ON THE EXPANSION OF FIBONACCI AND LUCAS POLYNOMIALS

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Abstract. Recently, Belbachir and Bencherif [2] have expanded Fibonacci and Lucas polynomials using bases of Fibonacci and Lucas like polynomials. Here, we provide simplified proofs of the expansion formulæ, that in essence a computer can do. Furthermore, for 2 of the 5 instances, we find $q$-analogues.

1. Introduction

In [2], Fibonacci and Lucas polynomials were studied:

\[ U_0 = 0, \quad U_1 = 1, \quad U_n = xU_{n-1} + yU_{n-2}, \]
\[ V_0 = 2, \quad V_1 = x, \quad V_n = xV_{n-1} + yV_{n-2}. \]

We prefer the modified polynomials

\[ u_0 = 0, \quad u_1 = 1, \quad u_n = u_{n-1} + zu_{n-2}, \]
\[ v_0 = 2, \quad v_1 = 1, \quad v_n = v_{n-1} + zv_{n-2}, \]

so that

\[ U_n(x, y) = x^n u_n \left( \frac{y}{x^2} \right), \quad V_n(x, y) = x^n v_n \left( \frac{y}{x^2} \right). \]

Then, with

\[ \lambda_{1,2} = \frac{1 \pm \sqrt{1 + 4z}}{2}, \]

\[ u_n = \frac{1}{\sqrt{1 + 4z}} (\lambda_1^n - \lambda_2^n), \quad v_n = \lambda_1^n + \lambda_2^n. \]

Substituting $z = t/(1 - t)^2$, these formulæ become particularly nice:

\[ u_n = \frac{1 - (-t)^n}{(1 + t)(1 - t)^{n-1}}, \quad v_n = \frac{1 + (-t)^n}{(1 - t)^n}. \]

The main result of [2] are the following 5 formulæ:

\[ 2u_{2n+1} = \sum_{k=0}^{n} a_{n,k} v_{2n-k}, \quad a_{n,k} = 2 \sum_{j=0}^{n} (-1)^{j+k} \binom{j}{k} - (-1)^{n+k} \binom{n}{k}. \]  \hfill (1.1)

\[ u_{2n} = \sum_{k=1}^{n} b_{n,k} u_{2n-k}, \quad b_{n,k} = (-1)^{k+1} \binom{n}{k}. \]  \hfill (1.2)
\[ v_{2n-1} = \sum_{k=1}^{n} c_{n,k} u_{2n-k}, \quad c_{n,k} = 2(-1)^{k+1} \binom{n}{k} - [k = 1]. \quad (1.3) \]

\[ 2v_{2n-1} = \sum_{k=1}^{n} d_{n,k} v_{2n-1-k}, \quad d_{n,k} = (-1)^{k+1} \frac{2n-k}{n} \binom{n}{k}. \quad (1.4) \]

\[ 2u_{2n} = \sum_{k=1}^{n} e_{n,k} v_{2n-1-k}, \quad (1.5) \]

\[ e_{n,k} = (-1)^{k+1} \frac{2n-k}{2n} \binom{n}{k} + \sum_{j=0}^{n-1} (-1)^{j+k-1} \binom{j}{k-1} - \frac{1}{2} (-1)^{n+k} \binom{n-1}{k-1}. \]

But the proofs of all these, using the simple forms for \( u_n \) and \( v_n \), can be done by a computer! To give the reader an idea, let us do the last one, which seems to be the most complicated:

\[
\begin{align*}
\sum_{k=1}^{n} e_{n,k} v_{2n-1-k} &= \sum_{k=1}^{n} (-1)^{k+1} \frac{2n-k}{2n} \binom{n}{k} v_{2n-1-k} \\
&\quad + \sum_{j=0}^{n-1} \sum_{k=1}^{j+1} (-1)^{j+k-1} \binom{j}{k-1} v_{2n-1-k} - \sum_{k=1}^{n} \frac{1}{2} (-1)^{n+k} \binom{n-1}{k-1} v_{2n-1-k} \\
&= \frac{1 - t^{2n-1}}{(1-t)^{2n-1}} + \frac{1 + t^{2n-1}}{(1-t)^{2n-2}(1+t)} - \frac{(-1)^n t^n}{(1-t)^{2n-2} + (1-t)^{2n-2}} \\
&= \frac{2(1-t^{2n})}{(1-t)^{2n-1}(1+t)} = u_{2n}.
\end{align*}
\]

The other proofs are similar/easier:

\[
\begin{align*}
\sum_{k=0}^{n} a_{n,k} v_{2n-k} &= \frac{2((-t)^n (1+t) + 1 + t^{2n+1})}{(1-t)^{2n}(1+t)} - \frac{2(-t)^n}{(1-t)^{2n}} \\
&= \frac{2[1 + t^{2n+1}]}{(1-t)^{2n}(1+t)} = 2u_{2n+1}.
\end{align*}
\]

\[
\begin{align*}
\sum_{k=1}^{n} c_{n,k} u_{2n-k} &= \frac{2(1-t^{2n})}{(1-t)^{2n-1}(1+t)} - \frac{1 + t^{2n-1}}{(1+t)(1-t)^{2n-2}} \\
&= \frac{1 - t^{2n-1}}{(1-t)^{2n-1}} = v_{2n-1}.
\end{align*}
\]

2. \( q \)-ANALOGUES

Now we are interested in \( q \)-analogues. For this, we replace \( u_n \) by

\[ \text{Fib}_n = \sum_{0 \leq k \leq \frac{n-1}{2}} q^{\binom{k+1}{2}} \binom{n-k-1}{k}_q z^k. \]
and \(v_n\) by

\[ \text{Luc}_n = \sum_{0 \leq k \leq n} q^{(k)} \binom{n - k}{k} \frac{[n]_q}{[n - k]_q} z^k ; \]

as suggested by Cigler [3]. We use standard \(q\)-notation here:

\[ [n]_q := 1 + q + \cdots + q^{n-1}, \quad [n]_q! := [1]_q[2]_q \cdots [n]_q, \quad \left[ \begin{array}{c} n \\ k \end{array} \right]_q := \frac{[n]_q!}{[k]_q! [n-k]_q!} , \]

compare [1]; the notions of the Introduction are the special instance \(q = 1\).

**Theorem 1.**

\[ \text{Luc}_{2n-1} = \sum_{k=1}^{n} d_{n,k} \text{Luc}_{2n-1-k}, \]

with

\[ d_{n,k} = (-1)^{k-1} \frac{q^{(2)}}{1 + q^{n-1}} \left( \left[ \begin{array}{c} n - 1 \\ k \end{array} \right]_q + q^{n-1} \left[ \begin{array}{c} n \\ k \end{array} \right]_q \right) . \]

**Proof.** We must prove that

\[
\sum_{0 \leq k \leq n-1} q^{(k)} \binom{2n - 1 - k}{k} \frac{[2n - 1]_q}{[2n - 1 - k]_q} z^k \\
= \sum_{j=1}^{n} (-1)^{j-1} \frac{q^{(2)}}{1 + q^{n-1}} \left( \left[ \begin{array}{c} n - 1 \\ j \end{array} \right]_q + q^{n-1} \left[ \begin{array}{c} n \\ j \end{array} \right]_q \right) \\
	imes \sum_{0 \leq k \leq 2n-j-1} q^{(k)} \binom{2n - j - 1 - k}{k} \frac{[2n - j - 1]_q}{[2n - j - 1 - k]_q} z^k.
\]

Comparing coefficients, we have to prove that

\[
q^{(2)} \binom{2n - 1 - k}{k} \frac{[2n - 1]_q}{[2n - 1 - k]_q} \\
= \sum_{j=1}^{n} (-1)^{j-1} \frac{q^{(2)}}{1 + q^{n-1}} \left( \left[ \begin{array}{c} n - 1 \\ j \end{array} \right]_q + q^{n-1} \left[ \begin{array}{c} n \\ j \end{array} \right]_q \right) q^{(2)} \binom{2n - j - 1 - k}{k} \frac{[2n - j - 1]_q}{[2n - j - 1 - k]_q}.
\]

Simplifying, we must prove that

\[
\sum_{j=0}^{n} (-1)^{j} q^{(2)} \left( \left[ \begin{array}{c} n - 1 \\ j \end{array} \right]_q + q^{n-1} \left[ \begin{array}{c} n \\ j \end{array} \right]_q \right) \left[ \begin{array}{c} 2n - j - 2 - k \\ k - 1 \end{array} \right]_q [2n - j - 1]_q = 0.
\]

Another form of this is

\[
\sum_{j=0}^{n} (-1)^{j} q^{(2)} \left( 1 - q^{2n-1} - q^{n-j} + q^{n-1} \right) \left( 1 - q^{2n-j-1} \right) \left[ \begin{array}{c} n \\ j \end{array} \right]_q \left[ \begin{array}{c} 2n - j - 2 - k \\ k - 1 \end{array} \right]_q = 0.
\]

Notice that

\[
\sum_{j=0}^{n} (-1)^{j} q^{(2)} \left[ \begin{array}{c} n \\ j \end{array} \right]_q q^{-aj} = 0
\]
for $0 \leq a \leq n - 1$. This follows from Rothe's formula [1, p. 490]

$$\sum_{j=0}^{n} (-1)^j q^{\binom{j}{2}} q^{n-j} x^j = (1 - x)(1 - xq) \ldots (1 - q^{n-1}).$$

We write the desired identity as

$$\sum_{j=0}^{n} (-1)^j q^{\binom{j}{2}} (A + Bq^{-j} + Cq^{-2j}) \binom{n}{j} q^{n-j} q^{\binom{n-j-1}{2}} q^{2n-j-k} = 0.$$

Therefore, for $k \leq n - 2$, the identity holds. For $k = n - 1$,

$$\sum_{j=0}^{1} (-1)^j q^{\binom{j}{2}} (1 - q^{2n-1} - q^{n-j} + q^{n-1}) \binom{n}{j} q^{n-j} = 0$$

can be shown by inspection, and for $k = n$, the identity holds, since the sum is empty. □

**Theorem 2.**

$$\text{Fib}_{2n} = \sum_{k=1}^{n} b_{n,k} \text{Fib}_{2n-k}$$

with

$$b_{n,k} = (-1)^{k-1} q^{\binom{k}{2}} \binom{n}{k} q.$$

**Proof.** We must prove that

$$\sum_{0 \leq k \leq n-1} q^{\binom{k+1}{2}} \binom{2n-k-1}{k} q^k = \sum_{j=1}^{n} (-1)^{j-1} q^{\binom{j}{2}} \binom{n}{j} q^{\binom{k+1}{2}} \binom{2n-j-k-1}{k} q^k.$$

Comparing coefficients, this means

$$\sum_{j=0}^{n} (-1)^j q^{\binom{j}{2}} \binom{n}{j} q^{2n-j-k-1} = 0,$$

which follows by a similar but simpler argument than before. □

3. **Conclusion**

We found 2 $q$-analogues; for the remaining 3 instances we were not successful and leave this as a challenge for anybody who is interested.
REFERENCES


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