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On the maximum dimensions of subalgebras of $\mathrm{M}_n(K)$ satisfying two related identities

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ABSTRACT

For an arbitrary $q \geq 2$, we find an upper bound for the dimension of a subalgebra of the full matrix algebra $\mathrm{M}_n(K)$ over an arbitrary field K satisfying the identity

$$\big[[x_1, y_1], z_1\big] \cdot \big[[x_2, y_2], z_2\big] \cdot \cdots \cdot \big[[x_q, y_q], z_q\big] = 0,$$

and we show that this upper bound is sharp by presenting an example in block triangular form of a subalgebra of $\mathrm{M}_n(K)$ with dimension equal to the obtained upper bound. We apply this result to Lie solvable algebras of index 2, i.e., algebras satisfying the identity $\big[[x_1, y_1], [x_2, y_2]\big] = 0$. To be precise, for $n \leq 4$, we find the sharp upper bound for the dimension of a Lie solvable subalgebra of $\mathrm{M}_n(K)$ of index 2, and for $n > 4$, we obtain the relatively tight (at least for small values of $n > 4$) interval

$$\left[2 + \left\lfloor \frac{3n^2}{8} \right\rfloor,\ 2 + \left\lfloor \frac{5n^2}{12} \right\rfloor\right]$$

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for the maximum dimension of a Lie solvable subalgebra of $M_n(K)$ of index 2, the exact value of which is not known.

1. Introduction

Throughout the paper, all algebras are assumed to be associative, unital, and over a field K. By a subalgebra of the full $n \times n$ matrix algebra $M_n(K)$ we mean a K-subalgebra of $M_n(K)$.

The identity

$$[x_1, y_1] \cdot [x_2, y_2] \cdot \quad \cdots \quad \cdot [x_q, y_q] = 0 \tag{1.1}$$

features prominently in numerous papers, e.g., [1], [2], [4–7], [11] and [12].

Mal'tsev proved in [4] that all the polynomial identities of the upper triangular $q \times q$ matrix algebra over K, denoted by $U_q(K)$, are consequences of the identity in (1.1). See [5] for an explicit form of a finite set of generators of an ideal of identities of the algebra $U_q^*(R)$ over a commutative integral domain R, with $U_q^*(R)$ denoting the R-subalgebra of $U_q(R)$ comprising all the matrices in $U_q(R)$ with constant main diagonal.

If, in the context of some class of subalgebras of a finite dimensional algebra, we say that an algebra $\mathcal{A}$ has maximum dimension, then we mean that $\mathcal{A}$ has maximum possible dimension in the considered class.

The maximum dimension of a subalgebra of $M_n(K)$ satisfying the identity in (1.1), found by Domokos in [2] to be

$$\max_{n_1+n_2+\cdots+n_q=n} \left\{ q + \frac{n^2}{2} - \sum_{i=1}^{q} \left(\frac{n_i^2}{2} - \left\lfloor \frac{n_i^2}{4} \right\rfloor \right) \right\}, \tag{1.2}$$

was refined in [11], where an explicit q-tuple $(n_1, n_2, \ldots, n_q)$, realizing the above maximum, was exhibited. (In (1.2), every n_i, $i = 1, 2, \ldots, q$, is a non-negative integer, and $\lfloor \ \rfloor$ denotes the integer floor function.) In particular, the sharp upper bound (see, for example, [11, Theorem 14 and Proposition 16]) for the dimension of a subalgebra of $M_n(K)$ satisfying the identity $[x_1, y_1] \cdot [x_2, y_2] = 0$ is

$$2 + \left\lfloor \frac{3n^2}{8} \right\rfloor. \tag{1.3}$$

Recently, in [9], an identity similar to the one in (1.1), namely the identity

$$\big[[x_1, y_1], z_1\big] \cdot \big[[x_2, y_2], z_2\big] \cdot \quad \cdots \quad \cdot \big[[x_q, y_q], z_q\big] = 0, \tag{1.4}$$

which is a typical identity of the upper triangular $q \times q$ matrix ring $\mathrm{U}_q(R)$, with R a ring satisfying the identity $\big[[x,y],z\big] = 0$ (i.e., Lie nilpotency of index 2), was studied, and a generalization of the Cayley-Hamilton Theorem was obtained for an $n \times n$ matrix A over a unital ring R satisfying the identity in (1.4), namely the "power" Cayley-Hamilton identity

$$\Big(I_n\lambda_0^{(2)} + A\lambda_1^{(2)} + \cdots + A^{n^2-1}\lambda_{n^2-1}^{(2)} + A^{n^2}\lambda_{n^2}^{(2)}\Big)^q = 0$$

with certain right coefficients

$$\lambda_i^{(2)} \in R,\ 0 \le i \le n^2-1, \quad \text{and} \quad \lambda_{n^2}^{(2)} = n\big\{(n-1)!\big\}^{1+n}.$$

Definition 1.1. If an algebra $\mathcal{A}$ satisfies the identity in (1.1) (respectively, (1.4)) for some positive integer q, then we say that $\mathcal{A}$ is D_q (respectively, LD_q).

In [6], the structure, conjugation and isomorphism problems of maximal D_q subalgebras of $\mathrm{M}_n(K)$ is studied, in which it is shown that a maximal D_q subalgebra $\mathcal{A}$ of $\mathrm{M}_n(K)$ is conjugated with a block triangular subalgebra of $\mathrm{M}_n(K)$ with maximal commutative diagonal blocks. By analysis of conjugations, the sizes of the obtained diagonal blocks are uniquely determined. The isomorphism problem in a certain class of maximal D_q subalgebras of $\mathrm{M}_n(K)$ which contain all D_q subalgebras of $\mathrm{M}_n(K)$ with maximum dimension is also studied in [6].

Guided by the typical algebras constructed in [2] and [8] to find the maximum dimension of a D_q subalgebra of $\mathrm{M}_n(K)$ and the maximum dimension of a Lie nilpotent subalgebra of $\mathrm{M}_n(K)$, respectively, and inspired by typical examples of LD_q subalgebras of $\mathrm{M}_n(K)$ in Section 5 of the present paper, we prove in one of the main results of the present paper, partially using the structure results obtained in [6], that the maximum dimension of a subalgebra of $\mathrm{M}_n(K)$ satisfying the identity in (1.4) is

$$\max_{n_1+n_2+\cdots+n_q=n}\left\{q + \frac{n^2}{2} - \sum_{i=1}^{q}\left(\frac{n_i^2}{2} - \left\lfloor\frac{n_i^2}{3}\right\rfloor\right)\right\},$$

where the n_i's, for $i = 1,2,\ldots,q$, are non-negative integers. Indeed, the mentioned concrete examples serve to showing that the upper bound in (1.4) is sharp.

We also provide another motivation for studying LD_q algebras. To this end, recall that the Lie central series and the Lie derived series of an algebra $\mathcal{A}$ are defined inductively as follows:

$$\mathfrak{C}^0(\mathcal{A}) = \mathcal{A},\ \mathfrak{C}^{q+1}(\mathcal{A}) = [\mathfrak{C}^q(\mathcal{A}), \mathcal{A}] \quad \text{(Lie central series)}$$

and

$$\mathfrak{D}^0(\mathcal{A}) = \mathcal{A},\ \mathfrak{D}^{q+1}(\mathcal{A}) = [\mathfrak{D}^q(\mathcal{A}), \mathfrak{D}^q(\mathcal{A})] \quad \text{(Lie derived series)}.$$

Definition 1.2. An algebra $\mathcal{A}$ is called Lie nilpotent (respectively, Lie solvable) of index q if $\mathfrak{C}^q(\mathcal{A}) = 0$ (respectively, $\mathfrak{D}^q(\mathcal{A}) = 0$), or, for short, $\mathcal{A}$ is Ln_q (respectively, Ls_q).

In [8], the authors determined the maximum dimension of a Lie nilpotent subalgebra of $\mathrm{M}_n(K)$ of any (fixed) index, and they suggested a study for finding the maximum dimension of a Lie solvable subalgebra of $\mathrm{M}_n(K)$ of any (fixed) index.

Towards an answer to finding the latter maximum dimension, it was shown in [12, Theorem 4] that if a structural R-subalgebra $\mathcal{A}$ of $\mathrm{U}_n(R)$, with R any commutative unital ring, is Ls_{q+1} (for some $q \geq 1$), then $\mathcal{A}$ is D_{2^q}. However, the result is not true in general because of the construction in [7, Corollary 2.2] of an R-subalgebra of $\mathrm{U}_9(R)$, namely the algebra $\mathrm{U}_3^*\big(\mathrm{U}_3^*(R)\big)$, which is Ls_2, but not D_2. Of course, $\mathrm{U}_3^*\big(\mathrm{U}_3^*(R)\big)$ is not a structural R-subalgebra of $\mathrm{U}_9(R)$. For the ease of the reader, we recall (see, for example, [12]) that a structural R-subalgebra of the full $n \times n$ matrix algebra $\mathrm{M}_n(R)$, R any commutative unital ring, comprises all matrices having zero in certain prescribed positions and any elements of R in the other positions. To be more precise, for a reflexive and transitive binary relation θ on the set $\{1, 2, ..., n\}$, the structural R-subalgebra $\mathrm{M}_n(\theta, R)$ of $\mathrm{M}_n(R)$ is defined as follows:

$$\mathrm{M}_n(\theta, R) = \{A \in \mathrm{M}_n(R) \mid A_{i,j} = 0 \text{ if } (i, j) \notin \theta\}.$$

We will prove that every Ls_2 algebra is LD_2, and that

$$2 + \left\lfloor \frac{5n^2}{12} \right\rfloor \tag{1.5}$$

is a sharp upper bound for the dimension of an LD_2 subalgebra of $\mathrm{M}_n(K)$. As an application, keeping in mind that every D_2 algebra is Ls_2, we conclude, in conjunction with the mentioned (see (1.3)) sharp upper bound for the dimension of a D_2 subalgebra of $\mathrm{M}_n(K)$, that if $\mathcal{A}$ is an Ls_2 subalgebra of $\mathrm{M}_n(K)$ with maximum dimension, then

$$2 + \left\lfloor \frac{3n^2}{8} \right\rfloor \leq \dim_K \mathcal{A} \leq 2 + \left\lfloor \frac{5n^2}{12} \right\rfloor. \tag{1.6}$$

Since $2 + \left\lfloor \frac{3n^2}{8} \right\rfloor = 2 + \left\lfloor \frac{5n^2}{12} \right\rfloor$ for $n = 2, 3, 4$, it means that, for these values of n, the maximum dimension of an Ls_2 subalgebra of $\mathrm{M}_n(K)$ in coincides with the maximum dimension of an LD_2 subalgebra of $\mathrm{M}_n(K)$ in (1.5) and the maximum dimension of a D_2 subalgebra of $\mathrm{M}_n(K)$ in (1.3). However, the question of whether the inequalities in (1.6) in general (i.e., for $n > 4$) are strict, remains open. Nevertheless, the interval

$$\left[2 + \left\lfloor \frac{3n^2}{8} \right\rfloor, \; 2 + \left\lfloor \frac{5n^2}{12} \right\rfloor \right] \tag{1.7}$$

is relatively tight for small values of $n > 4$.

The paper is laid out as follows. Firstly, in Section 2, we prove that every Lie solvable algebra of index 2 is LD_2, and we show that the converse does not hold. Further analysis shows that every LD_{2^q} algebra is Ls_{q+2}. In Section 3, we determine an upper bound for the dimension of an LD_q subalgebra of $\mathrm{M}_n(K)$. Next, in Section 4, we present an equivalent formula for the upper bound obtained in Section 3, which enables us, on the one hand, to find the interval in (1.7) for the maximum dimension of a Lie solvable subalgebra of $\mathrm{M}_n(K)$ of index 2, and on the other hand, in Section 5, to show that the obtained upper bound for the dimension of an LD_q subalgebra of $\mathrm{M}_n(K)$ is sharp by exhibiting an example of an LD_q subalgebra of $\mathrm{M}_n(K)$ with dimension equal to the obtained upper bound.

2. Ls_2 algebras are LD_2

First, let us again recall the example of an Ls_2 subalgebra of $\mathrm{M}_9(K)$ in [7, Corollary 2.2] (see also [11, Example 3] and the remarks following it) which is not D_2. Therefore, it justifies that we look for some modification, which can possibly be satisfied by Lie solvable algebras of index 2.

Indeed, in the first theorem in this section we will show that every Ls_2 algebra is LD_2. We show that the converse is not true by exhibiting an example of an LD_2 subalgebra of $\mathrm{M}_6(K)$ which is not Ls_2.

We will also prove that every LD_{2^q} algebra is Ls_{q+2}, which implies that every LD_2 algebra is Ls_3, but interestingly, the mentioned subalgebra of $\mathrm{M}_6(K)$ shows that the index of Lie solvability of 3 in the latter implication cannot be lowered to 2.

We found inspiration for the proof of the following theorem in the self-contained proof in [10, Theorem 2.2] of a classical result of Jennings (see [3]).

Theorem 2.1. *If $\mathcal{A}$ is an Ls_2 algebra, then $\mathcal{A}$ is LD_2.*

Proof. We use the identity

$$[x, yz] = [x, y]z + y[x, z]$$

several times to get

$$\begin{aligned} 0 &= \big[[x_1, y_1], [z_2, z_1 r]\big] = \big[[x_1, y_1], [z_2, z_1]r + z_1[z_2, r]\big] \\ &= \big[[x_1, y_1], [z_2, z_1]r\big] + \big[[x_1, y_1], z_1[z_2, r]\big] \\ &= \big[[x_1, y_1], [z_2, z_1]\big]r + [z_2, z_1] \cdot \big[[x_1, y_1], r\big] + \big[[x_1, y_1], z_1\big] \cdot [z_2, r] + z_1\big[[x_1, y_1], [z_2, r]\big]. \end{aligned}$$

Hence, Lie solvability of index 2 implies that

$$[z_2, z_1] \cdot \big[[x_1, y_1], r\big] + \big[[x_1, y_1], z_1\big] \cdot [z_2, r] = 0.$$

Taking $r = [x_2, y_2]$, and again using the Lie solvability, we get $\big[[x_1, y_1], z_1\big] \cdot \big[z_2, [x_2, y_2]\big] = 0$. Therefore, $\big[[x_1, y_1], z_1\big] \cdot \big[[x_2, y_2], z_2\big] = 0$, and so $\mathcal{A}$ is LD_2. $\square$

It would be interesting to know if Ls_q algebras for $q > 2$ also satisfy some similar identity. However, one of the possible natural generalizations of Theorem 2.1, namely that an Ls_q algebra is $\mathrm{LD}_{2^{q-1}}$ for every $q \geq 2$, does not even hold for $q = 3$:

Example 2.2. We claim that the subalgebra

$$\mathcal{A} = \left\{ \begin{pmatrix} a & b & c & d & e \\ & f & g & h & i \\ & & a & b & c \\ & & & f & g \\ & & & & a \end{pmatrix} : a, b, c, d, e, f, g, h, i \in K \right\}$$

of $\mathrm{U}_5(K)$ is Ls_3, but not LD_4.

To show that $\mathcal{A}$ is Ls_3, first note that we have the inclusion

$$[\mathcal{A}, \mathcal{A}] \subseteq \left\{ \begin{pmatrix} 0 & b & c & d & e \\ & 0 & g & h & i \\ & & 0 & b & c \\ & & & 0 & g \\ & & & & 0 \end{pmatrix} : b, c, d, e, g, h, i \in K \right\},$$

which implies that

$$\big[[\mathcal{A}, \mathcal{A}], [\mathcal{A}, \mathcal{A}]\big] \subseteq \left\{ \begin{pmatrix} 0 & 0 & c & d & e \\ & 0 & 0 & h & i \\ & & 0 & 0 & c \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix} : c, d, e, h, i \in K \right\}.$$

Now it is easy to verify that matrices from $\big[[\mathcal{A}, \mathcal{A}], [\mathcal{A}, \mathcal{A}]\big]$ commute, and so $\mathcal{A}$ is Ls_3.

To verify that $\mathcal{A}$ is not LD_4, notice that in an LD_4 algebra, every linear combination of elements in the form

$$\big[[a_1, b_1], c_1\big] \cdot \big[[a_2, b_2], c_2\big] \cdot \big[[a_3, b_3], c_3\big] \cdot \big[[a_4, b_4], c_4\big], \tag{2.1}$$

where $a_i, b_i, c_i \in \mathcal{A}$, for $i \in \{1, 2, 3, 4\}$, is zero. Consider the following four matrices in $\mathcal{A}$:

$$x_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 \\ & & & 0 & 0 \\ & & & & 1 \end{pmatrix}, \quad y_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ & & 0 & 1 & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix},$$

$$x_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 0 \end{pmatrix}, \; y_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ & 0 & 1 & 0 & 0 \\ & & 0 & 0 & 0 \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}.$$

Direct calculations show that $[x_1, y_1] = y_1$, and so $\big[[x_1, y_1], -x_1\big] = [y_1, -x_1] = y_1$. Similarly, we find that $\big[[x_2, y_2], -x_2\big] = y_2$. Therefore,

$$(\big[[x_1, y_1], -x_1\big] + \big[[x_2, y_2], -x_2\big])^4 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ & 0 & 1 & 0 & 0 \\ & & 0 & 1 & 0 \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix} \neq 0.$$

However, by expanding $(\big[[x_1, y_1], -x_1\big] + \big[[x_2, y_2], -x_2\big])^4$, we obtain a linear combination of products in form (2.1). Consequently, $\mathcal{A}$ is not LD_4.

Proposition 2.3. *If $\mathcal{A}$ is an LD_{2^q} algebra, then $\mathcal{A}$ is Ls_{q+2}.*

Proof. Start with $q = 1$, i.e., assume that $\mathcal{A}$ is an LD_2 algebra. If we substitute $z_1 = [s_1, t_1]$ and $z_2 = [s_2, t_2]$ in the identity $\big[[x_1, y_1], z_1\big] \cdot \big[[x_2, y_2], z_2\big] = 0$, then we get

$$\big[[x_1, y_1], [s_1, t_1]\big] \cdot \big[[x_2, y_2], [s_2, t_2]\big] = 0.$$

It implies that, in the algebra $\mathcal{A}$, the product of any elements in form $\big[[x, y], [z, w]\big]$ is 0. Hence, $\mathcal{A}$ satisfies the identity

$$\Big[\big[[x_1, y_1], [s_1, t_1]\big], \big[[x_2, y_2], [s_2, t_2]\big]\Big] = 0,$$

i.e., $\mathcal{A}$ is Ls_3.

Similarly, if $q > 1$ and $\mathcal{A}$ is LD_{2^q}, then in (1.4) we take $z_i = [s_i, t_i]$, for $i = 1, 2, \ldots, 2^q$. It follows that $\mathcal{A}$ satisfies the identity

$$\big[[x_1, y_1], [s_1, t_1]\big] \cdot \big[[x_2, y_2], [s_2, t_2]\big] \cdot \quad \cdots \quad \cdot \big[[x_{2^q}, y_{2^q}], [s_{2^q}, t_{2^q}]\big] = 0. \tag{2.2}$$

Define $\mathcal{B} = \big[[\mathcal{A}, \mathcal{A}], [\mathcal{A}, \mathcal{A}]\big] := \big\{\big[[x, y], [z, w]\big] : x, y, z, w \in \mathcal{A}\big\}$. By (2.2),

$$\mathcal{B}^{2^q} := \{b_1 \cdot b_2 \cdot \quad \cdots \quad \cdot b_{2^q} : b_i \in \mathcal{B} \text{ for } i = 1, 2, \ldots, 2^q\} = 0,$$

and so $\mathfrak{D}^q(\mathcal{B}) = 0$, since every element of $\mathfrak{D}^q(\mathcal{B})$ is a linear combination of elements of $\mathcal{B}^{2^q}$. We get $\mathfrak{D}^{q+2}(\mathcal{A}) = 0$, because $\mathcal{B} = \mathfrak{D}^2(\mathcal{A})$. This completes the proof. □

We conclude the section with an example of an LD_2 subalgebra of $\mathrm{M}_6(K)$ which is not Ls_2.

Henceforth, we denote the $n \times n$ zero matrix by 0_n, the $n \times n$ identity matrix by I_n and the matrix unit of $\mathrm{M}_n(K)$ with 1 in position (i, j) and zeroes elsewhere by E_{ij}.

Example 2.4. Consider the following subalgebra of $\mathrm{M}_6(K)$ in block form:

$$\mathcal{A} = \begin{pmatrix} \mathrm{U}_3^*(K) & \mathrm{M}_3(K) \\ 0_3 & \mathrm{U}_3^*(K) \end{pmatrix}.$$

By the construction from Section 5, $\mathcal{A}$ is an LD_2 subalgebra of $\mathrm{M}_6(K)$ with maximum dimension (equal to 17).

To show that $\mathcal{A}$ is not Ls_2, consider the commutators

$$\left(\begin{array}{c|c} E_{13} & 0_3 \\ \hline 0_3 & 0_3 \end{array}\right) = \left[\left(\begin{array}{c|c} E_{12} & 0_3 \\ \hline 0_3 & 0_3 \end{array}\right), \left(\begin{array}{c|c} E_{23} & 0_3 \\ \hline 0_3 & 0_3 \end{array}\right)\right]$$

and

$$\left(\begin{array}{c|c} 0_3 & E_{31} \\ \hline 0_3 & 0_3 \end{array}\right) = \left[\left(\begin{array}{c|c} I_3 & 0_3 \\ \hline 0_3 & 0_3 \end{array}\right), \left(\begin{array}{c|c} 0_3 & E_{31} \\ \hline 0_3 & 0_3 \end{array}\right)\right].$$

Then

$$\left[\left(\begin{array}{c|c} E_{13} & 0_3 \\ \hline 0_3 & 0_3 \end{array}\right), \left(\begin{array}{c|c} 0_3 & E_{31} \\ \hline 0_3 & 0_3 \end{array}\right)\right] = \left(\begin{array}{c|c} 0_3 & E_{11} \\ \hline 0_3 & 0_3 \end{array}\right) \neq 0_6,$$

and so $\mathcal{A}$ is not Ls_2.

3. The maximum dimension of an LD_q subalgebra of $\mathrm{M}_n(K)$

In this section, we find (in Theorem 3.3) an upper bound for the dimension of an LD_q subalgebra of $\mathrm{M}_n(K)$. In Section 5, we will show that this upper bound is sharp.

We begin by proving LD_q counterparts of two results in [6] about D_q subalgebras of $\mathrm{M}_n(K)$, namely [6, Proposition 2.4] and [6, Theorem 3.1].

Lemma 3.1. *Let $\mathcal{A}$ be an LD_q algebra. Then the ideal I of $\mathcal{A}$ generated by the set $\{[[x, y], z] \colon x, y, z \in \mathcal{A}\}$ is nilpotent with $I^q = 0$. Moreover, if ℓ is the nilpotency index of I, i.e., $I^\ell = 0$ and $I^{\ell-1} \neq 0$, and if, additionally, $\mathcal{A}$ is a subalgebra of $M_n(K)$, then $\ell \leq n$.*

Proof. To show that $I^q = 0$, it suffices to replace all instances of x_i appearing in the commutator $[x_i, y_i]$ in the proof of [6, Proposition 2.4] with $[u_i, v_i]$, $i = 1, 2, \ldots, q$, since it then follows from the equality

$$\big[[u_1, v_1], y_1\big] \cdot \big[[u_2, v_2], y_2\big] \cdot \; \cdots \; \cdot \big[[u_q, v_q], y_q\big] = 0,$$

for all u_i, v_i, y_i, $i = 1, 2, \ldots, q$, that $I^q = 0$.

The second part of the lemma can also be proved by a slight modification of the arguments given in the last two paragraphs of the proof of [6, Proposition 2.4]. Since it is, in the present paper, convenient not to assume that an LD_q algebra is not necessarily LD_{q-1}, the index of nilpotency ℓ of the ideal I might possibly be strictly lower than q. Replacing q with ℓ in the mentioned arguments completes the proof. □

It follows from Lemma 3.1 that if $q > n$, then an LD_q subalgebra of $\mathrm{M}_n(K)$ is already LD_n. Note also that when $q = 1$, then we get an Ln_2 subalgebra of $\mathrm{M}_n(K)$, the maximum dimension of which was already determined in [8]. So, if it is not stated otherwise, if we refer to an LD_q subalgebra of $\mathrm{M}_n(K)$, then we assume that $1 < q \leq n$.

Proposition 3.2. *Let $\mathcal{A}$ be an LD_q subalgebra of $M_n(K)$, and let ℓ be the nilpotency index of the (nilpotent) ideal I generated by the set $\{[[x, y], z]\colon x, y, z \in \mathcal{A}\}$. Then there are positive integers $n_1, n_2, \ldots, n_\ell$ satisfying $\sum_{i=1}^{\ell} n_i = n$, an invertible matrix $X \in M_n(K)$ and an LD_ℓ subalgebra $\mathcal{B}$ of $M_n(K)$ in block triangular form*

$$\begin{pmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} & \ldots & \mathcal{B}_{1\ell} \\ & \mathcal{B}_{22} & \ldots & \mathcal{B}_{2\ell} \\ & & \ddots & \vdots \\ & & & \mathcal{B}_{\ell\ell} \end{pmatrix}, \tag{3.1}$$

such that $X^{-1}\mathcal{A}X \subseteq \mathcal{B}$, where $\mathcal{B}_{ij} = \mathrm{M}_{n_i \times n_j}(K)$ for all $1 \leq i < j \leq \ell$ and every $\mathcal{B}_{ii}$ is an Ln_2 subalgebra of $M_{n_i}(K)$.

Proof. We will point how to adjust the proof of [6, Theorem 3.1]. In the mentioned proof we deal with the (nilpotent of index q) ideal $\mathcal{C}_{\mathcal{A}}$, generated by all the commutators of $\mathcal{A}$. Instead, now take the ideal I in the statement of Proposition 3.2, which by Lemma 3.1 is nilpotent with nilpotency index ℓ for some $\ell \leq q$. The argument in the first paragraph of [6, Theorem 3.1] ensures that there exist an invertible matrix $X \in \mathrm{M}_n(K)$ and positive integers $n_1, n_2, \ldots, n_\ell$ satisfying $\sum_{i=1}^{\ell} n_i = n$ such that every matrix A' in the algebra $\mathcal{A}' = X^{-1}\mathcal{A}X$ can be written in the block triangular form

$$\begin{pmatrix} A'_{11} & A'_{12} & \ldots & A'_{1\ell} \\ & A'_{22} & \ldots & A'_{2\ell} \\ & & \ddots & \vdots \\ & & & A'_{\ell\ell} \end{pmatrix}, \tag{3.2}$$

where $A'_{ij} \in \mathrm{M}_{n_i \times n_j}(K)$ for all i and j such that $1 \le i \le j \le \ell$ (and all other entries are zero). Moreover, in the case when $A' \in X^{-1}IX$, then every A'_{ii} in (3.2) above is the zero $n_i \times n_i$ matrix, $i = 1, 2, \dots, \ell$. If we additionally notice that the ideal $X^{-1}IX$ of $\mathcal{A}'$ is equal to the ideal I' generated by the set $\{[[x', y'], z]\colon x', y', z' \in \mathcal{A}'\}$, then

$$\overline{\mathcal{A}'}_{ii} = \left\{ A'_{ii} \in \mathrm{M}_{n_i}(K) : \begin{pmatrix} A'_{11} & \dots & A'_{1i} & \dots & A'_{1\ell} \\ & \ddots & \vdots & \ddots & \vdots \\ & & A'_{ii} & \dots & A'_{i\ell} \\ & & & \ddots & \vdots \\ & & & & A'_{\ell\ell} \end{pmatrix} \in \mathcal{A}' \right\}$$

is an Ln_2 subalgebra of $\mathrm{M}_{n_i}(K)$ for every $i,\ i = 1, 2, \dots \ell$.

Now, by defining $\mathcal{B}$ to be the block triangular subalgebra

$$\left\{ \begin{pmatrix} X'_{11} & X'_{12} & \dots & X'_{1\ell} \\ & X'_{22} & \dots & X'_{2\ell} \\ & & \ddots & \vdots \\ & & & X'_{\ell\ell} \end{pmatrix} : X'_{ii} \in \overline{\mathcal{A}'}_{ii} \text{ for } i = 1, 2, \dots, \ell \text{ and } X'_{ij} \in \mathrm{M}_{n_i \times n_j}(K) \text{ if } i < j \right\}$$

of $\mathrm{M}_n(K)$, it follows that $\mathcal{B}$ is in the form in (3.1) and $\mathcal{A}' = X^{-1}\mathcal{A}X \subseteq \mathcal{B}$. Moreover, it is straightforward to verify that $\mathcal{B}$ is LD_ℓ. □

Before we state and prove the main result of this section, we first recall results from [8] about the maximum dimension of a Lie nilpotent subalgebra of $\mathrm{M}_n(K)$ of a (fixed) index, say q.

For positive integers q and n, define the function

$$M(q, n) = \max \left\{ 1 + \frac{1}{2}(n^2 - \sum_{i=1}^{q} {k_i}^2) \right\}, \tag{3.3}$$

where the maximum is taken over non-negative integers $k_1, k_2, \dots, k_q$ such that $\sum_{i=1}^{q} k_i = n$.

The main result in [8] is the following: if $\mathcal{A}$ is a Lie nilpotent subalgebra of $\mathrm{M}_n(K)$ of index q, then $\dim_K \mathcal{A} \le M(q+1, n)$, and the inequality is sharp.

For $1 \le q \le 7$, there is a simplified formula for the function $M(q, n)$ (see [8, Theorem 32, page 4578]):

$$M(q, n) = \left\lfloor \frac{n^2(q-1)}{2q} \right\rfloor + 1. \tag{3.4}$$

It follows from the main result in [8] and from (3.4) that the maximum dimensions of Lie nilpotent subalgebras of $\mathrm{M}_n(K)$ of indexes 2 and 5 are

$$M(3,n)=\left\lfloor \frac{n^2}{3}\right\rfloor+1 \qquad \text{and} \qquad M(6,n)=\left\lfloor \frac{5n^2}{12}\right\rfloor+1, \tag{3.5}$$

respectively.

We are now in a position to state and prove the main result in this section, and we wish to highlight the fact that the construction in Section 5 shows that the inequality in (3.6) is sharp.

Theorem 3.3. *Let $\mathcal{A}$ be an LD_q subalgebra of $M_n(K)$. Then*

$$\dim_K \mathcal{A} \le \max_{n_1+n_2+\cdots+n_q=n}\left\{q+\frac{n^2}{2}-\sum_{i=1}^{q}\left(\frac{n_i^2}{2}-\left\lfloor\frac{n_i^2}{3}\right\rfloor\right)\right\}, \tag{3.6}$$

where the n_i's are non-negative integers, $i=1,2,\ldots,q$.

Proof. Since the dimensions of the algebra $\mathcal{A}$ and the conjugation $X^{-1}\mathcal{A}X$ (for any invertible matrix $X \in \mathrm{M}_n(K)$) are equal, it follows from Proposition 3.2 that $\dim_K \mathcal{A} \le \dim_K \mathcal{B}$, where $\mathcal{B}$ is an LD_ℓ subalgebra of $\mathrm{M}_n(K)$ as in (3.1). Recall from the definition of ℓ in the statement of Lemma 3.1 that $\ell \le q$. We will show that

$$\dim_K \mathcal{B} \le \ell+\frac{n^2}{2}-\sum_{i=1}^{\ell}\left(\frac{n_i^2}{2}-\left\lfloor\frac{n_i^2}{3}\right\rfloor\right). \tag{3.7}$$

As $\dim_K \mathcal{A} \le \dim_K \mathcal{B}$, it will provide the inequality in (3.6), since we can extend (if necessary) the ℓ-tuple $(n_1,n_2,\ldots,n_\ell)$ to the q-tuple $(n_1,n_2,\ldots,n_\ell,0,\ldots,0)$.

Remember that $\mathcal{B}$ is a block triangular subalgebra of $\mathrm{M}_n(K)$ such that the diagonal block $\mathcal{B}_{ii}$ is an Ln_2 subalgebra of $\mathrm{M}_{n_i}(K)$ for every i, $i=1,2,\ldots,\ell$, and $\mathcal{B}_{ij}=\mathrm{M}_{n_i\times n_j}(K)$ if $1\le i<j\le q$. So, using the first part of (3.5), we obtain the inequality

$$\begin{aligned}\dim_K \mathcal{B} &= \sum_{1\le i\le j\le \ell}\dim_K \mathcal{B}_{ij}=\sum_{i=1}^{\ell}\dim_K \mathcal{B}_{ii}+\sum_{1\le i<j\le\ell}\dim_K \mathcal{B}_{ij}\le \\ &\le \sum_{i=1}^{\ell}\left(1+\left\lfloor\frac{n_i^2}{3}\right\rfloor\right)+\sum_{1\le i<j\le \ell}n_in_j. \end{aligned}\tag{3.8}$$

Since $\sum_{i=1}^{\ell} n_i = n$, it follows immediately that

$$\sum_{1\le i<j\le\ell} n_in_j=\frac{n^2}{2}-\sum_{i=1}^{\ell}\frac{n_i^2}{2},$$

and so we conclude from (3.8) that

$$\dim_K \mathcal{B} \leq \sum_{i=1}^{\ell}\left(1+\left\lfloor\frac{n_i^2}{3}\right\rfloor\right)+\frac{n^2}{2}-\sum_{i=1}^{\ell}\frac{n_i^2}{2}=\ell+\frac{n^2}{2}-\sum_{i=1}^{\ell}\left(\frac{n_i^2}{2}-\left\lfloor\frac{n_i^2}{3}\right\rfloor\right),$$

which, by the remark immediately following (3.7), completes the proof. □

4. An equivalent formula for the maximum dimension of an LD_q subalgebra of $\mathrm{M}_n(K)$

Invoking the function $M(q, n)$ in equation (3.3), we derive in this section a formula for the maximum dimension of an LD_q subalgebra of $\mathrm{M}_n(K)$ which is equivalent to the formula proven in Theorem 3.3. From the proven result, we obtain an upper bound on the dimension of a Lie solvable subalgebra of $\mathrm{M}_n(K)$ of index 2.

The mentioned equivalent formula (derived in this section) for the maximum dimension of an LD_q subalgebra of $\mathrm{M}_n(K)$ will be used in Section 5 in the construction of an LD_q subalgebra of $\mathrm{M}_n(K)$ with this maximum dimension.

Proposition 4.1. *Let q and n be positive integers such that $q \leq n$. Then*

$$\max_{n_1+n_2+\cdots+n_q=n}\left\{q+\frac{n^2}{2}-\sum_{i=1}^{q}\left(\frac{n_i^2}{2}-\left\lfloor\frac{n_i^2}{3}\right\rfloor\right)\right\}=q-1+M(3q,n),$$

where the sum is taken over non-negative integers $n_1, n_2, \ldots, n_q$.

Proof. Firstly, choose a $3q$-tuple of non-negative integers $(\ell_1, \ell_2, \ldots, \ell_{3q})$ such that

$$\sum_{i=1}^{3q}\ell_i = n \quad \text{and} \quad M(3q,n)=1+\frac{1}{2}\left(n^2-\sum_{i=1}^{3q}\ell_i^2\right).$$

Now define the q-tuple $(n'_1, n'_2, \ldots, n'_q)$, where $n'_i = \ell_{3i-2}+\ell_{3i-1}+\ell_{3i}$ for $i = 1, 2, \ldots, q$. Applying the relevant part of formula (3.5), we have that

$$M(3, n'_i) = 1 + \left\lfloor\frac{{n'_i}^2}{3}\right\rfloor,$$

and so

$$\frac{1}{2}({n'_i}^2-\ell_{3i-2}^2-\ell_{3i-1}^2-\ell_{3i}^2)\leq\left\lfloor\frac{{n'_i}^2}{3}\right\rfloor,$$

for each $i, i = 1, 2, \ldots, q$. Therefore,

$$q-1+M(3q,n)=q+\frac{1}{2}(n^2-\sum_{i=1}^{3q}\ell_i^2)=q+\frac{n^2}{2}+\sum_{i=1}^{q}\frac{1}{2}(-\ell_{3i-2}^2-\ell_{3i-1}^2-\ell_{3i}^2)$$

$$= q + \frac{n^2}{2} + \sum_{i=1}^{q} \frac{1}{2}(-{n'_i}^2 + {n'_i}^2 - \ell_{3i-2}^2 - \ell_{3i-1}^2 - \ell_{3i}^2) \leq q + \frac{n^2}{2} - \sum_{i=1}^{q} \left(\frac{{n'_i}^2}{2} - \left\lfloor \frac{{n'_i}^2}{3} \right\rfloor \right).$$

Since

$$\sum_{i=1}^{q} n'_i \;=\; \sum_{i=1}^{q} (\ell_{3i-2} + \ell_{3i-1} + \ell_{3i}) \;=\; \sum_{i=1}^{3q} \ell_i \;=\; n,$$

we conclude from the latter inequality that

$$q - 1 + M(3q, n) \leq \max_{n_1+n_2+\cdots+n_q=n} \left\{ q + \frac{n^2}{2} - \sum_{i=1}^{q} \left(\frac{n_i^2}{2} - \left\lfloor \frac{n_i^2}{3} \right\rfloor \right) \right\}.$$

To complete the proof, we have to prove the opposite inequality. To this end, choose a q-tuple of non-negative integers $(n'_1, n'_2, \ldots, n'_q)$ such that $\sum_{i=1}^{q} n'_i = n$ and

$$\max_{n_1+n_2+\cdots+n_q=n} \left\{ q + \frac{n^2}{2} - \sum_{i=1}^{q} \left(\frac{n_i^2}{2} - \left\lfloor \frac{n_i^2}{3} \right\rfloor \right) \right\} = q + \frac{n^2}{2} - \sum_{i=1}^{q} \left(\frac{{n'_i}^2}{2} - \left\lfloor \frac{{n'_i}^2}{3} \right\rfloor \right). \quad (4.1)$$

Again, form the relevant part of formula (3.5),

$$M(3, n'_i) = 1 + \left\lfloor \frac{{n'_i}^2}{3} \right\rfloor,$$

for $i = 1, 2, \ldots, q$. Hence, for each i, there exist non-negative integers ℓ_{3i-2}, ℓ_{3i-1} and ℓ_{3i} such that

$$\ell_{3i-2} + \ell_{3i-1} + \ell_{3i} = n'_i \quad \text{and} \quad 1 + \frac{1}{2}\left({n'_i}^2 - \ell_{3i-2}^2 - \ell_{3i-1}^2 - \ell_{3i}^2\right) = 1 + \left\lfloor \frac{{n'_i}^2}{3} \right\rfloor,$$

which implies that

$$\begin{aligned} q + \frac{n^2}{2} - \sum_{i=1}^{q} \left(\frac{{n'_i}^2}{2} - \left\lfloor \frac{{n'_i}^2}{3} \right\rfloor \right) &= q + \frac{n^2}{2} - \sum_{i=1}^{q} \left(\frac{{n'_i}^2}{2} - \frac{1}{2}({n'_i}^2 - \ell_{3i-2}^2 - \ell_{3i-1}^2 - \ell_{3i}^2) \right) \\ &= q + \frac{n^2}{2} - \sum_{i=1}^{3q} \frac{\ell_i^2}{2}. \end{aligned} \quad (4.2)$$

It follows from the definition of numbers n'_i and ℓ_i that $\sum_{i=1}^{3q} \ell_i = n$. Finally, from equations (4.1) and (4.2), and by the definition of the function $M(3q, n)$,

$$\max_{n_1+n_2+\ldots n_q=n} \left\{ q + \frac{n^2}{2} - \sum_{i=1}^{q} \left(\frac{n_i^2}{2} - \left\lfloor \frac{n_i^2}{3} \right\rfloor \right) \right\} = q + \frac{n^2}{2} - \sum_{i=1}^{q} \left(\frac{{n'_i}^2}{2} - \left\lfloor \frac{{n'_i}^2}{3} \right\rfloor \right) =$$

$$= q + \frac{n^2}{2} - \sum_{i=1}^{3q} \frac{\ell_i^2}{2} \leq q - 1 + M(3q, n),$$

which completes the proof. □

Invoking the upper bound from Theorem 3.3, which by the construction given in Section 5 is sharp, we therefore immediately have the following result:

Corollary 4.2. *Let $\mathcal{A}$ be an LD_q subalgebra of $M_n(K)$. Then*

$$\dim_K \mathcal{A} \leq q - 1 + M(3q, n),$$

and the inequality is sharp.

In the case $q = 2$ in Corollary 4.2, we get the inequality $\dim_K \mathcal{A} \leq 1 + M(6, n)$. Using Theorem 2.1 and the formula $M(6, n) = \left\lfloor \frac{5n^2}{12} \right\rfloor + 1$ in (3.5), we obtain the following upper bound for the dimension of a Lie solvable subalgebra of $\mathrm{M}_n(K)$ of index 2:

Corollary 4.3. *Let $\mathcal{A}$ be an Ls_2 subalgebra of $M_n(K)$. Then*

$$\dim_K \mathcal{A} \leq 2 + \left\lfloor \frac{5n^2}{12} \right\rfloor.$$

In contrast to Corollary 4.2, we do not know if the upper bound in Corollary 4.3 is sharp. Since every D_2 algebra is Ls_2, we conclude, invoking the sharp upper bound $2 + \left\lfloor \frac{3n^2}{8} \right\rfloor$ (see, for example, [11, Proposition 16]) for the dimension of a D_2 subalgebra of $\mathrm{M}_n(K)$, that:

Proposition 4.4. *Let $\mathcal{A}$ be an Ls_2 subalgebra of $M_n(K)$ with maximum dimension. Then*

$$2 + \left\lfloor \frac{3n^2}{8} \right\rfloor \leq \dim_K \mathcal{A} \leq 2 + \left\lfloor \frac{5n^2}{12} \right\rfloor.$$

Corollary 4.5. *Let $\mathcal{A}$ be an Ls_2 subalgebra of $M_n(K)$ with maximum dimension. Then*

$$\dim_K \mathcal{A} = \begin{cases} 3 & \textit{if} \quad n = 2, \\ 5 & \textit{if} \quad n = 3, \\ 8 & \textit{if} \quad n = 4. \end{cases}$$

5. Construction of an LD_q subalgebra of $\mathrm{M}_n(K)$ with maximum dimension

In this section we will construct an example of an LD_q subalgebra of $\mathrm{M}_n(K)$ with maximum dimension. The construction will be based on the example of a Lie nilpotent subalgebra of $\mathrm{M}_n(K)$ of (some fixed) index q (say) with maximum dimension presented in [8], which we recall here for the ease of the reader:

Definition 5.1 *(see [8], pages 4554-4556 and pages 4575-4577).* Let $k_1, k_2, \ldots, k_{q+1}$ be a sequence of positive integers such that $k_1 + k_2 + \cdots + k_{q+1} = n$. For each $p \in \{1, 2, \ldots, q\}$, define the rectangular array

$$B_p :=$$

$$\begin{cases} \{(i,j) \in \mathbb{N} \times \mathbb{N} : 1 \le i \le k_1 < j \le n\} & \text{if } p = 1, \\ \{(i,j) \in \mathbb{N} \times \mathbb{N} : k_1 + k_2 + \cdots + k_{p-1} < i \le k_1 + k_2 + \cdots + k_p < j \le n\} & \text{if } p > 1. \end{cases}$$

Put $B := \bigcup_{p=1}^{q} B_p$, and consider the subset

$$J := \left\{ \sum_{(i,j) \in B} b_{ij} E_{ij} : \ b_{ij} \in K \text{ for all } (i,j) \in B \right\}$$

of $\mathrm{M}_n(K)$. We define $\mathcal{A} = KI_n + J$ and call it the *algebra of $n \times n$ matrices over K of type* $(k_1, k_2, \ldots, k_{q+1})$. As $J^{q+1} = 0$, the algebra $\mathcal{A}$ is Lie nilpotent of index q (i.e., Ln_q (see Definition 1.2)).

Note that $q + 1 \le n$ in the above definition. Let $r < q + 1$ be the non-negative integer such that $r \equiv n \bmod (q + 1)$, and consider the positive integers

$$k_i := \begin{cases} \left\lfloor \frac{n}{q+1} \right\rfloor & \text{if} \quad 1 \le i \le q + 1 - r, \\ \left\lfloor \frac{n}{q+1} \right\rfloor + 1 & \text{if} \quad q + 2 - r \le i \le q + 1. \end{cases}$$

Then the algebra of $n \times n$ matrices of type $(k_1, k_2, \ldots, k_{q+1})$ is an Ln_q subalgebra of $\mathrm{M}_n(K)$ with maximum dimension.

Example 5.2. The algebra

$$KI_4 + \begin{pmatrix} 0 & K & K & K \\ & 0 & K & K \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}$$

of 4×4 matrices of type $(1, 1, 2)$ is an Ln_2 subalgebra of $\mathrm{M}_4(K)$ with maximum dimension, and the algebra

$$KI_6 + \begin{pmatrix} 0 & K & K & K & K & K \\ & 0 & K & K & K & K \\ & & 0 & 0 & K & K \\ & & & 0 & K & K \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}$$

of 6×6 matrices of type $(1, 1, 2, 2)$ is an Ln_3 subalgebra of $\mathrm{M}_6(K)$ with maximum dimension.

We will now construct an LD_q subalgebra $\mathcal{A}$ of $\mathrm{M}_n(K)$ with maximum dimension (recall that we assume that $q \leq n$ (see the paragraph immediately following the proof of Lemma 3.1)) in the form

$$\mathcal{A} = K\left(\sum_{i=1}^{n_1} E_{ii}\right) + K\left(\sum_{i=n_1+1}^{n_1+n_2} E_{ii}\right) + \cdots + K\left(\sum_{i=n_1+\cdots+n_{q-1}+1}^{n} E_{ii}\right) + J, \tag{5.1}$$

where we will define positive integers n_i such that $\sum_{i=1}^{q} n_i = n$ and a subset J of the strictly upper triangular $n \times n$ matrices with different formulas depending on two cases. In both these cases we will show that $\mathcal{A}$ is in block triangular form

$$\begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \ldots & \mathcal{A}_{1q} \\ & \mathcal{A}_{22} & \ldots & \mathcal{A}_{2q} \\ & & \ddots & \vdots \\ & & & \mathcal{A}_{qq} \end{pmatrix}, \tag{5.2}$$

where $\mathcal{A}_{ii}$, $i = 1, 2, \ldots, q$, is an Ln_2 or a commutative subalgebra of $\mathrm{M}_{n_i}(K)$, and $\mathcal{A}_{ij} = \mathrm{M}_{n_i \times n_j}(K)$ for every $i < j$.

It is evident that such a block triangular subalgebra is LD_q, and so, after showing that $\mathcal{A}$ is in form (5.2), we will only have to verify the dimension of $\mathcal{A}$.

It follows from Theorem 3.3 and Proposition 4.1 that every LD_q subalgebra of $\mathrm{M}_n(K)$ has dimension less than or equal to $q-1+M(3q, n)$ (see formula (3.3)). The constructed subalgebra $\mathcal{A}$ of $\mathrm{M}_n(K)$ will have dimension precisely equal to $q-1+M(3q, n)$, and so $\mathcal{A}$ is an LD_q subalgebra of $\mathrm{M}_n(K)$ with maximum dimension.

Case 1. $q \leq n/3$: Consider the $3q$-tuple $(\ell_1, \ell_2, \ldots, \ell_{3q})$ defined by the formula

$$\ell_i = \begin{cases} \left\lfloor \frac{n}{3q} \right\rfloor & \text{if} \quad 1 \leq i \leq 3q - r, \\ \left\lfloor \frac{n}{3q} \right\rfloor + 1 & \text{if} \quad 3q - r + 1 \leq i \leq 3q, \end{cases}$$

where $r < 3q$ is the non-negative integer such that $n \equiv r \pmod{3q}$. Let J be the subset of $\mathrm{M}_n(K)$ such that $KI_n + J$ is the algebra of $n \times n$ matrices of type $(\ell_1, \ell_2, \ldots, \ell_{3q})$ (see Definition 5.1), and set

$$n_i = \ell_{3i-2} + \ell_{3i-1} + \ell_{3i}$$

for $i = 1, 2, \ldots, q$. Note that

$$\dim_K \mathcal{A} = q + \dim_K J = q - 1 + M(3q, n),$$

since $KI_n + J$ is an Ln_{3q-1} subalgebra of $\mathrm{M}_n(K)$ with maximum dimension equal to $M(3q, n)$.

As already mentioned, the last equality and the block triangular form in (5.2) show that $\mathcal{A}$ is an LD_q subalgebra of $\mathrm{M}_n(K)$ with maximum dimension.

Case 2. $n/3 < q \leq n$: Then $\left\lfloor \frac{n}{3q} \right\rfloor = 0$, and so we cannot directly adopt the construction from the previous case, since the algebra of $n \times n$ matrices of type

$$\left(\left\lfloor \frac{n}{3q} \right\rfloor, \ldots, \left\lfloor \frac{n}{3q} \right\rfloor, \left\lfloor \frac{n}{3q} \right\rfloor + 1, \ldots, \left\lfloor \frac{n}{3q} \right\rfloor + 1 \right) \tag{5.3}$$

with at least one appearance of $\left\lfloor \frac{n}{3q} \right\rfloor$ in the $3q$-tuple in (5.3) does not make sense. Instead, define

$$n_i = \begin{cases} \left\lfloor \frac{n}{q} \right\rfloor & \text{if} \quad 1 \leq i \leq q - r, \\ \left\lfloor \frac{n}{q} \right\rfloor + 1 & \text{if} \quad q - r + 1 \leq i \leq q, \end{cases}$$

where $r < q$ is the non-negative integer such that $n \equiv r \pmod{q}$, and let J be the set of all the strictly upper triangular $n \times n$ matrices.

Note that $1 \leq \left\lfloor \frac{n}{q} \right\rfloor < 3$, and so every n_i satisfies the inequality $1 \leq n_i \leq 3$. Moreover

$$\sum_{i=1}^{q} n_i = (q - r) \left\lfloor \frac{n}{q} \right\rfloor + r \left(\left\lfloor \frac{n}{q} \right\rfloor + 1 \right) = q \left\lfloor \frac{n}{q} \right\rfloor + r = n.$$

It follows that $\mathcal{A}$ defined by formula (5.1) can be written in block form (5.2), with

$$\mathcal{A}_{ii} = \begin{cases} K & \text{if } n_i = 1, \\ \mathrm{U}_2^*(K) & \text{if } n_i = 2, \\ \mathrm{U}_3^*(K) & \text{if } n_i = 3. \end{cases}$$

The algebras K and $\mathrm{U}_2^*(K)$ are commutative, and the algebra $\mathrm{U}_3^*(K)$ is Ln_2. Next, we verify the dimension of $\mathcal{A}$. The construction of $\mathcal{A}$ shows that

$$\dim_K \mathcal{A} = q + \frac{n^2-n}{2} = q - 1 + \left(1 + \frac{n^2-n}{2}\right).$$

Keeping in mind that $n < 3q$ in the case we are dealing with now, we invoke [8, Corollary 27]:

$$M(3q, n) = 1 + \frac{n^2-n}{2}.$$

Therefore, $\dim_K \mathcal{A} = q - 1 + M(3q, n)$.

As in the previous case, the obtained dimension and the block triangular form ensure that $\mathcal{A}$ is an LD_q subalgebra of $\mathrm{M}_n(K)$ with maximum dimension.

Note that if $q = n$, then $\mathcal{A}$ is the entire upper triangular matrix algebra $\mathrm{U}_n(K)$.

Example 5.3. (i) The LD_2 subalgebra $\mathcal{A}$ of $\mathrm{M}_7(K)$ with maximum dimension described in Case 1 above is the following:

$$K \cdot \begin{pmatrix} I_3 & 0_{3\times 4} \\ 0_{4\times 3} & 0_4 \end{pmatrix} + K \cdot \begin{pmatrix} 0_3 & 0_{3\times 4} \\ 0_{4\times 3} & I_4 \end{pmatrix} + \begin{pmatrix} 0 & K & K & K & K & K & K \\ & 0 & K & K & K & K & K \\ & & 0 & K & K & K & K \\ & & & 0 & K & K & K \\ & & & & 0 & K & K \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{pmatrix}.$$

Note that $\dim \mathcal{A} = 22$, which agrees with the formula $2 + \left\lfloor \frac{5\cdot 7^2}{12} \right\rfloor$ obtained right after Corollary 4.2.

We observe that the algebra $\mathcal{A}$ above is related to some Ln_2 and Ln_5 subalgebras of the relevant full matrix algebras. Firstly, notice that

$$KI_7 + \begin{pmatrix} 0 & K & K & K & K & K & K \\ & 0 & K & K & K & K & K \\ & & 0 & K & K & K & K \\ & & & 0 & K & K & K \\ & & & & 0 & K & K \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{pmatrix} \tag{5.4}$$

is the algebra of 7×7 matrices of type $(1, 1, 1, 1, 1, 2)$, which is an Ln_5 subalgebra of $\mathrm{M}_7(K)$ with maximum dimension. The only difference between $\mathcal{A}$ and the algebra in (5.4) is the entries of matrices in the respective algebras on the main diagonal. Secondly, let

$$\mathcal{A}_{11} = KI_3 + \begin{pmatrix} 0 & K & K \\ & 0 & K \\ & & 0 \end{pmatrix} = \mathrm{U}_3^*(K) \quad \text{and} \quad \mathcal{A}_{22} = KI_4 + \begin{pmatrix} 0 & K & K & K \\ & 0 & K & K \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}.$$

Then $\mathcal{A}_{11}$ is an Ln_2 subalgebra of $\mathrm{M}_3(K)$ with maximum dimension, and $\mathcal{A}_{22}$ is an Ln_2 subalgebra of $\mathrm{M}_4(K)$ with maximum dimension (see Example 5.2). These algebras appear in the following representation of $\mathcal{A}$ in block triangular form:

$$\begin{pmatrix} \mathcal{A}_{11} & \mathrm{M}_{3\times 4}(K) \\ & \mathcal{A}_{22} \end{pmatrix}.$$

Note also that the LD_2 subalgebra of $\mathrm{M}_6(K)$ with maximum dimension constructed in Case 1 above is the algebra

$$\mathcal{A} = K \cdot \begin{pmatrix} I_3 & 0_3 \\ 0_3 & 0_3 \end{pmatrix} + K \cdot \begin{pmatrix} 0_3 & 0_3 \\ 0_3 & I_3 \end{pmatrix} + \begin{pmatrix} 0 & K & K & K & K & K \\ & 0 & K & K & K & K \\ & & 0 & K & K & K \\ & & & 0 & K & K \\ & & & & 0 & K \\ & & & & & 0 \end{pmatrix},$$

which is precisely the block triangular subalgebra $\begin{pmatrix} \mathrm{U}_3^*(K) & \mathrm{M}_3(K) \\ 0_3 & \mathrm{U}_3^*(K) \end{pmatrix}$ of $\mathrm{M}_6(K)$ in Example 2.4.

(ii) The LD_3 subalgebra $\mathcal{A}$ of $\mathrm{M}_7(K)$ with maximum dimension constructed in Case 2 above (note that $\frac{7}{3} < 3 < 7$) yields the following:

$$\mathcal{A} = K \cdot \begin{pmatrix} I_2 & 0_{2\times 5} \\ 0_{5\times 2} & 0_5 \end{pmatrix} + K \cdot \begin{pmatrix} 0_2 & 0_2 & 0_{2\times 3} \\ 0_2 & I_2 & 0_{2\times 3} \\ 0_{3\times 2} & 0_{3\times 2} & 0_3 \end{pmatrix} + K \cdot \begin{pmatrix} 0_4 & 0_{4\times 3} \\ 0_{3\times 4} & I_3 \end{pmatrix} +$$

$$+ \begin{pmatrix} 0 & K & K & K & K & K & K \\ & 0 & K & K & K & K & K \\ & & 0 & K & K & K & K \\ & & & 0 & K & K & K \\ & & & & 0 & K & K \\ & & & & & 0 & K \\ & & & & & & 0 \end{pmatrix}.$$

In this case we can write $\mathcal{A}$ in the block triangular form

$$\begin{pmatrix} \mathcal{A}_{11} & \mathrm{M}_2(K) & \mathrm{M}_{2\times 3}(K) \\ & \mathcal{A}_{22} & \mathrm{M}_{2\times 3}(K) \\ & & \mathcal{A}_{33} \end{pmatrix},$$

where

$$\mathcal{A}_{11} = KI_2 + \begin{pmatrix} 0 & K \\ & 0 \end{pmatrix}, \quad \mathcal{A}_{22} = KI_2 + \begin{pmatrix} 0 & K \\ & 0 \end{pmatrix}, \quad \mathcal{A}_{33} = KI_3 + \begin{pmatrix} 0 & K & K \\ & 0 & K \\ & & 0 \end{pmatrix}.$$

Note that $\mathcal{A}_{11}$ and $\mathcal{A}_{22}$ are commutative algebras.

Declaration of competing interest

None declared.

Data availability

No data was used for the research described in the article.

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