NOTE

A SHORT PROOF FOR A PARTITION IDENTITY OF HWANG AND WEI

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In Volume 46 of this journal [1] Hwang and Wei prove the following identity:

$$\sum_{p \in P} \prod_{i=1}^{m} \left( \frac{n_i + 1 - k_i}{k_i} \right) = \sum_{i=0}^{j} \binom{i + m - 2}{m - 2} \left( \frac{n + 1 - k - 2j}{k - 2j} \right)$$

for integers $n_i$ with $n_1 + n_2 + \cdots + n_m = n$ ($m \geq 2$), where $p = (k_1, \ldots, k_m)$ runs through the set $P$ of all partitions $k = k_1 + k_2 + \cdots + k_m$ of the nonnegative integer $k$ into $m$ non-negative integers $k_i$. We give here a very short proof for this identity, using generating functions.

We set

$$A_n(z) = \sum_{k=0}^{\infty} \binom{n + 1 - k}{k} z^k.$$ 

Then it follows from Riordan [2, p. 154, 2c; note the typos!], that

$$A_n(z) = \frac{(-2z)}{1 - \mu}^{n+2} = \frac{1 + \mu}{2}^{n+2} \quad \text{with } \mu = (1 + 4z)^{1/2}. \quad (1)$$

The identity in question is equivalent to

$$\prod_{i=1}^{m} A_{n_i}(z) = \sum_{j=0}^{\infty} \binom{i + m - 2}{m - 2} z^{2j} A_{n-4j}(z)$$

$$= \sum_{j=0}^{\infty} \binom{-m + 1}{j} (-z^2)^j A_{n-4j}(z).$$

With $n = n_1 + n_2 + \cdots + n_m$ and formula (1), we have to show

$$\mu^{-m} \left( \frac{1 + \mu}{2} \right)^{n+2m} = \mu^{-1} \left( \frac{1 + \mu}{2} \right)^{n+2} \left( 1 - \frac{(1 - \mu)^4}{16z^2} \right)^{-n+1}.$$
Regarding \( 4z = -(1 + \mu)(1 - \mu) \) it follows that

\[
1 - \frac{(1 - \mu)^2}{16z^2} = 1 - \frac{(1 - \mu)^2}{(1 + \mu)^2} = \frac{4\mu}{(1 + \mu)^2}
\]

from which the desired identity is immediate.

References