IDENTITIES INVOLVING RATIONAL SUMS BY INVERSION AND PARTIAL FRACTION DECOMPOSITION

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Abstract. Identities appearing recently in [2] are treated by inverting them; the resulting sums are evaluated using partial fraction decomposition, following Wenchang Chu [1]. This approach produces a general formula, not only special cases.

1. Introduction

In [2], we find the sums
\[
\sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \frac{1}{x^2+(i+j)x+ij} = \frac{n}{(x+n)^3},
\]
\[
\sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \text{complicated}(k) = \frac{n}{(x+n)^4}.
\]

Here, we will discuss an alternative approach to such identities, which will produce the general formula.

It is based on two principles: inverse pairs and partial fraction decomposition.

I am sure that many other approaches will also work, but I have chosen one that I find useful and appealing. Of course, it is not limited to the sums treated in this paper.

2. Inverse pairs

The following inverse relations are well-known:
\[
b_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} a_k \quad \longleftrightarrow \quad a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} b_k.
\]

They are also easy to prove, e.g., with the use of exponential generating functions.

So, if we want a “nice” answer, like \( b_n = \frac{n}{(x+n)^2} \), we must compute
\[
a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} b_k
\]
to find the “complicated” term.

A technical comment: We will treat \( x = 0 \) as a limiting case, otherwise we would have trouble with \( b_0 \), and we would have to artificially define it as 0.

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The computation of 
\[ a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} \frac{k}{(x+k)^2} \]
(and similar quantities) will be treated in the next section.

3. Partial fraction decomposition

The following approach is based on [1]. Consider (for \( n \geq 1 \))
\[ T := \frac{n!}{z(z-1)\ldots(z-n)(x+z)^2} \]
and perform partial fraction decomposition:
\[ T = \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} \frac{k}{(x+k)^2} \left[ \frac{1}{z-k} + \frac{\lambda}{(x+z)^2} + \frac{\mu}{x+z} \right]. \]
Now we multiply this relation by \( z \) and let \( z \to \infty \) to find
\[ 0 = \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} \frac{k}{(x+k)^2} + \mu. \]
This evaluates the sum:
\[ \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \frac{k}{(x+k)^2} = (-1)^n \mu. \]
Now
\[ (-1)^n \mu = (-1)^n [(x+z)^{-1}] \frac{n!}{z(z-1)\ldots(z-n)(x+z)^2} \frac{z}{(z-1)\ldots(z-n)} \]
\[ = (-1)^n [(x+z)^{1}] \frac{n!}{(z-1)\ldots(z-n)} \]
\[ = [z^1] \frac{n!}{(1+x-z)\ldots(n+x-z)} \]
\[ = \frac{n!}{(1+x)\ldots(n+x)} \sum_{k=1}^{n} \frac{1}{k+x}. \]
This produces the identity
\[ \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \frac{1}{(x+k)^2} \sum_{j=1}^{k} \frac{1}{j+x} = \frac{n}{(x+n)^2}. \]
This instance was the warm-up for the general instance \( b_n = \frac{n}{(x+n)^{d+1}} \), which is not much more complicated.
Analogous computations lead to

\[ \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \frac{k}{(x+k)^{d+1}} = (-1)^n \mu \]

with

\[ (-1)^n \mu = (-1)^n [(x+z)^{-1}] \frac{n!}{z(z-1) \ldots (z-n)} \frac{z}{(x+z)^{d+1}} \]

\[ = (-1)^n [(x+z)^d] \frac{n!}{z(z-1) \ldots (z-n)} \]

\[ = [z^d] \frac{n!}{(1+x-z) \ldots (n+x-z)} \]

\[ = \frac{1}{(1+x) \ldots (n+x)} \frac{n!}{[z^d]} \frac{1}{(1-z) \ldots (1-z/n+x)} \]

\[ = \frac{1}{(x+n)} \frac{n!}{[z^d]} \exp \left( \log \frac{1}{1-x/z} + \ldots + \log \frac{1}{1-z/n+x} \right) \]

\[ = \frac{1}{(x+n)} \frac{n!}{[z^d]} \exp \left( \sum_{k=1}^{n} \sum_{j \geq 1} \frac{z^j}{j(k+x)^j} \right) \]

\[ = \frac{1}{(x+n)} \frac{n!}{[z^d]} \exp \left( \sum_{j \geq 1} \frac{s_{n,j} z^j}{j} \right) \]

\[ = \frac{1}{(x+n)} \frac{n!}{[z^d]} \prod_{j \geq 1} \sum_{l \geq 0} \frac{s_{n,j}^l z^{jl}}{l!j^l} , \]

with

\[ s_{n,j} = \sum_{k=1}^{n} \frac{1}{(k+x)^j} . \]

Consequently

\[ (-1)^n \mu = \frac{1}{(x+n)} \sum_{l_1+2l_2+3l_3+\ldots=d} \frac{s_{l_1}^{l_1} s_{l_2}^{l_2} \ldots}{l_1! l_2! \ldots l_1! 2^{l_2} \ldots} . \]

**Theorem 1.**

\[ \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \frac{k}{(x+k)^{d+1}} = \frac{1}{(x+n)} \sum_{l_1+2l_2+3l_3+\ldots=d} \frac{s_{l_1}^{l_1} s_{l_2}^{l_2} \ldots}{l_1! l_2! \ldots l_1! 2^{l_2} \ldots} \]

with

\[ s_{n,j} = \sum_{k=1}^{n} \frac{1}{(k+x)^j} . \]
For instance, we recover the formula for $d = 2$, since
\[
\frac{s_{n,1} + s_{n,2}}{2} = \sum_{1 \leq i < j \leq n} \frac{1}{x^2 + (i + j)x + ij},
\]
as one can easily check. The other instance $d = 3$, given in [2] evaluates here handily as $s_{n,1}^3/6 + s_{n,1}s_{n,2}/2 + s_{n,3}/3$.

REFERENCES


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