A CONTINUED FRACTION EXPANSION FOR A q-TANGENT FUNCTION:
AN ELEMENTARY PROOF

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ABSTRACT. We prove a continued fraction expansion for a certain q-tangent function that was conjectured by the present writer, then proved by Fulmek, now in a completely elementary way.

1. Introduction

In [3], the present writer defined the following q-trigonometric functions

\[ \sin_q(z) = \sum_{n \geq 0} \frac{(-1)^n z^{2n+1}}{[2n+1]_q!} q^{n^2}, \]
\[ \cos_q(z) = \sum_{n \geq 0} \frac{(-1)^n z^{2n}}{[2n]_q!} q^{n^2}. \]

Here, we use standard q-notations:

\[ [n]_q := \frac{1 - q^n}{1 - q}, \quad [n]_q! := [1]_q [2]_q \ldots [n]_q. \]

These q-functions are variations of Jackson’s [2] q-sine and q-cosine functions.

For the q-tangent function \( \tan_q(z) = \frac{\sin_q(z)}{\cos_q(z)} \), the following continued fraction expansion was conjectured in [3]:

\[ z \tan_q(z) = \frac{z^2}{[1]_q q^0 - \frac{z^2}{[3]_q q^{-2} - \frac{z^2}{[5]_q q^1 - \frac{z^2}{[7]_q q^{-9} - \ldots}}}}. \]

Here, the powers of q are of the form \((-1)^{n-1} n(n-1)/2 - n + 1\).

In [1], this statement was proven using heavy machinery from q-analysis.

Happily, after about 8 years, I was now successful to provide a complete elementary proof that I will present in the next section.
2. The proof

We write
\[
\frac{z \sin_q(z)}{\cos_q(z)} = \frac{z^2}{N_0} = \frac{z^2}{C_1 - \frac{z^2}{N_1}} = \frac{z^2}{C_2 - \frac{z^2}{N_2}} = \ldots,
\]
and set
\[
N_i = \frac{a_i}{b_i}.
\]
This means
\[
N_i = C_{i+1} - \frac{z^2}{N_{i+1}}
\]
or
\[
\frac{z^2}{N_{i+1}} = C_{i+1} - N_i
\]
and
\[
\frac{b_{i+1}z^2}{a_{i+1}} = C_{i+1} - \frac{a_i}{b_i} = \frac{C_{i+1}b_i - a_i}{b_i}.
\]
Therefore \(a_i = b_{i-1}\) and
\[
b_{i+1}z^2 = C_{i+1}b_i - b_{i-1}.
\]
The initial conditions are
\[
b_{-1} = \cos_q(z) \quad \text{and} \quad b_0 = \sum_{n \geq 0} \frac{(-1)^n q^{n^2} z^{2n}}{[2n + 1]_q!}.
\]
The constants \(C_i\) guarantee that all the \(b_i\) are power series, i.e., they make the constant term in \(C_{i+1}b_i - b_{i-1}\) disappear. Our goal is to show that \(C_i = [2i - 1]_q q^{(i-1)(i+1)/2 - i+1}\) are the (unique) numbers that do this. We are proving the claim by proving the following explicit formula for \(b_i\):
\[
b_i = \sum_{n \geq 0} \frac{(-1)^n z^{2n}}{[2n + 2i + 1]_q!} \left( \prod_{j=1}^{i} [2n + 2j]_q \right) q^{(n + \lfloor \frac{i+1}{2} \rfloor)^2 + [i \text{ odd}] \lfloor \frac{i+1}{2} \rfloor}.
\]
Note that the \(C_i\) are uniquely determined by the imposed condition, and since the \(b_i\) are power series, we are done once we prove this formula by induction. The first two instances satisfy this, and we do the induction step now:
\[
C_{i+1}b_i - b_{i-1} = [2i + 1] q^{(-1)^i \lfloor \frac{i+1}{2} \rfloor - i} \sum_{n \geq 0} \frac{(-1)^n z^{2n}}{[2n + 2i + 1]_q!} \left( \prod_{j=1}^{i} [2n + 2j]_q \right) q^{(n + \lfloor \frac{i+1}{2} \rfloor)^2 + [i \text{ odd}] \lfloor \frac{i+1}{2} \rfloor - i} - \sum_{n \geq 0} \frac{(-1)^n z^{2n}}{[2n + 2i - 1]_q!} \left( \prod_{j=1}^{i-1} [2n + 2j]_q \right) q^{(n + \lfloor \frac{i}{2} \rfloor)^2 + [i-1 \text{ odd}] \lfloor \frac{i}{2} \rfloor} = \sum_{n \geq 0} \frac{(-1)^n z^{2n}}{[2n + 2i + 1]_q!} \left( [2i + 1] q \left( \prod_{j=1}^{i} [2n + 2j]_q \right) q^{(n + \lfloor \frac{i+1}{2} \rfloor)^2 + [i \text{ odd}] \lfloor \frac{i+1}{2} \rfloor + (-1)^i \lfloor \frac{i+1}{2} \rfloor - i}
\]
\[-[2n + 2i + 1]_q \left( \prod_{j=1}^{i} [2n + 2j]_q \right) q^{(n + \lfloor \frac{i}{2} \rfloor)^2 + [i-1 \text{ odd}] \binom{i}{2}} \]

\[= \frac{1}{1 - q} \sum_{n \geq 0} \frac{(-1)^n z^{2n}}{[2n + 2i + 1]_q !} \left( \prod_{j=1}^{i} [2n + 2j]_q \right) \times \left( (1 - q^{2i + 1}) q^{(n + \lfloor \frac{i}{2} \rfloor)^2 + [i \text{ even}] \binom{i+1}{2} - i} - (1 - q^{2n + 2i + 1}) q^{(n + \lfloor \frac{i}{2} \rfloor)^2 + [i \text{ even}] \binom{i}{2}} \right). \]

The last bracket in this expression can be simplified for \( i \) even:

\[-q^{(n + \frac{i}{2})^2 + (i) + 2i + 1}(1 - q^{2n}) \]

and for \( i \) odd:

\[-q^{(n + \frac{i-1}{2})^2}(1 - q^{2n}). \]

Putting everything together, we arrive at

\[C_{i+1} b_i - b_{i-1} = \sum_{n \geq 0} \frac{(-1)^{n-1} z^{2n}}{[2n + 2i + 1]_q !} \left( \prod_{j=0}^{i} [2n + 2j]_q \right) q^{(n + \lfloor \frac{i}{2} \rfloor)^2 + [i \text{ even}] \binom{i+2}{2}}. \]

Notice that the constant term vanishes, whence

\[b_{i+1} = \sum_{n \geq 1} \frac{(-1)^{n-1} z^{2n-2}}{[2n + 2i + 1]_q !} \left( \prod_{j=0}^{i} [2n + 2j]_q \right) q^{(n + \lfloor \frac{i}{2} \rfloor)^2 + [i \text{ even}] \binom{i+2}{2}} \]

\[= \sum_{n \geq 0} \frac{(-1)^n z^{2n}}{[2n + 2(i + 1) + 1]_q !} \left( \prod_{j=1}^{i+1} [2n + 2j]_q \right) q^{(n + \lfloor \frac{i+2}{2} \rfloor)^2 + [i + 1 \text{ odd}] \binom{i+2}{2}}, \]

which is the announced formula.

**References**


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