A GENERALIZED FILBERT MATRIX

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Abstract. A generalized Filbert matrix is introduced, sharing properties of the Hilbert matrix and Fibonacci numbers. Explicit formulae are derived for the LU-decomposition, their inverses, and the Cholesky factorization. The approach is to use $q$-analysis and to leave the justification of the necessary identities to the $q$-version of Zeilberger’s celebrated algorithm.

1. Introduction

The Filbert matrix $H_n = (h_{ij})_{i,j=1}^n$ is defined by $h_{ij} = \frac{1}{F_{i+j-1}}$ as an analogue of the Hilbert matrix where $F_n$ is the $n$th Fibonacci number. It has been defined and studied by Richardson [1].

In this paper we will study the generalized matrix with entries $\frac{1}{F_{i+j+r}}$, where $r \geq -1$ is an integer parameter. The size of the matrix does not really matter, and we can think about an infinite matrix $\mathcal{F}$ and restrict it whenever necessary to the first $n$ rows resp. columns and write $\mathcal{F}_n$.

Our approach will be as follows. We will use the Binet form

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^{n-1} - q^n}{1-q},$$

with $q = \beta/\alpha = -\alpha^{-2}$, so that $\alpha = i/\sqrt{q}$. All the identities we are going to derive hold for general $q$, and results about Fibonacci numbers come out as corollaries for the special choice of $q$.

Throughout this paper we will use the following notations: $(x;q)_n = (1-x)(1-xq)\ldots(1-xq^{n-1})$ and the Gaussian $q$-binomial coefficients

$$\left[ \begin{array}{c} n \\ k \end{array} \right] = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$ 

Furthermore, we will use $Fibonomial$ coefficients

$$\left\{ \begin{array}{c} n \\ k \end{array} \right\} = \frac{F_n F_{n-1} \ldots F_{n-k+1}}{F_1 \ldots F_k}.$$ 

The link between the two notations is

$$\left\{ \begin{array}{c} n \\ k \end{array} \right\} = \alpha^{k(n-k)} \left[ \begin{array}{c} n \\ k \end{array} \right] \text{ with } q = -\alpha^{-2}.$$ 

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We will obtain the LU-decomposition \( \mathcal{F} = L \cdot U \):

**Theorem 1.** For \( 1 \leq d \leq n \) we have

\[
L_{n,d} = q^{\frac{n-d}{2}} \mathbf{1}^{n+d}(-1)^n \begin{bmatrix} n - 1 & 2d + r & n + d + r \\ d & d & d \\ 1 & 1 & 1 \end{bmatrix}^{-1}
\]

and its Fibonacci corollary

\[
L_{n,d} = \left\{ \begin{array}{c} n - 1 \\ d - 1 \end{array} \right\} \left\{ \begin{array}{c} 2d + r \\ d \\ n + d + r \end{array} \right\}^{-1}.
\]

**Theorem 2.** For \( 1 \leq d \leq n \) we have

\[
U_{d,n} = q^{\frac{n-d-r-1}{2} + d^2 + rd} \mathbf{1}^{n+d+1}(-1)^{n+d+r} \begin{bmatrix} 2d + r - 1 & n + d + r - 1 & n \\ d - 1 & d & 1 - q^n \end{bmatrix}^{-1}
\]

and its Fibonacci corollary

\[
U_{d,n} = (-1)^{r(d+1)} \left\{ \begin{array}{c} 2d + r - 1 \\ d - 1 \end{array} \right\}^{-1} \left\{ \begin{array}{c} n + d + r \\ d \end{array} \right\}^{-1} \left\{ \begin{array}{c} n \\ d \end{array} \right\} \frac{1}{F_n}.
\]

We could also determine the inverses of the matrices \( L \) and \( U \):

**Theorem 3.** For \( 1 \leq d \leq n \) we have

\[
L_{n,d}^{-1} = q^{\frac{(n-d)^2}{2}} \mathbf{1}^{n+d}(-1)^d \begin{bmatrix} n + r \\ d + r \\ n - 1 \end{bmatrix} \begin{bmatrix} n + d - 1 + r \\ 2n + r - 1 \end{bmatrix}^{-1}
\]

and its Fibonacci corollary

\[
L_{n,d}^{-1} = (-1)^{(n+1)d + \frac{n(n+1)}{2}} + \frac{d(n+1)}{2} \begin{bmatrix} n + r \\ d + r \\ n - 1 \end{bmatrix} \begin{bmatrix} n + d - 1 + r \\ 2n + r - 1 \end{bmatrix}^{-1}.
\]

**Theorem 4.** For \( 1 \leq d \leq n \) we have

\[
U_{d,n}^{-1} = q^{\frac{n^2 + d^2 + r + 1}{2} - (d+r)n} \mathbf{1}^{n+d-1+r}(-1)^{n-d} \begin{bmatrix} 2n + r \\ n \end{bmatrix} \begin{bmatrix} n + d + r - 1 \\ d + r \end{bmatrix} \begin{bmatrix} n - 1 \\ d - 1 \end{bmatrix} \frac{1 - q^n}{1 - q}
\]

and its Fibonacci corollary

\[
U_{d,n}^{-1} = (-1)^{\frac{n(n+1)}{2} + \frac{d(d+1)}{2} - dn - r + 1} \begin{bmatrix} 2n + r \\ n \end{bmatrix} \begin{bmatrix} n + d + r - 1 \\ d + r \end{bmatrix} \begin{bmatrix} n - 1 \\ d - 1 \end{bmatrix} F_n.
\]

As a consequence we can compute the determinant of \( \mathcal{F}_n \), since it is simply evaluated as \( U_{1,1} \cdots U_{n,n} \) (we only state the Fibonacci version):

**Theorem 5.**

\[
\det \mathcal{F}_n = (-1)^{r n(n-1)} \prod_{d=1}^{n} \begin{bmatrix} 2d + r - 1 \\ d - 1 \end{bmatrix}^{-1} \begin{bmatrix} 2d + r \\ d \end{bmatrix}^{-1} \frac{1}{F_d}.
\]
Now we determine the inverse of the matrix $\mathcal{F}$. This time it depends on the dimension, so we compute $(\mathcal{F}_n)^{-1}$.

**Theorem 6.** For $1 \leq i, j \leq n$:

$$(\mathcal{F}_n)^{-1} = q^{\frac{r^2 + r + 1}{2} + (i + j + r)n} \frac{1 + q^{r + (i + j + r) - 1} + q^{r - 1} + n - 1}{n} \left[ \begin{array}{c} n + r + i \\ n \\ i - 1 \\ j - 1 \end{array} \right] \left[ \begin{array}{c} n - 1 \\ n - 1 \\ n - 1 \\ n \end{array} \right] \frac{(1 - q^n)^2}{(1 - q^{r + i + j})(1 - q)}$$

and its Fibonacci corollary

$$(\mathcal{F}_n)^{-1} = (-1)^{\frac{i(i-1)}{2} + \frac{j(j-1)}{2} + n(i + j + r) + n} \left\{ \begin{array}{c} n + r + i \\ n \\ n - 1 \\ n - 1 \\ n - 1 \end{array} \right\} \left\{ \begin{array}{c} n + r + j \\ n \\ n - 1 \\ n - 1 \\ n - 1 \end{array} \right\} \frac{F_n^2}{F_{r+i+j}}.$$

We can also find the Cholesky decomposition $\mathcal{F} = \mathcal{C} \cdot \mathcal{C}^T$ with a lower triangular matrix $\mathcal{C}$:

**Theorem 7.** For $n \geq d$:

$$\mathcal{C}_{n,d} = (-1)^{\frac{n^2 + r + 1}{2} + \frac{d(d-1)}{2} + \frac{d + 1}{2} + \frac{q^{r - 1} + q^{r + (i + j + r) - 1} + q}{2}} \sqrt{\frac{(1 - q^{2d+r})(1 - q)}{1 - q^{2n+r}}} \left\{ \begin{array}{c} 2n + r \\ n - d \end{array} \right\} \left\{ \begin{array}{c} 2n + r - 1 \\ n - 1 \end{array} \right\}^{-1}$$

and its Fibonacci corollary

$$\mathcal{C}_{n,d} = (-1)^{\frac{d(d-1)}{2} + \frac{r(d+1)}{2} + \frac{F_{2d+r}}{F_{2n+r}}} \left\{ \begin{array}{c} 2n + r \\ n - d \end{array} \right\} \left\{ \begin{array}{c} 2n + r + 1 \\ n - 1 \end{array} \right\}^{-1}.$$

**Notice that for odd $r$, even the Fibonacci version may contain complex numbers.**

2. **Proofs**

In order to show that indeed $\mathcal{F} = L \cdot U$, we need to show that for any $m, n$:

$$\sum_d L_{m,d} \mathcal{U}_{d,n} = \mathcal{F}_{m,n} = \alpha^{-m-n-r+1} \frac{1 - q}{1 - q^{m+n+r}}.$$

In rewritten form the formula to be proved reads

$$\sum_d \left( q^{d^2 + (r-1)d - r} - q^{d^2 + (r+1)d} \right) \left[ \begin{array}{c} 2m + r \\ m - d \end{array} \right] \left[ \begin{array}{c} 2n + r \\ n - d \end{array} \right] = \frac{(1 - q^{2n+r})(1 - q^{2m+r})}{1 - q^{m+n+r}} \left[ \begin{array}{c} 2m + r - 1 \\ m - 1 \end{array} \right] \left[ \begin{array}{c} 2n + r - 1 \\ n - 1 \end{array} \right].$$

Nowadays, such identities are a routine verification using the $q$-Zeilberger algorithm, as described in the book [2].

For interest, we also state (as a corollary) the corresponding Fibonacci identity:

$$\sum_d (-1)^{(d-1)} F_{2d+r} \left\{ \begin{array}{c} 2m + r \\ m - d \end{array} \right\} \left\{ \begin{array}{c} 2n + r \\ n - d \end{array} \right\} = \frac{F_{2m+r} F_{2m+r}}{F_{m+n+r}} \left\{ \begin{array}{c} 2m + r - 1 \\ m - 1 \end{array} \right\} \left\{ \begin{array}{c} 2n + r - 1 \\ n - 1 \end{array} \right\}.$$
Now we move to the inverse matrices. Since $L$ and $L^{-1}$ are lower triangular matrices, we only need to look at the entries indexed by $(m, n)$ with $m \geq n$:

$$\sum_{n \leq d \leq m} L_{m,d}L_{d,n}^{-1}$$

$$= \sum_{n \leq d \leq m} q^{m+d}i^{m+d}(-1)^{m} \left[ \frac{m-1}{2d} \right] \left[ \frac{m+d+r}{d} \right]^{-1}$$

$$\times q^{m}i^{m}(-1)^{m} \left[ \frac{d+r}{n+r} \right] \left[ \frac{n+d-1+r}{n-1} \right] \left[ \frac{2d+1+r}{d-1} \right]^{-1}$$

$$= \frac{1}{1-q^{2m+r}} \left[ \frac{2m+r-1}{m-1} \right]^{-1} \left[ \frac{n+m+n}{m-1} \right] \left[ \frac{2m+r}{m-d} \right] \left[ \frac{n+d-1+r}{n-r} \right] \left[ \frac{d-1}{d-1} \right]^{-1}$$

Now we turn to the inverse matrices. Since $U$ and $U^{-1}$ are lower triangular matrices, the argument for $U \cdot U^{-1}$ is similar:

$$\sum_{m \leq d \leq n} U_{m,d}U_{d,n}^{-1}$$

$$= (-1)^{m+n}i^{m+n}q^{-\frac{m}{2}+m^2+r}n^{-\frac{m}{2}-rn} \left[ \frac{2m+r-1}{m-1} \right]^{-1} \left[ \frac{2n+r-1}{n-1} \right] \frac{1-q^{2m+r}}{1-q^{n+m+r}}$$

$$\times \sum_{m \leq d \leq n} (-1)^{d}q^{\frac{(d+1)-d}{2}n} \left[ \frac{d+r+1}{d-m} \right] \left[ \frac{n+m+r}{n-d} \right]$$

Again, the q-Zeilberger algorithm evaluates this to $[m = n]$.

Now we turn to the inverse matrix:

$$(\mathcal{F}^{-1}_{n})_{i,k} = i^{-k}(-1)^{i}q^{\frac{2k}{2}-(i+r)n+r}(1-q)^{2i} \left[ \frac{n+r+i}{n} \right] \left[ \frac{n-1}{i-1} \right]$$

$$\times \sum_{j=1}^{n} q^{\frac{(j+1)-jn}{2}}(-1)^{j} \left[ \frac{n+r+j}{n} \right] \left[ \frac{n-1}{j-1} \right] \frac{1}{(1-q^{j+k+r})(1-q^{r+i+j})}$$

And the q-Zeilberger algorithm evaluates this again to $[i = k]$.

The Cholesky verification goes like this:

$$\sum_{n \in \mathbb{N}} \mathcal{C}_{m,d} \mathcal{C}_{n,d}$$

$$= (-1)^{m+n+r}i^{m+n+r+1}q^{\frac{m+n+r-1}{2}} \left[ \frac{1-q}{1-q^{2m+r}} \right] \left[ \frac{2n+r-1}{n-1} \right]^{-1} \left[ \frac{2m+r-1}{m-1} \right]^{-1}$$

$$\times \sum_{d} q^{d(d-1)+rd}(1-q^{2d+r}) \left[ \frac{2m+r}{m-d} \right] \left[ \frac{2n+r}{n-d} \right].$$
And again the \( q \)-Zeilberger algorithm evaluates this to be
\[
\frac{1 - q}{1 - q^{m+n+r}} i^{m+n+r-1} q^{m+n+r-1}^{\frac{m+n+r-1}{2}},
\]
as it should.

References