function

$$\prod_{j=1}^{\infty}(1 - ux^j)^{-1},$$

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where the exponent of u marks the length. Differentiating with respect to u and setting $u = 1$, one obtains the generating function

$$\prod_{j=1}^{\infty}(1-x^j)^{-1} \cdot \sum_{j=1}^{\infty} \frac{x^j}{1-x^j}$$

for the total length, summed over all partitions. The coefficient of x^n, divided by the number $p(n)$ of partitions of n, yields the average length of a random partition of n. This was used by Kessler and Livingston [11] to obtain rather precise asymptotics for the average length: it is given by

$$\frac{\sqrt{6n}}{2\pi} \cdot \Big(\log n + 2\gamma - \log(\pi^2/6)\Big) + \mathcal{O}\Big((\log n)^3\Big).$$

In section 3, we will use our results to compute further terms of the expansion. The length asymptotically follows a Gumbel distribution, as was shown by Erdős and Lehner [4]. It is well known that the largest part follows the same distribution (as can be seen by conjugation of the Ferrers diagram), and so its mean is the same as well.

Number of distinct parts. The number of distinct parts is known to follow a normal distribution in the limit, as proved by Goh and Schmutz [7]. If one is only interested in an asymptotic expansion for the mean, one has to consider a bivariate generating function again:

$$\prod_{j=1}^{\infty} \left(1 + \frac{ux^j}{1-ux^j}\right)$$

Differentiating and setting $u = 1$ yields

$$\prod_{j=1}^{\infty}(1-x^j)^{-1} \cdot \frac{x}{1-x},$$

which is of the form $P(x)F(x)$ as well. A generalisation of this parameter, namely the sum of the m-th powers of all distinct parts, was studied by Hwang and Yeh [10]. The associated generating function is

$$\prod_{j=1}^{\infty}(1-x^j)^{-1} \cdot \sum_{j=1}^{\infty} j^m x^j,$$

which also belongs to our scheme.

Moments of a partition. For a partition $(c_1, c_2, \ldots, c_\ell)$ of an integer n, consider the k-th moment

$$\sum_{j=1}^{\ell} c_j^k.$$

The case $k = 0$ clearly corresponds to the length, while the above sum is always equal to n for $k = 1$ by definition. As above, differentiating the bivariate generating function

$$\prod_{j=1}^{\infty}(1-u^{j^k}x^j)^{-1}$$

with respect to u and setting $u = 1$ yields

$$\prod_{j=1}^{\infty}(1-x^j)^{-1} \cdot \sum_{j=1}^{\infty} \frac{j^k x^j}{1-x^j}.$$

See [9, Corollary 6.5] for an interesting connection to so-called hook lengths, and [17] for an occurrence of the special case $k = 2$.

Number of parts of a given size. The bivariate generating function for the number of parts of size d is given by

$$\frac{1-x^d}{1-ux^d}\cdot\prod_{j=1}^{\infty}(1-x^j)^{-1},$$

so that one has to study

$$\frac{x^d}{1-x^d}\cdot\prod_{j=1}^{\infty}(1-x^j)^{-1}$$

for the average number of occurrences of the number d among the parts of a random partition of n. Notice that in the case $d=1$, this coincides with the aforementioned generating function for the number of distinct parts, which is known as Stanley's Theorem [15, Exercise 1.26].

Number of parts with given multiplicity. If u marks all parts that occur with some prescribed multiplicity d, one obtains the bivariate generating function

$$\prod_{j=1}^{\infty}\left(\frac{1}{1-x^j}+(u-1)x^{dj}\right),$$

so that the generating function for the total number of parts of multiplicity d is given by

$$\prod_{j=1}^{\infty}(1-x^j)^{-1}\cdot\sum_{j=1}^{\infty}x^{dj}(1-x^j)=\prod_{j=1}^{\infty}(1-x^j)^{-1}\cdot\frac{(1-x)x^d}{(1-x^d)(1-x^{d+1})}.$$

The asymptotic behaviour of this parameter was found by Corteel et al. [3] to be $\sim\frac{6n}{\pi d(d+1)}$. In Section 3, we will show how further terms in the asymptotic expansion can be determined. It is also worth mentioning that the number of parts of multiplicity d almost exactly follows the same distribution as the number of d-successions (i.e., occurrences of two subsequent parts whose difference is d), which can be seen by conjugation of the Ferrers diagram. Successions in partitions were investigated in another recent paper [12]. Another related parameter that also falls under our scheme is the number of ascents of size d or more, which was studied in [2]; a special case is the number of gaps (ascents of size 2 or more), see [13].

Largest repeated part. It was shown in [8] that the generating function for the sum of the largest repeated part sizes over all partitions of n is given by

$$\prod_{j=1}^{\infty}(1-x^j)^{-1}\cdot\sum_{j=1}^{\infty}\frac{x^{2j}}{1-x^{2j}},$$

and this can easily be generalised to yield the generating function for the sum of the largest part sizes that are repeated at least d times (if no such part exists, one defines it to be 0):

$$\prod_{j=1}^{\infty}(1-x^j)^{-1}\cdot\sum_{j=1}^{\infty}\frac{x^{dj}}{1-x^{dj}}.$$

Note that $d=1$ corresponds to the maximum of a partition. Results on the limiting distributions of the largest repeated part and related parameters can be found in [18].

Longest run. The longest run (maximum multiplicity) was shown to follow a rather unusual limit law in [14]; see also [19], where the mean of the longest run was found to be asymptotically equal to $\left(4\sqrt{2}-\frac{6\sqrt{6}}{\pi}\right)\sqrt{n}$. In Section 3 we will considerably improve on this. Since the generating function for the number of partitions whose longest run is $< k$ is easily found to be

$$\prod_{j=1}^{\infty}\frac{1-x^{jk}}{1-x^j}=P(x)P(x^k)^{-1},$$

one has to study the generating function

$$P(x) \cdot \sum_{k=1}^{\infty} k \left(P(x^{k+1})^{-1} - P(x^k)^{-1}\right) = P(x) \cdot \sum_{k=1}^{\infty} \left(1 - P(x^k)^{-1}\right),$$

or, more generally for the m-th moment,

$$P(x) \cdot \sum_{k=1}^{\infty} (k^m - (k-1)^m) \left(1 - P(x^k)^{-1}\right).$$

A related question concerns the probability that a certain integer k is one of the parts of largest multiplicity. The generating function for the number of partitions for which this is the case is

$$\sum_{\ell=1}^{\infty} \frac{x^{k\ell}(1-x^k)}{1-x^{k(\ell+1)}} \cdot \prod_{j=1}^{\infty} \frac{1-x^{(\ell+1)j}}{1-x^j} = P(x) \cdot \sum_{\ell=1}^{\infty} \frac{x^{k\ell}(1-x^k)}{(1-x^{k(\ell+1)})P(x^{\ell+1})},$$

which is once again of the form $P(x)F(x)$. In a similar manner, one obtains the following generating function for the number of partitions with the property that all parts have strictly smaller multiplicity than k:

$$\sum_{\ell=1}^{\infty} \frac{x^{k\ell}(1-x^k)}{1-x^{k\ell}} \cdot \prod_{j=1}^{\infty} \frac{1-x^{\ell j}}{1-x^j} = P(x) \cdot \sum_{\ell=1}^{\infty} \frac{x^{k\ell}(1-x^k)}{(1-x^{k\ell})P(x^{\ell})}.$$

2. A general asymptotic scheme

Let us now develop a rather general scheme that allows us to obtain asymptotic expansions for the coefficients of a generating function of the form $P(x)F(x)$. The technical conditions we impose on $F(x)$ for this purpose are rather mild. First of all, we express the coefficient of x^n by means of Cauchy's integral formula:

$$[x^n]P(x)F(x) = \frac{1}{2\pi i} \oint_{\mathcal{C}} z^{-n-1} P(z) F(z)\, dz \tag{3}$$

for a closed curve $\mathcal{C}$ around 0 inside the unit circle. If one takes $\mathcal{C}$ to be the circle of radius e^{-r} around 0, the change of variable $z = e^{-r+iu}$ yields

$$\frac{1}{2\pi i} \oint_{\mathcal{C}} z^{-n-1} P(z) F(z)\, dz = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{rn-inu} P(e^{-r+iu}) F(e^{-r+iu})\, du. \tag{4}$$

We will show that the main asymptotic information can be determined from the behaviour as z goes to 1. Essentially,

$$\frac{1}{p(n)} \cdot [x^n]P(x)F(x)$$

is given by the value of $F(z)$ at the saddle point $z = e^{-\pi/\sqrt{6n}}$. For this purpose, it is necessary that $F(z)$ does not grow too quickly as $|z| \to 1$. Specifically, we assume that

$$|F(z)| \ll e^{C/(1-|z|)^{\eta}} \text{ as } |z| \to 1 \text{ for some } C > 0 \text{ and } \eta < 1. \tag{5}$$

A simple sufficient condition for this to hold (that will actually be satisfied in all our examples) is that the coefficients of $F(z)$ grow at most polynomially.

Our main results read as follows:

Theorem 1. *Suppose that the function $F(z)$ satisfies (5) and $F(e^{-t}) = \mathcal{O}(f(|t|))$ as $t \to 0$. Then one has*

$$\frac{1}{p(n)}[x^n]P(x)F(x) = \mathcal{O}\left(\exp\left(-n^{1/2-\epsilon}\right) + f\left(\frac{\pi}{\sqrt{6n}} + \mathcal{O}\left(n^{-1/2-\epsilon}\right)\right)\right)$$

for any fixed $0 < \epsilon < \frac{1-\eta}{2}$ as $n \to \infty$.

Theorem 2. *Suppose that the function $F(z)$ satisfies* (5) *and that*

$$\frac{F(e^{-t+iu})}{F(e^{-t})} \to 1$$

if $|u| \le At^{1+\epsilon}$ for some $A > 0$ and for some $\epsilon < \frac{1-\eta}{2}$, uniformly in u as $t \to 0$. Then one has

$$\frac{1}{p(n)}[x^n]P(x)F(x) = F\left(e^{-\pi/\sqrt{6n}}\right)(1+o(1)) + \mathcal{O}\left(\exp\left(-Bn^{1/2-\epsilon}\right)\right)$$

as $n \to \infty$ for some $B > 0$.

Remark 1. The required technical condition on F is necessary to avoid pathological examples (e.g., zeros of $F(z)$ accumulating as $z \to 1$). It is easy to check them for all the examples described in the introduction. A sufficient condition is that

$$\left|\frac{t^{1+\epsilon_0}F'(e^{-t})}{F(e^{-t})}\right|$$

remains bounded as $t \to 0$ for some $\epsilon_0 < \epsilon$.

To obtain more precise asymptotic formulae in the case that $F(e^{-t})$ can be expanded into powers of t around $t = 0$, we also need the following result:

Theorem 3. *Suppose that the function $F(z)$ satisfies* (5) *and $F(e^{-t}) = at^b + \mathcal{O}(f(|t|))$ as $t \to 0$ for real numbers a, b. Then one has*

$$\frac{1}{p(n)}[x^n]P(x)F(x) = a\left(\frac{2\pi}{\sqrt{24n-1}}\right)^b \cdot \frac{I_{|b+3/2|}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)}{I_{3/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)} + \mathcal{O}\left(\exp\left(-n^{1/2-\epsilon}\right) + f\left(\frac{\pi}{\sqrt{6n}} + \mathcal{O}\left(n^{-1/2-\epsilon}\right)\right)\right)$$

as $n \to \infty$ for any $0 < \epsilon < \frac{1-\eta}{2}$, where I_ν denotes a modified Bessel function of the first kind. Similarly, if $F(z)$ satisfies (5) *and $F(e^{-t}) = at^b \log\frac{1}{t} + \mathcal{O}(f(|t|))$ as $t \to 0$, then*

$$\frac{1}{p(n)}[x^n]P(x)F(x) = a\left(\frac{2\pi}{\sqrt{24n-1}}\right)^b \cdot \left(\log\left(\frac{\sqrt{24n-1}}{2\pi}\right)\frac{I_{|b+3/2|}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)}{I_{3/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)} + \sum_{k=1}^{2K}\frac{1}{k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{I_{|b+j+3/2|}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)}{I_{3/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)}\right) + \mathcal{O}\left(n^{-(b+K+1)/2} + f\left(\frac{\pi}{\sqrt{6n}} + \mathcal{O}\left(n^{-1/2-\epsilon}\right)\right)\right)$$

for any nonnegative integer K.

Remark 2. Of course this theorem generalises to asymptotic expansions of the form $F(e^{-t}) = \sum_{j=1}^{J} a_j t^{b_j} + \mathcal{O}(f(|t|))$, or even to mixed expansions involving logarithms.

The absolute value in the index of the Bessel function can actually be dropped, since the difference $I_\nu(z) - I_{-\nu}(z)$ decreases exponentially as $z \to \infty$. Note further that the modified Bessel function $I_{|b+3/2|}$ can be written in terms of elementary functions for integer values of b (see [20]): for nonnegative integer h, one has

$$I_{h+1/2}(z) = \frac{1}{\sqrt{2\pi z}}\sum_{j=0}^{h}\left(-\frac{1}{2z}\right)^j \cdot \frac{(h+j)!}{j!(h-j)!}\left(e^z - (-1)^{h-j}e^{-z}\right),$$

to the effect that the quotient simplifies, with $h = |b+3/2| - 1/2$ and $m = \sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}$, to

$$\frac{I_{|b+3/2|}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)}{I_{3/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)} = \frac{m}{m-1}\cdot\sum_{j=0}^{h}\frac{(h+j)!}{j!(h-j)!}\left(-\frac{1}{2m}\right)^j + \mathcal{O}(e^{-2m}).$$

For non-integer values of b, this is at least asymptotically correct, in the sense that

$$\frac{I_{|b+3/2|}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)}{I_{3/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)} = \frac{m}{m-1}\cdot\sum_{j=0}^{J}\binom{h+j}{2j}\frac{(2j)!}{j!}\left(-\frac{1}{2m}\right)^j + \mathcal{O}(m^{-J-1})$$

for any fixed J, with the same abbreviations as above. This also shows that in the sum

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{I_{|b+j+3/2|}(m)}{I_{3/2}(m)},$$

the power $m^{-\ell}$ in the asymptotic expansion vanishes for $\ell \le \lfloor\frac{k-1}{2}\rfloor$, since its coefficient is a polynomial of order 2ℓ, and the alternating sum above amounts to taking k-th differences. Therefore, this sum is of asymptotic order $m^{-\lfloor(k+1)/2\rfloor}$.

For the proofs we need the following technical lemma; similar estimates are frequently used in the theory of partitions, see for instance [14].

Lemma 4. *Uniformly as $|u| \le \pi$, one has*

$$\frac{|P(e^{-r+iu})|}{P(e^{-r})} \le \exp\left(-\frac{u^2}{r(u^2+(\pi r/2)^2))} + \mathcal{O}(r)\right)$$

as $r \to 0$.

Proof. The proof essentially follows the ideas of [14]. First we get

$$\begin{aligned}
|P(e^{-r+iu})| &= \left|\exp\left(-\sum_{j=1}^{\infty}\log\left(1-e^{-jr+iju}\right)\right)\right| = \left|\exp\left(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}e^{-jkr+ijku}\right)\right| \\
&= \exp\left(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}e^{-jkr}\cos(jku)\right) = P(e^{-r})\exp\left(-\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}e^{-jkr}(1-\cos(jku))\right) \\
&\le P(e^{-r})\exp\left(-\sum_{j=1}^{\infty}e^{-jr}(1-\cos(ju))\right) \\
&= P(e^{-r})\exp\left(-\coth\left(\frac{r}{2}\right)\cdot\frac{1-\cos u}{2(\cosh r-\cos u)}\right),
\end{aligned}$$

where the last step simply follows from summing the geometric series. Now, making use of the inequality $1-\cos u \ge 2(u/\pi)^2$, the exponent can be estimated below by

$$\begin{aligned}
\coth\left(\frac{r}{2}\right)\cdot\frac{1-\cos u}{2(\cosh r-\cos u)} &= \frac{1}{2}\coth\left(\frac{r}{2}\right)\cdot\left(1+\frac{\cosh r-1}{1-\cos u}\right)^{-1} \\
&\ge \frac{1}{2}\coth\left(\frac{r}{2}\right)\cdot\left(1+\frac{\cosh r-1}{2u^2/\pi^2}\right)^{-1} \\
&= \left(\frac{1}{r}+\mathcal{O}(r)\right)\cdot 2u^2\left(2u^2+\pi^2r^2/2+\mathcal{O}(r^4)\right)^{-1} \\
&= \frac{u^2}{r(u^2+\pi^2r^2/4)}\left(1+\mathcal{O}(r^2)\right)
\end{aligned}$$

$$= \frac{u^2}{r(u^2+\pi^2r^2/4)} + \mathcal{O}(r).$$

This completes the proof of our lemma. □

Proof of Theorem 1. We start with the integral representation (4), where we choose r to be the saddle point $r = \frac{\pi}{\sqrt{6n}}$. Now choose $\epsilon < \frac{1-\eta}{2}$ and split the integral into a central part for which $|u| \le 2r^{1+\epsilon}$ and the rest. For sufficiently large n and $2r^{1+\epsilon} \le |u| \le \pi$, one has

$$\frac{u^2}{r(u^2+\pi^2r^2/4)} = \frac{1}{r(1+\pi^2r^2/(4u^2))} \ge \frac{1}{r(1+\pi^2r^{-2\epsilon}/16)} \ge \frac{3}{2}r^{2\epsilon-1}$$

and thus, by Lemma 4 and the assumptions on F,

$$\begin{aligned}\left|\frac{1}{2\pi}\int_{2r^{1+\epsilon}\le|u|\le\pi} e^{rn-inu}P(e^{-r+iu})F(e^{-r+iu})\,du\right| &\le \sup_{2r^{1+\epsilon}\le|u|\le\pi}\left|e^{rn-inu}P(e^{-r+iu})F(e^{-r+iu})\right| \\ &\ll e^{rn}P(e^{-r})\exp\left(-\frac{3}{2}r^{2\epsilon-1} + C(1-e^{-r})^{-\eta} + \mathcal{O}(r)\right) \\ &= e^{rn}P(e^{-r})\exp\left(-\frac{3}{2}\left(\frac{\pi}{\sqrt{6n}}\right)^{2\epsilon-1} + \mathcal{O}(n^{\eta/2})\right).\end{aligned}$$

The functional equation (2) shows that

$$e^{nr}P(e^{-r}) = \sqrt{\frac{r}{2\pi}}\exp\left(nr+\frac{\pi^2}{6r}-\frac{r}{24}\right)P(e^{-4\pi^2/r}) \ll n^{3/4}p(n),$$

so that finally

$$\begin{aligned}\left|\frac{1}{2\pi}\int_{2r^{1+\epsilon}\le|u|\le\pi} e^{rn-inu}P(e^{-r+iu})F(e^{-r+iu})\,du\right| &\ll p(n)\exp\left(-\frac{3}{2}\left(\frac{\pi}{\sqrt{6n}}\right)^{2\epsilon-1} + \mathcal{O}(n^{\eta/2})\right) \\ &\ll p(n)\exp(-n^{1/2-\epsilon}).\end{aligned}$$

Hence it suffices to consider the remaining part of the integral. Since $|r-iu| = r + \mathcal{O}(r^{1+2\epsilon})$ for $|u| \le 2r^{1+\epsilon}$ by an easy calculation, one has

$$|F(e^{-r+iu})| \ll f(|r-iu|) = f\left(r+\mathcal{O}(r^{1+2\epsilon})\right) = f\left(\frac{\pi}{\sqrt{6n}} + \mathcal{O}\left(n^{-1/2-\epsilon}\right)\right)$$

within this interval. Hence it suffices to show that

$$\int_{|u|\le 2r^{1+\epsilon}} \left|e^{rn-inu}P(e^{-r+iu})\right|\,du \ll p(n).$$

To this end, note that by (2),

$$\begin{aligned}\left|e^{rn-inu}P(e^{-r+iu})\right| &= e^{rn}\sqrt{|r-iu|}\exp\left(\mathrm{Re}\left(\frac{\pi^2}{6(r-iu)} - \frac{r-iu}{24}\right)\right)\left|P(e^{-4\pi^2/(r-iu)})\right| \\ &\ll e^{rn}\sqrt{r}\exp\left(\frac{\pi^2 r}{6(r^2+u^2)}\right) \\ &= e^{rn}\sqrt{r}\exp\left(\frac{\pi^2}{6r} - \frac{\pi^2u^2}{6r(r^2+u^2)}\right) \\ &\le e^{rn}\sqrt{r}\exp\left(\frac{\pi^2}{6r} - \frac{\pi^2u^2}{30r^3}\right)\end{aligned}$$

for $|u| \le 2r^{1+\epsilon}$, so that

$$\begin{aligned}\int_{|u|\le 2r^{1+\epsilon}} \left|e^{rn-inu}P(e^{-r+iu})\right|\,du &\ll \sqrt{r}\exp\left(rn+\frac{\pi^2}{6r}\right)\int_{|u|\le 2r^{1+\epsilon}}\exp\left(-\frac{\pi^2u^2}{30r^3}\right)du \\ &\ll n^{-1/4}\exp\left(\pi\sqrt{\frac{2n}{3}}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{\pi^2u^2}{30r^3}\right)du\end{aligned}$$

$$= n^{-1/4} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)\sqrt{\frac{30r^3}{\pi}}$$
$$\ll n^{-1} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \ll p(n),$$

which completes the proof. □

Proof of Theorem 2. The proof proceeds in the same way as the previous one. One splits the integral (4) into the part for which $|u| \leq Ar^{1+\epsilon}$ and the remaining two intervals. The latter only contribute an error term of the form $\mathcal{O}\left(\exp\left(-Bn^{1/2-\epsilon}\right)\right)$, and the remaining part of the integral is given by

$$\frac{1}{2\pi}\int_{|u|\leq Ar^{1+\epsilon}} e^{rn-inu}P(e^{-r+iu})F(e^{-r+iu})\,du = F(e^{-r})\frac{1}{2\pi}\int_{|u|\leq Ar^{1+\epsilon}} e^{rn-inu}P(e^{-r+iu})\,du$$
$$+ o\left(|F(e^{-r})|\int_{|u|\leq Ar^{1+\epsilon}} \left|e^{rn-inu}P(e^{-r+iu})\right|\,du\right)$$
$$= F(e^{-r})p(n)(1+o(1)) + o\left(|F(e^{-r})|p(n)\right)$$

by the same estimates as before. □

Proof of Theorem 3. Recall the integral representation

$$[x^n]P(x)F(x) = \frac{1}{2\pi i}\oint_{\mathcal{C}} z^{-n-1}P(z)F(z)\,dz,$$

where we take $\mathcal{C}$ to be the circle of radius e^{-r} around 0. The change of variable $z = e^{-t}$ yields

$$[x^n]P(x)F(x) = \frac{1}{2\pi i}\int_{r-i\pi}^{r+i\pi} e^{nt}P(e^{-t})F(e^{-t})\,dt.$$

In view of Theorem 1 (and its proof), we may now restrict ourselves to the part of the integral where $|\operatorname{Im} t| \leq 2r^{1+\epsilon}$, and replace $F(e^{-t})$ by at^b there. We are left with an integral of the form

$$\frac{a}{2\pi i}\int_{r-2ir^{1+\epsilon}}^{r+2ir^{1+\epsilon}} e^{nt}P(e^{-t})t^b\,dt.$$

Now we make use of the functional equation (2) once again to obtain

$$\frac{a}{\sqrt{2\pi}}\cdot\frac{1}{2\pi i}\int_{r-2ir^{1+\epsilon}}^{r+2ir^{1+\epsilon}} t^{b+1/2}\exp\left(\left(n-\frac{1}{24}\right)t+\frac{\pi^2}{6t}\right)P(e^{-4\pi^2/t})\,dt.$$

Since $P(z) = 1 + \mathcal{O}(z)$ as $z \to 0$, one can replace the factor $P(e^{-4\pi^2/t})$ by 1 at the expense of an error term of order

$$\exp\left(-4\pi^2/r + \mathcal{O}(r^{-1+\epsilon})\right),$$

which is negligible. Hence one is left with an integral over an elementary function. Depending on b, one has to distinguish three cases:

- If $b < -\frac{3}{2}$, then we complete the integral to obtain

$$\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} t^{b+1/2}\exp\left(\left(n-\frac{1}{24}\right)t+\frac{\pi^2}{6t}\right)\,dt,$$

  which, after the change of variables $(n - 1/24)t = u$, yields

$$\left(n-\frac{1}{24}\right)^{-b-3/2}\cdot\frac{1}{2\pi i}\int_{L-i\infty}^{L+i\infty} u^{b+1/2}\exp\left(u+\frac{\pi^2(n-1/24)}{6u}\right)\,du$$

for $L = (n - 1/24)r$. This is a well-known integral representation for the modified Bessel function $I_{-b-3/2}$ (see [20]), so that we finally get

$$\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} t^{b+1/2}\exp\left(\left(n-\frac{1}{24}\right)t+\frac{\pi^2}{6t}\right)dt = \left(\frac{4\pi^2}{24n-1}\right)^{b/2+3/4} I_{-b-3/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right).$$

It remains to show that the parts of the integral that were added only contribute to the error term. Consider the integral

$$\mathcal{I} = \int_{r+2ir^{1+\epsilon}}^{r+i\infty} t^{b+1/2}\exp\left(\left(n-\frac{1}{24}\right)t+\frac{\pi^2}{6t}\right)dt,$$

which we estimate as follows:

$$\begin{aligned}
|\mathcal{I}| &\le \int_{r+2ir^{1+\epsilon}}^{r+i\infty}\left|t^{b+1/2}\exp\left(\left(n-\frac{1}{24}\right)t+\frac{\pi^2}{6t}\right)\right|dt\\
&= \int_{r+2ir^{1+\epsilon}}^{r+i\infty}|t|^{b+1/2}\exp\left(\left(n-\frac{1}{24}\right)r+\mathrm{Re}\left(\frac{\pi^2}{6t}\right)\right)dt\\
&\le \exp\left(\left(n-\frac{1}{24}\right)r\right)\int_{2r^{1+\epsilon}}^{\infty}u^{b+1/2}\exp\left(\mathrm{Re}\left(\frac{\pi^2}{6(r+iu)}\right)\right)du\\
&= \exp\left(\left(n-\frac{1}{24}\right)r\right)\int_{2r^{1+\epsilon}}^{\infty}u^{b+1/2}\exp\left(\frac{\pi^2 r}{6(r^2+u^2)}\right)du\\
&= \exp\left(\left(n-\frac{1}{24}\right)r+\frac{\pi^2}{6r}\right)\int_{2r^{1+\epsilon}}^{\infty}u^{b+1/2}\exp\left(-\frac{\pi^2u^2}{6r(r^2+u^2)}\right)du\\
&\le \exp\left(\left(n-\frac{1}{24}\right)r+\frac{\pi^2}{6r}-\frac{2\pi^2r^{2+2\epsilon}}{3r(r^2+4r^{2+2\epsilon})}\right)\int_{2r^{1+\epsilon}}^{\infty}u^{b+1/2}\,du\\
&= \exp\left(\left(n-\frac{1}{24}\right)r+\frac{\pi^2}{6r}-\frac{2\pi^2}{3}r^{2\epsilon-1}+\mathcal{O}\left(r^{4\epsilon-1}+\log(1/r)\right)\right)\\
&\ll p(n)\exp\left(-n^{1/2-\epsilon}\right),
\end{aligned}$$

completing the proof in this case.

- For $b = -\frac{3}{2}$, one has to be slightly more careful in the estimation of the integral, but otherwise the proof remains the same. We have to consider

$$\mathcal{I} = \int_{r+2ir^{1+\epsilon}}^{r+i\infty}\frac{1}{t}\exp\left(\left(n-\frac{1}{24}\right)t+\frac{\pi^2}{6t}\right)dt,$$

which we split into two integrals $\mathcal{I}_1$ and $\mathcal{I}_2$, corresponding to $\mathrm{Im}\,t \le 1$ and $\mathrm{Im}\,t \ge 1$ respectively. For $\mathcal{I}_1$, the same argument as above yields

$$\begin{aligned}
|\mathcal{I}_1| &\le \exp\left(\left(n-\frac{1}{24}\right)r+\frac{\pi^2}{6r}-\frac{2\pi^2r^{2+2\epsilon}}{6r(r^2+4r^{2+2\epsilon})}\right)\int_{2r^{1+\epsilon}}^{1}\frac{1}{u}\,du\\
&\ll p(n)\exp\left(-n^{1/2-\epsilon}\right),
\end{aligned}$$

while $\mathcal{I}_2$ can be estimated as follows:

$$\begin{aligned}
\mathcal{I}_2 &= \int_1^{\infty}(r+iu)^{-1}\exp\left(\left(n-\frac{1}{24}\right)(r+iu)+\frac{\pi^2}{6(r+iu)}\right)du\\
&= \exp\left(\left(n-\frac{1}{24}\right)r\right)\int_1^{\infty}(r+iu)^{-1}\exp\left(\left(n-\frac{1}{24}\right)iu+\frac{\pi^2}{6(r+iu)}\right)du\\
&= \exp\left(\left(n-\frac{1}{24}\right)r\right)\left(\int_1^{\infty}(iu)^{-1}\exp\left(\left(n-\frac{1}{24}\right)iu\right)du+\mathcal{O}\left(\int_1^{\infty}u^{-2}\,du\right)\right)\\
&= \exp\left(\left(n-\frac{1}{24}\right)r\right)\cdot\mathcal{O}(1)
\end{aligned}$$

$$\ll np(n)\exp\left(-\sqrt{\frac{\pi^2 n}{6}}\right).$$

- For $b > -\frac{3}{2}$, the same method would lead to non-convergent integrals, so that we have to use another change of variables first: in the integral
$$\frac{1}{2\pi i}\int_{r-iR}^{r+iR} t^{b+1/2}\exp\left(\left(n-\frac{1}{24}\right)t+\frac{\pi^2}{6t}\right)dt,$$
where $R = 2r^{1+\epsilon}$, set $\frac{\pi^2}{6t} = w$ to get
$$\left(\frac{\pi^2}{6}\right)^{b+3/2}\cdot\frac{1}{2\pi i}\int_{s-iS}^{s+iS} w^{-b-5/2}\exp\left(w+\frac{\pi^2(n-1/24)}{6w}\right)dw,$$
where $s = \frac{\pi^2 r}{6(r^2+R^2)}$ and $S = \frac{\pi^2 R}{6(r^2+R^2)}$. Now one completes the integral as in the first two cases (note that the exponent $-b-5/2$ is less than -1, so the same steps can be applied) to obtain
$$\left(\frac{4\pi^2}{24n-1}\right)^{b/2+3/4} I_{b+3/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)$$
plus an error term of the desired order. Note in particular that one obtains the first term of Rademacher's series (1) in the special case $b = 0$. Dividing by $p(n)$, one finally ends up with the stated formula.

The second part of the theorem can be reduced to the first part quite easily. We now have to consider the integral
$$\frac{1}{2\pi i}\int_{r-2ir^{1+\epsilon}}^{r+2ir^{1+\epsilon}} t^{b+1/2}\log\frac{1}{t}\exp\left(\left(n-\frac{1}{24}\right)t+\frac{\pi^2}{6t}\right)dt.$$
For our purposes, it is convenient to take $r = \frac{2\pi}{\sqrt{24n-1}}$ here rather than $r = \frac{\pi}{\sqrt{6n}}$, which is only a minor modification and does not alter the rest of the argument. Now write
$$\log\frac{1}{t} = \log\frac{1}{r} - \log\left(1-\left(1-\frac{t}{r}\right)\right) = \log\frac{1}{r} + \sum_{k=1}^{L}\frac{1}{k}\left(1-\frac{t}{r}\right)^k + \mathcal{O}\left(r^{\epsilon(L+1)}\right)$$
$$= \log\frac{1}{r} + \sum_{k=1}^{L}\frac{1}{k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{t^j}{r^j} + \mathcal{O}\left(r^{\epsilon(L+1)}\right).$$
Noting that the error term can be made arbitrarily small by expanding sufficiently far, we can now apply the first part of the theorem to each of the resulting summands, which completes the proof. □

3. Examples

Let us first consider some examples in which the asymptotic expansion of $F(e^{-t})$ can be obtained directly.

Number of distinct parts. Of all the examples presented in the introduction, this one is probably the simplest. One has
$$F(e^{-t}) = \frac{e^{-t}}{1-e^{-t}} = \frac{1}{e^t-1} = \frac{1}{t} - \frac{1}{2} + \frac{t}{12} + \mathcal{O}(t^3),$$
which, by Theorem 3, directly translates to the following formula for the average number of distinct parts in a random partition of n:
$$\frac{\sqrt{24n-1}}{2\pi}\cdot\frac{I_{1/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)}{I_{3/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)} - \frac{1}{2} + \frac{1}{12}\cdot\frac{2\pi}{\sqrt{24n-1}}\cdot\frac{I_{5/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)}{I_{3/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)} + \mathcal{O}(n^{-3/2}),$$

Making use of the asymptotic expansions for Bessel functions mentioned in Remark 2, this can be further simplified to

$$\frac{\sqrt{24n-1}}{2\pi}\cdot\frac{m}{m-1}-\frac{1}{2}+\frac{\pi}{6\sqrt{24n-1}}\cdot\frac{m}{m-1}\left(1-\frac{3}{m}+\frac{3}{m^2}\right)+\mathcal{O}(n^{-3/2}),$$

with $m=\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}$. Finally, this can be turned into an asymptotic expansion in powers of n (which is best done by means of computer algebra):

$$\frac{\sqrt{6n}}{\pi}+\frac{6-\pi^2}{2\pi^2}+\frac{2\pi^4-3\pi^2+216}{24\pi^3\sqrt{6n}}+\frac{54-\pi^4}{12\pi^4 n}+\mathcal{O}(n^{-3/2}).$$

Of course it is possible to determine arbitrarily many terms of the expansion in this and all later examples.

Number of parts of a given size. This is only a slight generalisation of the previous example; the expansion of $F(e^{-t})$ around $t=0$ is given by

$$F(e^{-t})=\frac{e^{-dt}}{1-e^{-dt}}=\frac{1}{e^{dt}-1}=\frac{1}{dt}-\frac{1}{2}+\frac{dt}{12}+\mathcal{O}(t^3),$$

to the effect that the average number of parts of size d in a random partition of n is

$$\frac{\sqrt{6n}}{\pi d}+\frac{6-\pi^2 d}{2\pi^2 d}+\frac{2\pi^4 d^2-3\pi^2+216}{24\pi^3 d\sqrt{6n}}+\frac{54-\pi^4 d^2}{12\pi^4 dn}+\mathcal{O}(n^{-3/2}).$$

Number of parts with given multiplicity. This is our last example in which the asymptotic expansion of $F(e^{-t})$ can be determined in such a direct way. One has

$$F(e^{-t})=\frac{(1-e^{-t})e^{-dt}}{(1-e^{-dt})(1-e^{-(d+1)t})}=\frac{1}{e^{dt}-1}-\frac{1}{e^{(d+1)t}-1}=\frac{1}{d(d+1)t}\frac{t}{2}+\mathcal{O}(t^3),$$

which leads to the following asymptotics for the average number of parts whose multiplicity is d:

$$\frac{\sqrt{6n}}{\pi d(d+1)}+\frac{3}{\pi^2 d(d+1)}+\frac{2\pi^4 d(d+1)-3\pi^2+216}{24\pi^3 d(d+1)\sqrt{6n}}+\frac{54+\pi^4 d(d+1)}{12\pi^4 d(d+1)n}+\mathcal{O}(n^{-3/2}).$$

As mentioned in the introduction, the main term of the mean has already been determined in [3, 12]. For the variance, the same technique yields

$$\frac{(4\pi^2d^3+5\pi^2d^2+\pi^2d-12d-6)\sqrt{6n}}{2\pi^3d^2(d+1)^2(2d+1)}+\frac{3(4\pi^2d^3+5\pi^2d^2+\pi^2d-24d-12)}{2\pi^4d^2(d+1)^2(2d+1)}+\mathcal{O}(n^{-1/2}).$$

Largest part, largest repeated part. Quite frequently, the Mellin transform comes to one's aid when it comes to the computation of asymptotic expansions. As our first example, let us study the sum

$$\sum_{j=1}^{\infty}\frac{e^{-jt}}{1-e^{-jt}}=\sum_{j=1}^{\infty}\frac{1}{e^{jt}-1}.$$

The Mellin transform of this function is easily found to be $\zeta(s)^2\Gamma(s)$, which has poles at 1, 0 and all negative odd integers. By means of a standard technique involving the Mellin inversion formula (see [5] for details), the behaviour at these poles can be translated to an asymptotic expansion around $t=0$:

$$\sum_{j=1}^{\infty}\frac{1}{e^{jt}-1}=\frac{\log(1/t)+\gamma}{t}+\frac{1}{4}-\frac{t}{144}+\mathcal{O}(t^3)$$

Now application of Theorem 3 immediately yields an asymptotic expansion for the mean of the largest part that is repeated at least d times in a random partition of n:

$$\frac{\sqrt{6n}}{2\pi d}\left(\log n+2\gamma-\log\left(\frac{\pi^2d^2}{6}\right)\right)+\frac{3}{2\pi^2 d}\log n+\frac{1}{4}+\frac{3(1+2\gamma-\log(\pi^2d^2/6))}{2\pi^2 d}+\mathcal{O}\left(\frac{\log n}{n}\right).$$

The result of Kessler and Livingston [11] that was mentioned in the introduction is included as a special case ($d = 1$), with an improved error term. For $d = 2$, the formula agrees with the asymptotic expansion found in [8].

Moments of a partition. While this example is perhaps less natural than the ones encountered so far, it shows an interesting phenomenon: one obtains an asymptotic expansion of $F(e^{-t})$ with a particularly strong error term, which in turn leads to strong asymptotic estimates for the mean of the k-th moment in the case that k is odd. First of all, one finds easily that the Mellin transform of

$$F(e^{-t}) = \sum_{j=1}^{\infty} \frac{j^k}{e^{jt} - 1}$$

is given by $\zeta(s)\zeta(s-k)\Gamma(s)$. For even k, this yields poles at $k+1$, 1 and all negative odd integers, so that

$$F(e^{-t}) = \frac{k!\zeta(k+1)}{t^{k+1}} + \frac{\zeta(1-k)}{t} + \frac{\zeta(-k-1)t}{12} - \frac{\zeta(-k-3)t^3}{720} + \dots$$

If k is odd, however, the zeros of $\zeta(s)\zeta(s-k)$ cancel with the negative poles of $\Gamma(s)$, so that $s = k+1$ and $s = 0$ are the only poles (if $k > 1$), and the Mellin transform yields an error term whose order can be any power of t. In fact, one can employ a known technique (see for instance [16]) to obtain a functional equation for $f(t) = F(e^{-t})$ as follows: shifting the path of integration in the Mellin inversion formula to the left, one obtains

$$\begin{aligned} f(t) &= \frac{1}{2\pi i} \int_{k+2-i\infty}^{k+2+i\infty} \zeta(s)\zeta(s-k)\Gamma(s)t^{-s}\,ds \\ &= \frac{k!\zeta(k+1)}{t^{k+1}} - \frac{\zeta(-k)}{2} + \frac{1}{2\pi i} \int_{-1-i\infty}^{-1+i\infty} \zeta(s)\zeta(s-k)\Gamma(s)t^{-s}\,ds. \end{aligned}$$

Now replace s by $k+1-s$ to obtain

$$f(t) = \frac{k!\zeta(k+1)}{t^{k+1}} - \frac{\zeta(-k)}{2} + \frac{1}{2\pi i} \int_{k+2-i\infty}^{k+2+i\infty} \zeta(k+1-s)\zeta(1-s)\Gamma(k+1-s)t^{s-k-1}\,ds.$$

The functional equations of the ζ-function and the Γ-function now yield

$$\begin{aligned} \zeta(k+1-s)\zeta(1-s) &= 2^{1+k-s}\pi^{k-s}\cos\left(\frac{\pi(s-k)}{2}\right)\Gamma(s-k)\zeta(s-k)\cdot 2^{1-s}\pi^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s) \\ &= 2^{2+k-2s}\pi^{k-2s}\sin\left(\frac{\pi k}{2}\right)\sin\left(\frac{\pi s}{2}\right)\cos\left(\frac{\pi s}{2}\right)\Gamma(s-k)\zeta(s)\zeta(s-k)\Gamma(s) \\ &= 2^{1+k-2s}\pi^{k-2s}\sin\left(\frac{\pi k}{2}\right)\sin(\pi s)\Gamma(s-k)\zeta(s)\zeta(s-k)\Gamma(s) \\ &= 2^{1+k-2s}\pi^{k-2s}\sin\left(\frac{\pi k}{2}\right)\sin(\pi s)\cdot\frac{\pi}{\sin(\pi(s-k))\Gamma(k+1-s)}\cdot\zeta(s)\zeta(s-k)\Gamma(s) \\ &= (2\pi)^{k+1-2s}\sin\left(\frac{\pi k}{2}\right)\sin(\pi s)\cdot\frac{1}{-\sin(\pi s)\Gamma(k+1-s)}\cdot\zeta(s)\zeta(s-k)\Gamma(s) \\ &= (-1)^{(k+1)/2}(2\pi)^{k+1-2s}\cdot\frac{1}{\Gamma(k+1-s)}\cdot\zeta(s)\zeta(s-k)\Gamma(s) \end{aligned}$$

so that finally

$$\begin{aligned} f(t) &= \frac{k!\zeta(k+1)}{t^{k+1}} - \frac{\zeta(-k)}{2} + \left(\frac{2\pi i}{t}\right)^{k+1}\cdot\frac{1}{2\pi i}\int_{k+2-i\infty}^{k+2+i\infty}\zeta(s)\zeta(s-k)\Gamma(s)\left(\frac{4\pi^2}{t}\right)^{-s} ds \\ &= \frac{k!\zeta(k+1)}{t^{k+1}} - \frac{\zeta(-k)}{2} + \left(\frac{2\pi i}{t}\right)^{k+1} f\left(\frac{4\pi^2}{t}\right). \end{aligned}$$

Since $f\left(\frac{4\pi^2}{t}\right) \ll \exp\left(-\frac{4\pi^2}{t}\right)$, Theorem 3 now shows that the average k-th moment (k odd, $k > 1$) of a random partition of n is given by

$$k!\zeta(k+1)\left(\frac{\sqrt{24n-1}}{2\pi}\right)^{k+1} \cdot \frac{I_{k-1/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)}{I_{3/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)} - \frac{\zeta(-k)}{2} + \mathcal{O}\left(\exp\left(-n^{1/2-\epsilon}\right)\right)$$

for any $\epsilon > 0$. Note that the error term is much stronger than in all previous examples.

Longest run. Our final example is also the most complicated one. As in the previous example, we encounter the situation that the Mellin transform of $F(e^{-t})$ has only finitely many poles. The technique to obtain a very precise asymptotic expansion is similar to the previous example, but the details are somewhat more intricate.

Recall the generating function for the m-th moment, which is given by

$$P(x) \cdot \sum_{k=1}^{\infty} (k^m - (k-1)^m)\left(1 - P(x^k)^{-1}\right).$$

We are therefore interested in the behaviour of functions of the form

$$\sum_{k=1}^{\infty} k^r \left(1 - P(e^{-kt})^{-1}\right)$$

as $t \to 0$. In the following, it will be convenient to replace t by $2\pi t$, so we study the function

$$G_r(t) = \sum_{k=1}^{\infty} k^r H(kt),$$

where

$$H(t) = 1 - P(e^{-2\pi t})^{-1} = 1 - \prod_{j=1}^{\infty}\left(1 - e^{-2\pi j t}\right).$$

Let us first consider the Mellin transform $H^*(s)$ of H: by (2), H satisfies the functional equation

$$1 - H(t) = \frac{1}{\sqrt{t}} \exp\left(\frac{\pi}{12}\left(t - \frac{1}{t}\right)\right)\left(1 - H\left(\frac{1}{t}\right)\right).$$

Therefore, for $\operatorname{Re} s > 0$,

$$\begin{aligned}
H^*(s) &= \int_0^{\infty} t^{s-1} H(t)\, dt = \int_0^1 t^{s-1} H(t)\, dt + \int_1^{\infty} t^{s-1} H(t)\, dt \\
&= \int_1^{\infty} t^{-s-1} H\left(\frac{1}{t}\right) dt + \int_1^{\infty} t^{s-1} H(t)\, dt \\
&= \int_1^{\infty} t^{-s-1}\left[1 - \sqrt{t}\exp\left(\frac{\pi}{12}\left(\frac{1}{t} - t\right)\right)(1 - H(t))\right] dt + \int_1^{\infty} t^{s-1} H(t)\, dt \\
&= \frac{1}{s} - \int_1^{\infty} t^{-s-1/2} \exp\left(\frac{\pi}{12}\left(\frac{1}{t} - t\right)\right) dt + \int_1^{\infty}\left[t^{-s-1/2}\exp\left(\frac{\pi}{12}\left(\frac{1}{t} - t\right)\right) + t^{s-1}\right] H(t)\, dt.
\end{aligned}$$

Since $H(t)$ decreases exponentially as $t \to \infty$, the integrals converge for any $s \in \mathbb{C}$, and so this gives us a meromorphic continuation of $H^*(s)$ to $\mathbb{C}$ (with a simple pole at $s = 0$). We may thus replace s by $1 - s$ to obtain

$$\begin{aligned}
H^*(1-s) &= \frac{1}{1-s} - \int_1^{\infty} t^{s-3/2}\exp\left(\frac{\pi}{12}\left(\frac{1}{t} - t\right)\right) dt + \int_1^{\infty}\left[t^{s-3/2}\exp\left(\frac{\pi}{12}\left(\frac{1}{t} - t\right)\right) + t^{-s}\right] H(t)\, dt \\
&= -\int_1^{\infty} t^{-s}\, dt - \int_1^{\infty} t^{s-3/2}\exp\left(\frac{\pi}{12}\left(\frac{1}{t} - t\right)\right) dt + \int_1^{\infty}\left[t^{s-3/2}\exp\left(\frac{\pi}{12}\left(\frac{1}{t} - t\right)\right) + t^{-s}\right] H(t)\, dt \\
&= \int_1^{\infty} t^{s-3/2}\exp\left(\frac{\pi}{12}\left(\frac{1}{t} - t\right)\right)(H(t) - 1)\, dt + \int_1^{\infty} t^{-s}\,(H(t) - 1)\, dt
\end{aligned}$$

$$= \int_1^\infty t^{s-3/2} \exp\left(\frac{\pi}{12}\left(\frac{1}{t}-t\right)\right)(H(t)-1)\,dt + \int_0^1 t^{s-2}\left(H\left(\frac{1}{t}\right)-1\right)\,dt$$
$$= \int_1^\infty t^{s-3/2} \exp\left(\frac{\pi}{12}\left(\frac{1}{t}-t\right)\right)(H(t)-1)\,dt + \int_0^1 t^{s-3/2} \exp\left(\frac{\pi}{12}\left(\frac{1}{t}-t\right)\right)(H(t)-1)\,dt$$
$$= \int_0^\infty t^{s-3/2} \exp\left(\frac{\pi}{12}\left(\frac{1}{t}-t\right)\right)(H(t)-1)\,dt$$

for $\operatorname{Re} s > 1$, which is exactly the Mellin transform $K^*(s)$ of

$$K(t) = \frac{1}{\sqrt{t}} \exp\left(\frac{\pi}{12}\left(\frac{1}{t}-t\right)\right)(H(t)-1) = \frac{1}{t}\left(H\left(\frac{1}{t}\right)-1\right) = -\frac{1}{t}\prod_{j=1}^\infty \left(1-e^{-2j\pi/t}\right).$$

Returning to $G_r(t)$, it is clear that the Mellin transform of this function is $G_r^*(s) = H^*(s)\zeta(s-r)$, and thus

$$G_r(t) = \frac{1}{2\pi i}\int_{r+1-i\infty}^{r+1+i\infty} H^*(s)\zeta(s-r)t^{-s}\,ds.$$

We shift the path of integration to the left, picking up residues at $s = r+1$ and $s = 0$:

- At $s = r+1$, the residue is $\frac{H^*(r+1)}{t^{r+1}}$; since $H(t)$ can also be written as

$$H(t) = \sum_{m=1}^\infty (-1)^{m-1}\left(e^{-m(3m-1)\pi t} + e^{-m(3m+1)\pi t}\right)$$

  by virtue of Euler's pentagonal theorem, one also has

$$H^*(s) = \pi^{-s}\Gamma(s)\sum_{m=1}^\infty (-1)^{m-1}\left(\frac{1}{(m(3m-1))^s} + \frac{1}{(m(3m+1))^s}\right)$$

  and thus

$$H^*(r+1) = \frac{r!}{\pi^{r+1}}\sum_{m=1}^\infty (-1)^{m-1}\left(\frac{1}{(m(3m-1))^{r+1}} + \frac{1}{(m(3m+1))^{r+1}}\right).$$

  In particular,

$$H^*(1) = \frac{2\sqrt{3}\pi - 9}{3\pi} \qquad \text{and} \qquad H^*(2) = \frac{54 - 8\sqrt{3}\pi - \pi^2}{2\pi^2}$$

  after a few manipulations.
- The residue of $H^*(s)$ at $s = 0$ is 1 by the above calculations, and so the residue of $H^*(s)\zeta(s-r)t^{-s}$ is $\zeta(-r)$.

Hence we have

$$G_r(t) = \frac{H^*(r+1)}{t^{r+1}} + \zeta(-r) + \frac{1}{2\pi i}\int_{-1-i\infty}^{-1+i\infty} H^*(s)\zeta(s-r)t^{-s}\,ds$$
$$= \frac{H^*(r+1)}{t^{r+1}} + \zeta(-r) + \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} H^*(1-s)\zeta(1-s-r)t^{s-1}\,ds$$
$$= \frac{H^*(r+1)}{t^{r+1}} + \zeta(-r) + \frac{1}{(2\pi)^r t}\cdot\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} 2\cos\left(\frac{\pi(s+r)}{2}\right)\Gamma(s+r)\zeta(s+r)K^*(s)\left(\frac{2\pi}{t}\right)^{-s}\,ds$$

by the functional equation of the zeta function and the equation for $H^*(1-s)$ that was deduced above. This can be written as

$$G_r(t) = \frac{H^*(r+1)}{t^{r+1}} + \zeta(-r) + \frac{1}{(2\pi)^r t}\sum_{k=1}^\infty k^{-r}\Phi_r\left(\frac{2\pi k}{t}\right),$$

where Φ_r is the function whose Mellin transform is

$$\Phi_r^*(s) = 2\cos\left(\frac{\pi(s+r)}{2}\right)\Gamma(s+r)K^*(s).$$

Noting that $\Gamma(s+r)$ is the Mellin transform of $t^r e^{-t}$, while $K^*(s)$ is the Mellin transform of $K(s) = -\frac{1}{t}\prod_{j=1}^{\infty}\left(1-e^{-2j\pi/t}\right)$, we find that $\Gamma(s+r)K^*(s)$ is the transform of the convolution

$$\begin{aligned} L_r(t) &= -\int_0^\infty x^r e^{-x} \frac{x}{t} \prod_{k=1}^{\infty}\left(1-e^{-2k\pi x/t}\right) \frac{dx}{x} \\ &= -\frac{1}{t}\int_0^\infty x^r e^{-x} \sum_{m\in\mathbb{Z}} (-1)^m e^{-m(3m-1)\pi x/t}\, dx \\ &= -\frac{r!}{t}\sum_{m\in\mathbb{Z}} \frac{(-1)^m}{(1+m(3m-1)\pi/t)^{r+1}} = -r!t^r \sum_{m\in\mathbb{Z}} \frac{(-1)^m}{(t+m(3m-1)\pi)^{r+1}}, \end{aligned}$$

making use of Euler's pentagonal theorem again. This can be simplified further:

$$\begin{aligned} L_r(t) &= -(-t)^r \cdot \frac{d}{dt^r} \sum_{m\in\mathbb{Z}} \frac{(-1)^m}{t+m(3m-1)\pi} = -\frac{(-t)^r}{3\pi}\cdot\frac{d}{dt^r}\sum_{m\in\mathbb{Z}}\frac{(-1)^m}{m^2-\frac{m}{3}+\frac{t}{3\pi}} \\ &= -\frac{(-t)^r}{3\pi}\cdot\frac{d}{dt^r}\sum_{m\in\mathbb{Z}}\frac{(-1)^m}{\left(m-\frac{1}{6}+\sqrt{\frac{1}{36}-\frac{t}{3\pi}}\right)\left(m-\frac{1}{6}-\sqrt{\frac{1}{36}-\frac{t}{3\pi}}\right)} \\ &= (-t)^r\cdot\frac{d}{dt^r}\frac{1}{\pi\sqrt{1-\frac{12t}{\pi}}}\sum_{m\in\mathbb{Z}}(-1)^m\left(\frac{1}{m-\frac{1}{6}+\sqrt{\frac{1}{36}-\frac{t}{3\pi}}}-\frac{1}{m-\frac{1}{6}-\sqrt{\frac{1}{36}-\frac{t}{3\pi}}}\right) \\ &= (-t)^r\cdot\frac{d}{dt^r}\frac{1}{\pi\sqrt{1-\frac{12t}{\pi}}}\left(\pi\csc\left(\frac{\pi}{6}+\frac{\pi}{6}\sqrt{1-\frac{12t}{\pi}}\right)-\pi\csc\left(\frac{\pi}{6}-\frac{\pi}{6}\sqrt{1-\frac{12t}{\pi}}\right)\right) \\ &= (-t)^r\cdot\frac{d}{dt^r}\frac{1}{\sqrt{1-\frac{12t}{\pi}}}\left(\csc\left(\frac{\pi}{6}+\frac{\pi}{6}\sqrt{1-\frac{12t}{\pi}}\right)-\csc\left(\frac{\pi}{6}-\frac{\pi}{6}\sqrt{1-\frac{12t}{\pi}}\right)\right) \end{aligned}$$

by virtue of the partial fraction decomposition of the cosecant. This function is meromorphic on the entire complex plane, in spite of the square root. Finally, since $2\cos(\frac{\pi(s+r)}{2}) = i^{-s-r}+(-i)^{-s-r}$, we find that $\Phi_r^*(s)$ is the Mellin transform of

$$\Phi_r(t) = i^{-r}L_r(it) + (-i)^{-r}L_r(-it).$$

To justify this formally, note first that the function L_r can be estimated by $L_r(z) \ll \exp(-C\sqrt{|z|})$ for some constant C inside the cone $\{z : |\arg z| \le 3\pi/4\}$. The Mellin transform of $L_r(it)$ is given by

$$\int_0^\infty t^{s-1}L_r(it)\,dt = i^{-s}\int_0^{i\infty} t^{s-1}L_r(t)\,dt = i^{-s}\lim_{R\to\infty}\int_0^{iR} t^{s-1}L_r(t)\,dt.$$

Now replace the line segment from 0 to iR by a line segment from 0 to R and a quarter-circle with parametrisation $t = Re^{i\theta}$. Then we get

$$\int_0^\infty t^{s-1}L_r(it)\,dt = i^{-s}\lim_{R\to\infty}\left(\int_0^R t^{s-1}L_r(t)\,dt + i^{1-s}R^s\int_0^{\pi/2} e^{i\theta s}f(Re^{i\theta})\,d\theta\right).$$

The integral along the quarter-circle tends to zero as $R\to\infty$ by the estimate given above. Hence we end up with $i^{-s}L_r^*(s)$, as claimed. The Mellin transform of $L_r(-it)$ is treated analogously.

It is not difficult to show now that $\Phi_r(t) = \mathcal{O}\left(t^{(r-1)/2}\exp\left(-\sqrt{\frac{\pi t}{6}}\right)\right)$ as $t\to\infty$, so that we finally obtain

$$G_r(t) = \frac{H^*(r+1)}{t^{r+1}} + \zeta(-r) + \mathcal{O}\left(t^{-(r+1)/2}\exp\left(-\sqrt{\frac{\pi^2}{3t}}\right)\right)$$

as $t \to 0$. Now we can return to the study of the moments of the longest run in integer partitions. As mentioned in the beginning, the generating function for the m-th moment is given by

$$P(x) \cdot \sum_{k=1}^{\infty} (k^m - (k-1)^m) \left(1 - \frac{1}{P(x^k)}\right) = P(x) \cdot \sum_{r=0}^{m-1} (-1)^{m-r-1} \binom{m}{r} \sum_{k \geq 1} k^r \left(1 - \frac{1}{P(x^k)}\right),$$

which is of the form $P(x)F(x)$ with

$$\begin{aligned} F(e^{-t}) &= \sum_{r=0}^{m-1} (-1)^{m-r-1} \binom{m}{r} G_r \left(\frac{t}{2\pi}\right) \\ &= \sum_{r=0}^{m-1} (-1)^{m-r-1} \binom{m}{r} \left(\frac{(2\pi)^{r+1} H^*(r+1)}{t^{r+1}} + \zeta(-r)\right) + \mathcal{O}\left(t^{-m/2} \exp\left(-\sqrt{\frac{\pi}{6t}}\right)\right). \end{aligned}$$

Now we are able to apply Theorem 3 and obtain a very strong error term again, as in the previous example; in particular, we find that the mean is given by

$$\frac{(2\sqrt{3}\pi - 9)(24n-1)}{3(\pi\sqrt{24n-1} - 6)} - \frac{1}{2} + \mathcal{O}\left(\exp\left(-Cn^{1/4}\right)\right),$$

and the variance by

$$\frac{(24n-1)^{3/2}((135\pi - 12\sqrt{3}\pi^2 - 7\pi^3)\sqrt{24n-1} + (18\pi^2 + 144\sqrt{3}\pi - 972))}{3\pi(\pi\sqrt{24n-1} - 6)^2} + \frac{1}{12} + \mathcal{O}\left(\exp\left(-Cn^{1/4}\right)\right)$$

for some constant C. Going on to higher moments, one also obtains an alternative approach to the limiting distribution of the longest run that was determined in [14]: let R_n denote the longest run in a random partition of n. Considering only the most significant term, we see that the m-th moment of R_n is

$$\mathbb{E}(R_n^m) = mH^*(m)(24n)^{m/2} + \mathcal{O}(n^{(m-1)/2}).$$

Therefore, the m-th moment of the renormalised random variable $n^{-1/2}R_n$ tends to $mH^*(m)(24)^{m/2}$. It is not difficult to show that these are exactly the moments associated with the distribution function

$$\Psi(x) = \prod_{j=1}^{\infty} \left(1 - e^{-\pi j x/\sqrt{6}}\right).$$

The moments grow sufficiently slowly for the distribution to be characterised by its moments. Hence convergence of moments implies weak convergence of $n^{-1/2}R_n$ to this limiting distribution (see for example [6, Theorem C.2]).

Let us finally consider the problem of determining the probability that k is one of the parts of largest multiplicity. Recall the generating function for all partitions that have this property, which is given by

$$P(x) \cdot \sum_{\ell=1}^{\infty} \frac{x^{k\ell}(1-x^k)}{(1-x^{k(\ell+1)})P(x^{\ell+1})}.$$

We have to study the behaviour of the second factor as $x \to 1$. Substitute $x = e^{-t}$ and simplify to obtain

$$F_k(e^{-t}) = \sum_{\ell=1}^{\infty} \frac{e^{-k\ell t}(1-e^{-kt})}{(1-e^{-k(\ell+1)t})P(e^{-(\ell+1)t})} = \sum_{\ell=1}^{\infty} \frac{e^{kt}-1}{(e^{k(\ell+1)t}-1)P(e^{-(\ell+1)t})} = (kt + \mathcal{O}(t^2)) \sum_{\ell=2}^{\infty} M_k(\ell t),$$

where $M_k(t) = \frac{1}{(e^{kt}-1)P(e^{-t})}$. The Mellin transform of $M_k(t)$,

$$M_k^*(s) = \int_0^\infty \frac{t^{s-1}}{(e^{kt}-1)P(e^{-t})}\, dt,$$

exists for all complex s, since $P(e^{-t})$ tends to 0 faster than any power of t as $t \to 0$ (by (2)) and to 1 as $t \to \infty$. Therefore, the Mellin transform of the sum $\sum_{\ell=2}^{\infty} M_k(\ell t)$ is $(\zeta(s)-1)M_k^*(s)$, which only has a simple pole at $s = 1$ with residue 1. Applying the inverse Mellin transform, we thus find that

$$\sum_{\ell=2}^{\infty} M_k(\ell t) \sim \frac{M_k^*(1)}{t},$$

so that finally

$$F_k(e^{-t}) \sim k M_k^*(1) = \int_0^\infty \frac{k}{(e^{kt}-1)P(e^{-t})}\,dt = \int_0^1 \frac{kx^{k-1}}{1-x^k}\prod_{j=1}^{\infty}(1-x^j)\,dx.$$

Hence, by Theorem 2, the probability that no part in a random partition of n has larger multiplicity than k tends to $p_k = kM_k^*(1)$. The same is true for the probability that all parts have strictly smaller multiplicity than k (by a similar argument). Hence the probability that there are two or more longest runs tends to 0. Finally, since the probabilities have to add up to 1, we obtain the rather curious identity

$$1 = \sum_{k=1}^{\infty} p_k = \sum_{k=1}^{\infty}\int_0^1 \frac{kx^{k-1}}{1-x^k}\prod_{j=1}^{\infty}(1-x^j)\,dx = \int_0^1 \left(\sum_{k=1}^{\infty}\sigma(k)x^{k-1}\right)\prod_{j=1}^{\infty}(1-x^j)\,dx,$$

where $\sigma(k)$ denotes the sum of all divisors of k. Numerical values of p_k are given in the following table:

k	1	2	3	4	5
p_k	0.51609432	0.21321189	0.10730957	0.05975045	0.03548875
k	6	7	8	9	10
p_k	0.02207159	0.01421668	0.00941619	0.00638121	0.00440862

It is natural to ask how the sequence p_k would behave for $k \to \infty$. This can be done by adopting ideas that are frequently used in the theory of partitions. We write

$$(6)\quad p_k = \int_0^\infty \frac{k}{(e^{kt}-1)P(e^{-t})}\,dt = \int_0^\infty \frac{ke^{-kt}}{P(e^{-t})}\,dt + \int_0^\infty \frac{ke^{-2kt}}{P(e^{-t})}\,dt + \int_0^\infty \frac{k}{e^{2kt}(e^{kt}-1)P(e^{-t})}\,dt.$$

The third integral can be estimated using the functional equation (2)

$$\int_0^\infty \frac{k}{e^{2kt}(e^{kt}-1)P(e^{-t})}\,dt = \int_0^\infty \frac{k}{e^{2kt}(e^{kt}-1)P(e^{-4\pi^2/t})}\sqrt{\frac{2\pi}{t}}\exp\left(-\frac{\pi^2}{6t}+\frac{t}{24}\right)dt$$
$$\le \int_0^\infty \sqrt{\frac{2\pi}{t^3}}\exp\left(-\frac{\pi^2}{6t}-\left(2k-\frac{1}{24}\right)t\right)dt = \mathcal{O}\left(\exp\left(-2\pi\sqrt{\frac{k}{3}}\right)\right).$$

The remaining integrals are again treated by using the functional equation (2):

$$\int_0^\infty ke^{-kt}\frac{1}{P(e^{-t})}\,dt = \int_0^\infty ke^{-kt}\frac{1}{P(e^{-4\pi^2/t})}\sqrt{\frac{2\pi}{t}}\exp\left(-\frac{\pi^2}{6t}+\frac{t}{24}\right)dt$$
$$= \int_0^\infty ke^{-kt}\sqrt{\frac{2\pi}{t}}\exp\left(-\frac{\pi^2}{6t}+\frac{t}{24}\right)dt + \int_0^\infty ke^{-kt}\sqrt{\frac{2\pi}{t}}\left(\frac{1}{P(e^{-4\pi^2/t})}-1\right)\exp\left(-\frac{\pi^2}{6t}+\frac{t}{24}\right)dt.$$

The first of these two integrals evaluates to

$$\frac{4\pi k e^{-\frac{1}{6}\pi\sqrt{24k-1}}}{\sqrt{8k-\frac{1}{3}}},$$

whereas the second integral is estimated as $\mathcal{O}\left(\sqrt{k}\exp\left(-5\pi\sqrt{2k/3}\right)\right)$ using $1 - 1/P(x) \le \frac{3}{2}x$. The middle integral in (6) can be treated analogously.

Thus we have found

$$p_k = \frac{4\pi\sqrt{3}ke^{-\frac{1}{6}\pi\sqrt{24k-1}}}{\sqrt{24k-1}} + \mathcal{O}\left(\sqrt{k}\exp\left(-2\pi\sqrt{\frac{k}{3}}\right)\right) = \pi\sqrt{2k}e^{-\pi\sqrt{\frac{2k}{3}}}\left(1+\mathcal{O}\left(\frac{1}{k}\right)\right).$$

It is clear that the above computations can be extended to yield an asymptotic expansion of p_k.

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