Decomposing the hypercube Q_n into n isomorphic edge-disjoint trees

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Abstract

We show that the edge set of the n-dimensional hypercube Q_n is the disjoint union of the edge sets of n isomorphic trees.

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1. Introduction

The problem of finding edge-disjoint trees in a hypercube arises for example in the context of parallel computing [3]. Independent of applications it is of high aesthetic appeal. The hypercube of dimension n, denoted by Q_n, comprises 2^n vertices each corresponding to a distinct binary string of length n. Two vertices are adjacent if and only if their corresponding binary strings differ in exactly one position. Since each vertex of Q_n has degree n, the number of edges is $n2^{n-1}$. A variety of decomposability options derive from this fact. In the remainder of the introduction we focus on three of them. The first two have been dealt with before in the literature, the third is the topic of this article.

1. Roskind and Tarjan [7] obtained a polynomial-time algorithm to find the maximum number of edge-disjoint spanning trees in a connected

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graph G. For Q_n, edge-counting yields $\lfloor \frac{n}{2} \rfloor$ as an upper bound; in fact equality holds and the answer is $\lfloor \frac{n}{2} \rfloor$ for many n-regular graphs (see [4]). For even n, an explicit construction of $n/2$ edge-disjoint spanning trees in Q_n appears in [1]. These trees are *not* isomorphic. There are $n/2$ leftover edges, which form a path. For odd n, an explicit construction has not yet been found.

2. So decomposing the edge set of Q_n into *spanning* trees is impossible. What about a decomposition into *isomorphic* trees? Surprisingly, for *every* tree T with n edges, the edge set of Q_n can by [2] be covered by 2^{n-1} trees isomorphic to T. This has been extended in [5, 6], and will be followed up in Section 4.
3. As opposed to decomposing Q_n into 2^{n-1} isomorphic trees of size n, in this article we decompose Q_n into n trees of size 2^{n-1} isomorphic to some tree T_n. Our tree T_n needs to have a very specific shape.

In Section 2, we present two equivalent definitions of T_n: a direct one (used in the proof of the main result in Section 3) and a recursive one.

2. Construction of T_n

Rather than dealing with $0,1$-strings, it will be more convenient to let the vertex set $V(Q_n)$ be the family of subsets of $[n]$, where $[n] := \{1, 2, \ldots, n\}$, with $X, X' \in V(Q_n)$ being adjacent if and only if their symmetric difference has size 1. Before tackling the formal definition of T_n, let us show some small cases (Figure 1); of course, we must have $|V(T_n)| = 2^{n-1} + 1$ for every n.

The following notation shall be employed throughout the article. Permutations π of $[n]$ will be written in cycle notation and to the *right* of the argument, separated by a dot. Thus if $\pi = (3,4)(2,5,7)$ (where $n \geq 7$), then $3 \cdot \pi = 4$ and $7 \cdot \pi = 2$. Each permutation of $[n]$ induces, in the obvious way, a permutation of the vertex set $V(Q_n)$. Using this notation, applying the permutation π above to the vertex (say) $X = \{1,2,4\}$ of Q_n yields $X \cdot \pi = \{1,5,3\}$. Furthermore, for $\{X_1, X_2, \ldots\} \subseteq V(Q_n)$ we set $\{X_1, X_2, \ldots\} \cdot \pi := \{X_1 \cdot \pi, X_2 \cdot \pi, \ldots\}$. If T is a subgraph of Q_n and π a permutation, then $T \cdot \pi$ is defined as the subgraph of Q_n that has vertex set $V(T) \cdot \pi$ and edge set $\{\{X \cdot \pi, Y \cdot \pi\} \ : \ \{X, Y\} \in E(T)\}$.

Let us resume the discussion of our trees T_n and put $n = 3$ and $\pi = (1,2,3)$. See Figure 2, where for clarity we drop brackets and commas in naming sets.

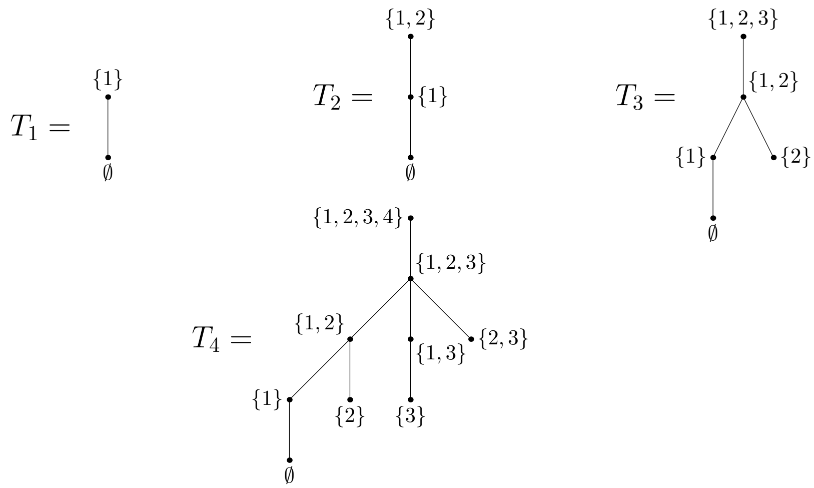

Figure 1: The trees T_1 to T_4.

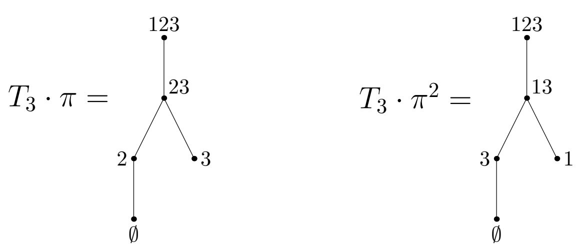

Figure 2: Application of permutations to a tree.

The edge-disjoint decomposition of Q_3 into the isomorphic rooted trees $T_3,\ T_3 \cdot \pi$, and $T_3 \cdot \pi^2$ is now apparent. The corresponding decomposition of Q_4 into the trees $T_4 \cdot \pi^i$ $(0 \leq i < 4,\ \pi := (1, 2, 3, 4))$ is already more surprising (see Figure 3).

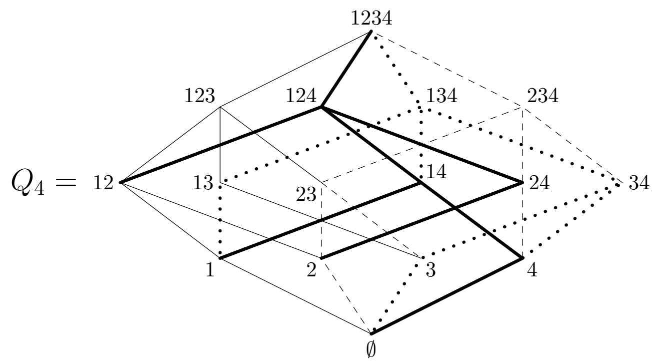

Figure 3: Decomposition of the four-dimensional hypercube.

Let us now define T_n in general. For $n \in \mathbb{N}$ let $V(T_n)$ consist of all subsets of $[n-1]$ plus the *root* vertex $[n]$. Furthermore, we assign a *parent vertex* $p(v)$ to every vertex v other than $[n]$, namely the set $v \cup \{x(v)\}$, where

$$x(v) \quad = \quad \min(\mathbb{N} \setminus v).$$

In other words, to obtain the parent vertex of v, we add the smallest positive integer that is not yet contained in the set v. If $v \neq [n-1]$, then this is an element of $[n-1]$; if $v = [n-1]$, then $x(v) = n$ and $p(v) = [n]$. Now the edge set $E(T_n)$ consists of all pairs $(v, p(v))$.

The fact that every vertex other than $[n]$ is adjacent to a unique superset implies that the resulting graph T_n has no cycles; since it explicitly has $2^{n-1} + 1$ vertices and 2^{n-1} edges, it therefore is a tree.

Let us show that T_n can be constructed in a recursive manner as well. For starters, observe that the subtrees $S_{5,1}, \ldots, S_{5,4}$ of T_5 in Figure 4 are isomorphic to $T_1, \ldots, T_4$ respectively. Generally for $n \geq 2$ and $1 \leq i \leq n-1$, define $S_{n,i}$ as the tree isomorphic to T_i obtained by adding the elements $i+1, \ldots, n-1$ to the names of all the vertices. In particular, all vertices of $S_{n,i}$ belong to $V(Q_{n-1})$ and $S_{n,i}$ has the root $[n-1]$.

Proposition 1. *The tree T_n $(n \geq 2)$ is isomorphic to the tree obtained by gluing $T_1, \ldots, T_{n-1}$ at their roots and then attaching a new root.*

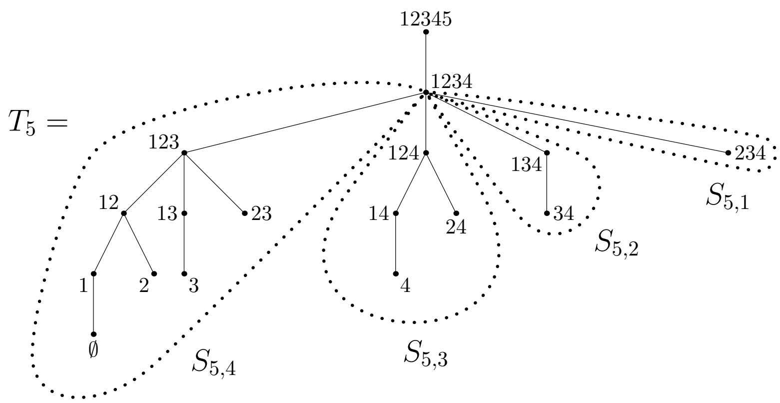

Figure 4: T_5 and its subtrees.

Proof. Because $T_i \simeq S_{n,i}$ $(1 \leq i \leq n-1)$ it suffices to show the following:

> Disregarding the common root $[n-1]$, the vertex sets of $S_{n,1}, \ldots, S_{n,n-1}$ are mutually disjoint and have union $V(Q_{n-1}) \setminus \{[n-1]\}$.

Indeed, disjointness follows from the fact that each vertex $v \neq [n-1]$ of $S_{n,i}$ has the property $\{i+1, i+2, \ldots, n-1\} \subseteq v$ but $i \notin v$. That the union is $V(Q_{n-1}) \setminus \{[n-1]\}$ follows from disjointness together with

$$(|V(S_{n,1})| - 1) + \cdots + (|V(S_{n,n-1})| - 1) = 1 + 2 + \cdots + 2^{n-2} = 2^{n-1} - 1.$$

□

3. Main result

Our main result states that the edges of the n-dimensional hypercube Q_n can be decomposed into isomorphic copies of T_n.

Theorem 2. *Let π be the cycle $(1, 2, \ldots, n)$. If $\{X, Y\} \in E(T_n)$, then $\{X, Y\} \notin E(T_n \cdot \pi^i)$ for all $0 < i < n$. In particular, the trees $T_n \cdot \pi^i$ $(0 < i < n)$ decompose Q_n.*

Proof. Obviously, if the edge $\{X, Y\}$ occurs in several trees, the larger of the two sets X and Y is the parent vertex in *all* these trees. Hence it suffices to show the following: if a set X belongs to the vertex sets of both T_n and

$T_n \cdot \pi^i$, then the parent vertices of X in T_n and $T_n \cdot \pi^i$ are distinct. This will imply that T_n and $T_n \cdot \pi^i$ have no common edges.

Since the vertex set of T_n consists of $[n]$ plus all subsets of $[n]$ that omit n, the vertex set of $T_n \cdot \pi^i$ consists of $[n]$ plus all subsets of $[n]$ that omit i.

Hence, a set X can only belong to the vertex sets of both T_n and $T_n \cdot \pi^i$ if $i, n \notin X$ or if $X = [n]$. Since $X = [n]$ has no parent vertex, it suffices to consider the case that $i, n \notin X$. Let $X \cup \{x\}$ and $X \cup \{y\}$ be the parent vertices of X in T_n and $T_n \cdot \pi^i$ respectively.

Since $i \notin X$, we have $1 \leq x \leq i$ by the definition of T_n. Let Z be the vertex of T_n such that $X = Z \cdot \pi^i$. From $n \notin X$ follows $(n-i) \notin Z$, and so the parent vertex $Z \cup \{z\}$ of Z satisfies $z \in \{1, 2, \ldots, n-i\}$. This forces $y = z \cdot \pi^i \in \{i+1, \ldots, n\}$. Altogether, this shows that $x \neq y$, and so the parents $X \cup \{x\}$ and $X \cup \{y\}$ are distinct, as claimed.

Therefore, the edge sets of T_n and $T_n \cdot \pi^i$ are disjoint for all $0 < i < n$. This also implies that the edge sets of $T_n \cdot \pi^i$ and $T_n \cdot \pi^j$ are disjoint for all $0 \leq i < j < n$: if they were not disjoint, the edge sets of T_n and $T_n \cdot \pi^{j-i}$ would not be disjoint either, a contradiction.

Since T_n has 2^{n-1} edges and the specified trees are pairwise edge-disjoint, it follows immediately that they decompose Q_n. $\square$

4. Three related matters

After discovering T_n in its recursive guise (Proposition 1), we noticed that T_n coincides with the *set enumeration tree* introduced in [8], except for the dual labeling of nodes (in [8] the root is ϕ, not $[n]$). Not the theoretical properties of T_n are considered in [8], but its usefulness as (to cite from the abstract) "a vehicle for representing sets and/or enumerating them in a best-first fashion". Many subsequent uses of the SE-tree, e.g. in data mining, can be surveyed with Google-Scholar.

Second, let F be a subset of $E(Q_n)$. Following Ramras, call F a *fundamental set* for Q_n with respect to group G if G is a subgroup of $\text{Aut}(Q_n)$ such that $\{g(E) : \ g \in G\}$ form an edge decomposition of Q_n. It is shown in [6] that if $|F| = n$ and the graph induced by F is connected with at most one cycle (e.g. a tree), then F is a fundamental set for Q_n with respect to some group. Results about fundamental $2n$-element sets are contained in [5]. Our tree T_n fits into this framework in that $E(T_n)$ is a fundamental size 2^{n-1} set for the group $G \subseteq \text{Aut}(Q_n)$ that is induced by the cyclic group $\langle \pi \rangle \subseteq S_n$.

Third, every rooted tree T (e.g. $T = T_n$) becomes a unique partially ordered set $(T, \leq)$ when the root is postulated as largest element. It is an open problem to find necessary or sufficient conditions for a rooted tree $(T, \leq)$ to be cover-preserving order-embeddable into $(Q_n, \subseteq)$. That is, we want a map $\phi : T \to Q_n$ that satisfies

1. $(\forall x, y \in T) \quad x \leq y \quad \Leftrightarrow \quad \phi(x) \subseteq \phi(y)$,
2. $(\forall x, y \in T) \quad x \prec y \quad \Rightarrow \quad \phi(x) \prec \phi(y)$.

Here $x \prec y$ means that $x < y$ and $(x < z \leq y \Rightarrow z = y)$. We mention that for posets $(P, \leq)$ with universal bounds 0 and 1 the problem is settled in [9] in terms of the chromatic number of some auxiliary graph. Notice that $(T_4, \leq)$ is *not* order-embedded in Q_4: while $\{2\} \subseteq \{2, 3\}$ in Q_4, the corresponding vertices are not comparable in $(T_4, \leq)$.

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References

[1] B. Barden, R. Libeskind-Hadas, J. Davis, W. Williams, On edge-disjoint spanning trees in hypercubes, Inform. Process. Lett. 70 (1999) 13–16.

[2] J.F. Fink, On the decomposition of n-cubes into isomorphic trees, J. Graph Theory 14 (1990) 405–411.

[3] F.T. Leighton, Introduction to parallel algorithms and architectures, Morgan Kaufmann, San Mateo, CA, 1992.

[4] E.M. Palmer, On the spanning tree packing number of a graph: a survey, Discrete Math. 230 (2001) 13–21. Paul Catlin memorial collection (Kalamazoo, MI, 1996).

[5] M. Ramras, Symmetric edge-decompositions of hypercubes, Graphs Combin. 7 (1991) 65–87.

[6] M. Ramras, Fundamental subsets of edges of hypercubes, Ars Combin. 46 (1997) 3–24.

[7] J. Roskind, R.E. Tarjan, A note on finding minimum-cost edge-disjoint spanning trees, Math. Oper. Res. 10 (1985) 701–708.

[8] R. Rymon, Search through systematic set enumeration, 1992. Tech. Report (CIS), University of Pennsylvania.

[9] M. Wild, Cover-preserving order embeddings into Boolean lattices, Order 9 (1992) 209–232.