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Research Paper

The structure of subalgebras of full matrix algebras over a field satisfying the identity $[x_1, y_1][x_2, y_2] \cdots [x_q, y_q] = 0$

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ABSTRACT

A subalgebra of the full matrix algebra $\mathrm{M}_n(K)$, K a field, satisfying the identity $[x_1, y_1][x_2, y_2] \cdots [x_q, y_q] = 0$ is called a D_q subalgebra of $\mathrm{M}_n(K)$. In the paper we deal with the structure, conjugation and isomorphism problems of maximal D_q subalgebras of $\mathrm{M}_n(K)$.
We show that a maximal D_q subalgebra $\mathcal{A}$ of $\mathrm{M}_n(K)$ is conjugated with a block triangular subalgebra of $\mathrm{M}_n(K)$ with maximal commutative diagonal blocks. By analysis of conjugations, the sizes of the obtained diagonal blocks are uniquely determined. It reduces the problem of conjugation of maximal D_q subalgebras of $\mathrm{M}_n(K)$ to the analogous problem in the class of commutative subalgebras of $\mathrm{M}_n(K)$. Further examining conjugations, in case $\mathcal{A}$ is contained in the upper triangular matrix algebra $\mathrm{U}_n(K)$, we prove that $\mathcal{A}$ is already in a block triangular form.
We consider the isomorphism problem in a certain class of maximal D_q subalgebras of $\mathrm{M}_n(K)$ which contain all D_q subalgebras of $\mathrm{M}_n(K)$ with maximum dimension. In case K is algebraically closed, we invoke Jacobson's characterization of maximal commutative subalgebras of $\mathrm{M}_n(K)$ with maximum (K-)dimension to show that isomorphic subalgebras in this

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class are already conjugated. To illustrate it, we invoke results from [19] and find all isomorphism (equivalently conjugation) classes of D_q subalgebras of $\mathrm{M}_n(K)$ with maximum possible dimension, in case K is algebraically closed.

Contents

1. Motivation, background and results on D_q subalgebras of full matrix algebras

Throughout the sequel, all algebras are assumed to be associative unital and over a field K. By a subalgebra of the full $n \times n$ matrix algebra $\mathrm{M}_n(K)$ we mean a K-subalgebra of $\mathrm{M}_n(K)$.

The main and direct motivation for the work presented here comes from [3], [19] and [20]. In [3], Domokos deals with the identity

$$[x_1, y_1]\,[x_2, y_2] \cdots [x_q, y_q] = 0 \tag{1.1}$$

in the context of subalgebras of $\mathrm{M}_n(K)$. Here, $[x, y]$ denotes the commutator Lie product $xy - yx$ (also called the Lie bracket in the literature), and q is a positive integer.

We note that $\mathrm{M}_n(K)$ with the commutator Lie product plays an exceptional role in the theory of finite-dimensional Lie algebras. The fundamental Ado-Iwasawa theorem (see [6]) asserts that every finite-dimensional Lie K-algebra can be embedded into $\mathrm{M}_n(K)$ for some $n \geq 1$.

Finite dimensional basic algebras over algebraically closed fields play an important role in the representation theory of Artinian algebras (see [1]). Such algebras satisfy (1.1) for some q. An Artinian ring R satisfies (1.1) for some q if and only if $R/\mathrm{rad}(R)$ is commutative, in which case the index of nilpotency of $\mathrm{rad}(R)$ is an upper bound for the least such q.

The identity in (1.1) has featured prominently in many other papers. See, for example, [2], [10], [11] and [12]. It was proved in [10] that all the polynomial identities of the upper triangular $q \times q$ matrix algebra $\mathrm{U}_q(K)$ over a field K of characteristic 0 are consequences of only one identity, namely the identity in (1.1). The case of any characteristic was handled in [13] and [16], where it was shown that over an infinite field of any characteristic the identities of $\mathrm{U}_q(K)$ follow from (1.1). When the field K is finite, then the identities of $\mathrm{U}_q(K)$ coincide with $T(K)^q$, where $T(K)$ is the T-ideal of the identities of the field K. (It is well known that $T(K)$ is generated by the identity $x^m - x$, when $|K| = m$.) For an explicit form of a finite set of generators of an ideal of identities of the algebra $\mathrm{U}_q^\star(R)$ over a commutative integral domain R, see [11]. Here $\mathrm{U}_q^\star(R)$ denotes the subalgebra of $\mathrm{U}_q(R)$ comprising all the matrices (in $\mathrm{U}_q(R)$) with constant main diagonal.

The 9×9 matrix algebra $\mathrm{U}_3^\star\big(\mathrm{U}_3^\star(R)\big)$ over any commutative ring R was exhibited in [12] as an example of an algebra satisfying the polynomial identity $[[x_1, y_1], [x_2, y_2]] = 0$ (Lie solvability index two), but none of the stronger identities $[x_1, y_1][x_2, y_2] = 0$ (the identity in (1.1), with $q = 2$) and $[[x, y], z] = 0$ (Lie nilpotency index two). A Cayley-Hamilton trace identity was exhibited in [12] for 2×2 matrices with entries in a ring R satisfying $[x_1, y_1][x_2, y_2] = 0$ and $\frac{1}{2} \in R$. See also [17].

The Cayley-Hamilton theorem and the corresponding trace identity play a crucial role (see [4] and [5]) in proving classical results about the polynomial and trace identities of $\mathrm{M}_n(K)$. In case $\mathrm{char}(K) = 0$, Kemer's pioneering work (see [8]) on the T-ideals of associative algebras revealed the significance of the identities satisfied by the $n \times n$ matrices over the Grassmann (exterior) algebra generated by an infinite sequence of anticommutative indeterminates.

If an algebra satisfies (1.1), then we say that it is D_q, and if a subalgebra of $\mathrm{M}_n(K)$ is D_q, then we say that it is a D_q subalgebra of $\mathrm{M}_n(K)$.

Considering D_1, i.e., when $q = 1$, we get exactly commutativity, which in the context of subalgebras of $\mathrm{M}_n(K)$, features prominently in the cited literature (see, for example, [7] and [15]). In particular, a classical result by Schur (see [15]) states that the maximum K-dimension of a commutative subalgebra of $\mathrm{M}_n(K)$, with K an algebraically closed field, is $\left\lfloor \frac{n^2}{4} \right\rfloor + 1$. Here $\lfloor\ \rfloor$ denotes the integer floor function. Schur's result was extended to an arbitrary field by Jacobson in [7]. We often write dimension instead of K-dimension.

If we say that an algebra $\mathcal{A}$ is maximal in an algebra $\mathcal{E}$ with respect to some conditions then we think about the inclusion relation. If, in the context of some class of subalgebras of a finite dimensional algebra $\mathcal{E}$, we say that $\mathcal{A}$ has maximum dimension, then we mean that $\mathcal{A}$ has maximum possible dimension in the considered class.

The mentioned maximum dimension $\left\lfloor \frac{n^2}{4} \right\rfloor + 1$ of a commutative subalgebra of $\mathrm{M}_n(K)$ is obtained by considering the subalgebra

$$KI_n + \begin{pmatrix} 0_\ell & \mathrm{M}_\ell(K) \\ 0_\ell & 0_\ell \end{pmatrix} \tag{1.2}$$

of $\mathrm{M}_n(K)$ if n is even (with $n = 2\ell$, for some integer ℓ), and by considering the subalgebra

$$KI_n + \begin{pmatrix} 0_\ell & \mathrm{M}_{\ell\times(\ell+1)}(K) \\ 0_{(\ell+1)\times\ell} & 0_{\ell+1} \end{pmatrix} \tag{1.3}$$

of $\mathrm{M}_n(K)$ if n is odd (with $n = 2\ell + 1$). Here, for example, 0_ℓ and $0_{(\ell+1)\times\ell}$ denote the $\ell \times \ell$ and $(\ell + 1) \times \ell$ zero matrices, respectively.

Henceforth, when we consider a D_q subalgebra $\mathcal{A}$ of $\mathrm{M}_n(K)$, then we always assume that $q \geq 2$ and that $\mathcal{A}$ is not D_{q-1} (and hence not D_1).

The main result in [3] is the following:

Theorem 1.1. *([3, Theorem 1]) Let K be a field, and $\mathcal{A}$ a finite dimensional K-algebra satisfying (1.1). If M is a finitely generated faithful module over $\mathcal{A}$, then*

$$\dim_K M \geq \sqrt{\frac{\dim_K \mathcal{A} - q}{\frac{1}{2} - \frac{1}{4q}}}. \tag{1.4}$$

In the proof of the above theorem, Domokos shows that, for the considered K-algebra $\mathcal{A}$,

$$\dim_K \mathcal{A} \leq \frac{1}{2}(\dim_K M)^2 + q - \sum_{i=1}^{q} \left(\frac{n_i^2}{2} - \left\lfloor \frac{n_i^2}{4} \right\rfloor \right) \tag{1.5}$$

for some positive integers $n_1, \dots, n_q$ such that $n_1 + \cdots + n_q = \dim_K M$.

If one takes $M = K^n$, then the right hand side in (1.5) takes the form

$$\frac{1}{2}\left(n_1 + \cdots + n_q\right)^2 + q - \sum_{i=1}^{q} \left(\frac{n_i^2}{2} - \left\lfloor \frac{n_i^2}{4} \right\rfloor \right), \tag{1.6}$$

which equals the expression in (1.13) below, implying that the inequality in (1.5) is sharp.

In [3], the n_i's are mentioned as any numbers which guarantee that

$$\frac{1}{2}(\dim_K M)^2 + q - \sum_{i=1}^{q} \left(\frac{n_i^2}{2} - \left\lfloor \frac{n_i^2}{4} \right\rfloor \right)$$

is a maximum. Evidently, such an q-tuple $(n_1, n_2, \dots, n_q)$ exists, but it is not exhibited in [3]. In this regard, we refer the reader to [19], where such an q-tuple is explicitly described and the maximum is exhibited precisely:

Theorem 1.2. *([19, Theorem 14]) Let $1 \leq q \leq n$, and let $n = q\left\lfloor \frac{n}{q} \right\rfloor + r$, $0 \leq r < q$ (with r as in the Division Algorithm). Then the precise sharp upper bound for the dimension of a D_q subalgebra of $\mathrm{M}_n(K)$ is*

$$\frac{1}{2}\left(n^2 - (q-r)\left\lfloor \frac{n}{q} \right\rfloor^2 - r\left(\left\lfloor \frac{n}{q} \right\rfloor + 1\right)^2\right) + q + (q-r)\left\lfloor \frac{\left\lfloor \frac{n}{q} \right\rfloor^2}{4} \right\rfloor + r\left\lfloor \frac{\left(\left\lfloor \frac{n}{q} \right\rfloor + 1\right)^2}{4} \right\rfloor, \tag{1.7}$$

which can be obtained by choosing $q-r$ *commutative subalgebras of* $\mathrm{M}_{\lfloor \frac{n}{q} \rfloor}(K)$ *of dimension* $\left\lfloor \frac{\lfloor \frac{n}{q} \rfloor^2}{4} \right\rfloor + 1$ *and* r *commutative subalgebras of* $\mathrm{M}_{\lfloor \frac{n}{q} \rfloor + 1}(K)$ *of dimension* $\left\lfloor \frac{\left(\lfloor \frac{n}{q} \rfloor + 1\right)^2}{4} \right\rfloor + 1$ *on the diagonal blocks for the algebra presented in (1.12) (see also [3, page 157]).*

In this vein we also draw the reader's attention to [18], where the maximum dimension of a Lie nilpotent subalgebra of $\mathrm{M}_n(K)$ of index m is obtained.

In general, if $\mathcal{A}$ is a subalgebra of $\mathrm{M}_n(K)$ and every matrix $A \in \mathcal{A}$ is seen in the block triangular form

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1q} \\ & A_{22} & \dots & A_{2q} \\ & & \ddots & \vdots \\ & & & A_{qq} \end{pmatrix},$$

where $A_{ij} \in \mathrm{M}_{n_i \times n_j}(K)$ for all $i \leq j$, then considering the set

$$\overline{\mathcal{A}}_{ii} = \left\{ A_{ii} \in \mathrm{M}_{n_i}(K) : \begin{pmatrix} A_{11} & \dots & A_{1i} & \dots & A_{1q} \\ & \ddots & \vdots & \ddots & \vdots \\ & & A_{ii} & \dots & A_{iq} \\ & & & \ddots & \vdots \\ & & & & A_{qq} \end{pmatrix} \in \mathcal{A} \right\}, \tag{1.8}$$

it is important to note that

$$\begin{pmatrix} 0_{n_1} & \cdots & 0_{n_1 \times n_{i-1}} & 0_{n_1 \times n_i} & 0_{n_1 \times n_{i+1}} & \cdots & 0_{n_1 \times n_q} \\ & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & 0_{n_{i-1}} & 0_{n_{i-1} \times n_i} & 0_{n_{i-1} \times n_{i+1}} & \cdots & 0_{n_{i-1} \times n_q} \\ & & & \overline{\mathcal{A}}_{ii} & 0_{n_i \times n_{i+1}} & \cdots & 0_{n_i \times n_q} \\ & & & & 0_{n_{i+1}} & \cdots & 0_{n_{i+1} \times n_q} \\ & & & & & \ddots & \vdots \\ & & & & & & 0_{n_q} \end{pmatrix} \tag{1.9}$$

need not be a subset of $\mathcal{A}$. If the set in (1.9) is indeed contained in $\mathcal{A}$ for every i, $i = 1, 2, \dots, q$, then we say that the algebras $\overline{\mathcal{A}}_{ii}$ are *independent*.

For example, every matrix A in the subalgebra

$$\left\{ \begin{pmatrix} a & b & c & e & f & g & t & u & v \\ 0 & a & d & 0 & e & h & 0 & t & w \\ 0 & 0 & a & 0 & 0 & e & 0 & 0 & t \\ 0 & 0 & 0 & a & b & c & p & q & r \\ 0 & 0 & 0 & 0 & a & d & 0 & p & s \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 0 & a & b & c \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & d \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix} : a,b,c,d,e,f,g,h,p,q,r,s,t,u,v,w \in K \right\}$$

of $\mathrm{U}_9(K)$ in [12, Corollary 2.2] can be written in the block triangular form

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ & A_{22} & A_{23} \\ & & A_{33} \end{pmatrix},$$

with $A_{ij} \in \mathrm{M}_3(K)$ for $1 \le i \le j \le q$. In this example the three algebras $\overline{\mathcal{A}}_{11}$, $\overline{\mathcal{A}}_{22}$ and $\overline{\mathcal{A}}_{33}$ are not independent.

We will consider block triangular subalgebras of $\mathrm{M}_n(K)$ where the sizes of the diagonal blocks play an important role:

Definition 1.3. For any positive integers $n_1, \ldots, n_q$, with $q \ge 2$, such that $n_1 + \cdots + n_q = n$, consider a block triangular subalgebra

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \ldots & \mathcal{A}_{1q} \\ & \mathcal{A}_{22} & \ldots & \mathcal{A}_{2q} \\ & & \ddots & \vdots \\ & & & \mathcal{A}_{qq} \end{pmatrix} \tag{1.10}$$

of $\mathrm{M}_n(K)$ where

(1) $\mathcal{A}_{ii}$ is a subalgebra of $\mathrm{M}_{n_i}(K)$ for every $i, i = 1, 2, \ldots, q$,
(2) $\mathcal{A}_{ij} = \mathrm{M}_{n_i \times n_j}(K)$ for all i and j such that $1 \le i < j \le q$, and
(3) all other entries are zero.

We call $\mathcal{A}$ a subalgebra of $\mathrm{M}_n(K)$ of *type* $(n_1, n_2, \ldots, n_q)$. If $\mathcal{A}_{ii} = \mathrm{M}_{n_i}(K)$ for all i, then we call $\mathcal{A}$ the *full subalgebra of* $M_n(K)$ *of type* $(n_1, n_2, \ldots, n_q)$.

It is important to note that the notation in (1.10) means that for every i, $i = 1, 2, \dots, q$,

$$\begin{pmatrix} 0_{n_1} & \cdots & 0_{n_1\times n_{i-1}} & 0_{n_1\times n_i} & 0_{n_1\times n_{i+1}} & \cdots & 0_{n_1\times n_q} \\ & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & 0_{n_{i-1}} & 0_{n_{i-1}\times n_i} & 0_{n_{i-1}\times n_{i+1}} & \cdots & 0_{n_{i-1}\times n_q} \\ & & & \mathcal{A}_{ii} & 0_{n_i\times n_{i+1}} & \cdots & 0_{n_i\times n_q} \\ & & & & 0_{n_{i+1}} & \cdots & 0_{n_{i+1}\times n_q} \\ & & & & & \ddots & \vdots \\ & & & & & & 0_{n_q} \end{pmatrix} \subseteq \mathcal{A}, \tag{1.11}$$

and similarly, for all i and j such that $1 \le i < j \le q$, the subset of $\mathrm{M}_n(K)$ having $\mathcal{A}_{ij}$ $(= \mathrm{M}_{n_i\times n_j}(K))$ in block (i, j), and zeroes elsewhere, is also contained in $\mathcal{A}$.

Remark 1.4. In the case when all the algebras $\mathcal{A}_{ii},\ i = 1, 2, \dots, q$, on the diagonal blocks of a subalgebra $\mathcal{A}$ of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \dots, n_q)$ are commutative, then for any $X, Y \in \mathcal{A}$, the commutator $[X, Y]$ is an element of

$$\begin{pmatrix} 0_{n_1} & \mathrm{M}_{n_1\times n_2}(K) & \mathrm{M}_{n_1\times n_3}(K) & \cdots & \mathrm{M}_{n_1\times n_q}(K) \\ & 0_{n_2} & \mathrm{M}_{n_2\times n_3}(K) & \cdots & \mathrm{M}_{n_2\times n_q}(K) \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \mathrm{M}_{n_{q-1}\times n_q}(K) \\ & & & & 0_{n_q} \end{pmatrix}.$$

It follows that the product of any q such commutators is zero, and so $\mathcal{A}$ is D_q.

This remark enables us to define three classes of D_q subalgebras $\mathcal{A}$ of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \dots, n_q)$ (for some q-tuple $(n_1, n_2, \dots, n_q)$). We stress that, for each of these classes, all the subalgebras in the diagonal blocks are assumed to be commutative. Keeping this in mind, and using “max-comm”, “max-dim” and “db’s” as abbreviations for “maximal commutative”, “maximum dimensional” and “diagonal blocks”, respectively, we now state:

Definition 1.5. Let $\mathcal{A}$ be a subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \dots, n_q)$ (see Definition 1.3), with every subalgebra $\mathcal{A}_{ii}$ of $\mathrm{M}_{n_i}(K)$ commutative, $i = 1, 2, \dots, q$. Then $\mathcal{A}$ is called a

(1) D_q subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \dots, n_q)$ with max-comm db's if every $\mathcal{A}_{ii}$ is a maximal commutative subalgebra of $\mathrm{M}_{n_i}(K)$, $i = 1, 2, \dots, q$;
(2) D_q subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \dots, n_q)$ with max-dim db's if every $\mathcal{A}_{ii}$ is a commutative subalgebra of $\mathrm{M}_{n_i}(K)$ with maximum dimension (equal to $\left\lfloor \frac{n_i^2}{4} \right\rfloor + 1$), $i = 1, 2, \dots, q$;
(3) max-dim D_q subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \dots, n_q)$ with max-dim db's if every $\mathcal{A}_{ii}$ is a commutative subalgebra of $\mathrm{M}_{n_i}(K)$ with maximum dimension (equal to $\left\lfloor \frac{n_i^2}{4} \right\rfloor + 1$), $i = 1, 2, \dots, q$, and $\mathcal{A}$ is a D_q subalgebra of $\mathrm{M}_n(K)$ of maximum dimension (as in (1.7)).

Note that, letting $n_1, \dots, n_q$ as in Theorem 1.2, we obtain an algebra $\mathcal{A}$ as in Definition 1.5(3) by taking a subalgebra

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \dots & \mathcal{A}_{1q} \\ & \mathcal{A}_{22} & \dots & \mathcal{A}_{2q} \\ & & \ddots & \vdots \\ & & & \mathcal{A}_{qq} \end{pmatrix} \tag{1.12}$$

of $\mathrm{M}_n(K)$ constructed in [3] with

$$\dim_K \mathcal{A} = q + \sum_{i=1}^{q} \left\lfloor \frac{n_i^2}{4} \right\rfloor + \sum_{1 \le i < j \le q} n_i n_j. \tag{1.13}$$

We draw the reader's attention to the fact that a max-dim D_2 subalgebra $\mathcal{A}$ of $\mathrm{M}_n(K)$ of some type (n_1, n_2) with max-dim db's such that $\mathcal{A} \subseteq \mathrm{U}_n(K)$ is called a typical D_2 subalgebra of $\mathrm{U}_n(K)$ in [20].

After preparatory results in Section 2, we prove in Section 3 and Section 4 that, up to conjugation, a subalgebra $\mathcal{A}$ of $\mathrm{M}_n(K)$ is a maximal D_q subalgebra of $\mathrm{M}_n(K)$ if and only if $\mathcal{A}$ is a D_q subalgebra of $\mathrm{M}_n(K)$ of (some) type $(n_1, n_2, \dots, n_q)$ with max-comm db's (see Theorem 3.2 and Theorem 4.3). Continuing our analysis of conjugations, we show in Corollary 4.9 that in case $\mathcal{A}$ is a maximal D_q subalgebra of $\mathrm{M}_n(K)$ contained in $\mathrm{U}_n(K)$, then $\mathcal{A}$ is a D_q subalgebra of $\mathrm{M}_n(K)$ of some type $(n_1, n_2, \dots, n_q)$ with max-comm db's. By examining in Theorem 4.10 when two D_q subalgebras of $\mathrm{M}_n(K)$ with max-comm db's are conjugated, we prove that the uniqueness of the mentioned tuple $(n_1, n_2, \dots, n_q)$ and the pairwise uniqueness (up to conjugation) of the algebras in the corresponding q diagonal blocks are necessary and sufficient conditions.

Next, we will deal with the isomorphism problem of D_q subalgebras of $\mathrm{M}_n(K)$ with max-dim db's. In Section 6, after giving an interpretation of these algebras in the light of results from Section 3 and Section 4, we describe necessary conditions for two D_q subalgebras of $\mathrm{M}_n(K)$ with max-dim db's to be isomorphic (see Theorem 6.2). Using results from Section 5, where we clarify the structure of commutative subalgebras of matrix algebras over algebraically closed fields, as discussed in [7], we also provide in Theorem 6.6 sufficient conditions for two D_q subalgebras of $\mathrm{M}_n(K)$ with max-dim db's to be isomorphic, in case the field K is algebraically closed. It turns out that such isomorphic subalgebras are already conjugated. In Section 7, we illustrate theorems obtained in the previous one section on D_q subalgebras of $\mathrm{M}_n(K)$ with maximum dimension. In order to do it, we recall results in [19] about the non-uniqueness of q-tuples $(n_1, n_2, n_2, \ldots, n_q)$ which give rise to a max-dim D_q subalgebra of $\mathrm{M}_n(K)$.

2. Block form of subalgebras of $\mathrm{M}_n(K)$ with nilpotent ideal

In this section we will show (in Lemma 2.1) a block triangular form of subalgebras of the matrix algebra $\mathrm{M}_n(K)$ containing a nonzero nilpotent ideal (see also [14, Theorem 1.5.1]). We provide a relatively detailed proof of Lemma 2.1 and illustrate it in Example 2.3.

Lemma 2.1 will be invoked in Section 3, where we will prove (in Theorem 3.2) that every maximal D_q subalgebra of $\mathrm{M}_n(K)$ is conjugated with a D_q subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \ldots, n_q)$ with max-comm db's.

We conclude the section by showing (in Proposition 2.4) that every D_q algebra contains a nonzero nilpotent ideal in a natural way.

Lemma 2.1. *Let $\mathcal{A}$ be a subalgebra of $M_n(K)$, and let I be a nonzero nilpotent ideal of $\mathcal{A}$ with nilpotency index q, i.e. $I^q = 0$ and $I^{q-1} \neq 0$. Then $q \leq n$, and there exist natural numbers $n_1, n_2, \ldots, n_q$ such that $\sum_{i=1}^{q} n_i = n$ and an invertible matrix $X \in \mathrm{M}_n(K)$ such that $X^{-1}\mathcal{A}X$ is a subalgebra of the full subalgebra of $M_n(K)$ of type $(n_1, n_2, \ldots, n_q)$. Moreover, the ideal $X^{-1}IX$ is contained in*

$$
\begin{pmatrix}
0_{n_1} & \mathrm{M}_{n_1\times n_2}(K) & \mathrm{M}_{n_1\times n_3}(K) & \cdots & \mathrm{M}_{n_1\times n_q}(K) \\
 & 0_{n_2} & \mathrm{M}_{n_2\times n_3}(K) & \cdots & \mathrm{M}_{n_2\times n_q}(K) \\
 & & \ddots & \ddots & \vdots \\
 & & & \ddots & \mathrm{M}_{n_{q-1}\times n_q}(K) \\
 & & & & 0_{n_q}
\end{pmatrix}.
$$

Proof. Denote the vector space K^n by V. For $i = 1, 2, \ldots, q$, let $n_i = \dim_K I^{q-i}V/I^{q-i+1}V$, with $I^0V := V$.

By definition, $n_1 = \dim_K I^{q-1}V/I^qV = \dim_K I^{q-1}V$, since $I^q = 0$ (and hence $I^qV = 0$). Next, $I^{q-1}V$ is a K-subspace of $I^{q-2}V$, and so $n_1 + n_2 = \dim_K I^{q-1}V +$

$\dim_K I^{q-2}V/I^{q-1}V = \dim_K I^{q-2}V$. Inductively, assume that $n_1 + n_2 + \cdots + n_i = \dim_K I^{q-i}V$ for some positive integer $i < q$. Since $I^{q-i}V$ is a K-subspace of $I^{q-i-1}V$, we conclude that

$$\begin{aligned} n_1 + n_2 + \cdots + n_{i+1} &= \dim_K I^{q-i}V + n_{i+1} \\ &= \dim_K I^{q-i}V + \dim_K I^{q-i-1}V/I^{q-i}V = \dim_K I^{q-i-1}V. \end{aligned}$$

Hence $n_1+n_2+\cdots+n_i = \dim_K I^{q-i}V$ for $i = 1, 2, \ldots, q$; in particular, $n_1+n_2+\cdots+n_q = \dim_K V = \dim_K K^n = n$.

We have the following sequence of K-subspaces of V:

$$0 \subseteq I^{q-1}V \subseteq I^{q-2}V \subseteq \cdots \subseteq IV \subseteq V. \tag{2.1}$$

Using the assumption that $I^q = 0$ and $I^{q-1} \neq 0$, we will show that all the inclusions in (2.1) are proper. Suppose that $I^{q-1}V = 0$ or $I^{q-j}V = I^{q-j-1}V$ for some j, $1 \leq j \leq q-1$. Then

$$0 = I^q V = I^j(I^{q-j}V) = I^j(I^{q-j-1}V) = I^{q-1}V,$$

and so from $V = K^n$ we conclude that $I^{q-1} = 0$ (otherwise some matrix in I^{q-1} would have a nonzero entry in some row, which would in turn imply that $I^{q-1}V \neq 0$); a contradiction. This establishes the proper inclusions. Thus,

$$1 \leq \dim_K I^{q-1}V,\ 2 \leq \dim_K I^{q-2}V,\ \cdots,\ q-1 \leq \dim_K IV,\ q \leq \dim_K V = n.$$

Now, using the sequence of K-subspaces in (2.1), we define a basis $B = (v_1, v_2, \ldots, v_n)$ for the K-space V in the following way:

Start with a basis $(v_1, v_2, \ldots, v_{n_1})$ for the K-space $I^{q-1}V$ (keeping in mind that, by definition, $\dim_K I^{q-1}V = n_1$). Next, $I^{q-1}V$ is a K-subspace of $I^{q-2}V$, and $\dim_K I^{q-2}V = n_1 + n_2$. So, take vectors $v_{n_1+1}, v_{n_1+2}, \ldots, v_{n_1+n_2}$ such that $(v_1, v_2, \ldots, v_{n_1+n_2})$ is a basis for $I^{q-2}V$. Continuing in this way, we construct a basis $B = (v_1, v_2, \ldots, v_n)$ for the K-space V, where $(v_1, v_2, \ldots, v_{n_1+n_2+\cdots+n_i})$ is a basis for the K-space $I^{q-i}V$, $i = 1, 2, \ldots, q-1$.

Now we take an arbitrary matrix $Y \in \mathcal{A}$. Let $\varphi\colon V \to V$ be a linear map such that the (transformation) matrix of φ with respect to the standard basis $E := (e_1, e_2, \ldots, e_n)$ for V is $M(\varphi)_E^E = Y$. (Note also that we have vectors v_1, v_2, $\ldots$, v_n written in terms of E). As I is an ideal of the algebra $\mathcal{A}$, we have $Y(I^{q-1}V) \subseteq \mathcal{A}(I^{q-1}V) \subseteq I^{q-1}V$. Since $(v_1, v_2, \ldots, v_{n_1})$ is a basis for $I^{q-1}V$, for any $i = 1, 2, \ldots, n_1$ we can write

$$\varphi(v_i) = Yv_i = y_{1i}v_1 + y_{2i}v_2 + \cdots + y_{n_1,i}v_{n_i},$$

for some scalars $y_{ji} \in K$ with $1 \leq i, j \leq n_1$. Similarly, $Y(I^{q-2}V) \subseteq \mathcal{A}(I^{q-2}V) \subseteq I^{q-2}V$, and $(v_1, v_2, \ldots, v_{n_1+n_2})$ is a basis of the K-space $I^{q-2}V$, and so for $j = n_1+1, n_1+2, \ldots, n_1+n_2$ we can write

$$\varphi(v_j) = Yv_j = y_{1j}v_1 + y_{2j}v_2 + \ldots + y_{n_1+n_2,j}v_{n_1+n_2},$$

for some scalars $y_{kj} \in K$ with $1 \leq k \leq n_1 + n_2$ (and $n_1 + 1 \leq j \leq n_1 + n_2$). Continuing in this way, we eventually simply have that $YV \subseteq V$, and with $(v_1, v_2, \ldots, v_n)$ being a basis for V, we can therefore, for $l = n_1 + n_2 + \cdots + n_{q-1} + 1, n_1 + n_2 + \cdots + n_{q-1} + 2, \ldots, n$, write

$$\varphi(v_l) = Yv_l = y_{1l}v_1 + y_{2l}v_2 + \ldots + y_{nl}v_n,$$

for some scalars y_{ml}, with $1 \leq m \leq n$ (and $n_1 + n_2 + \cdots + n_{q-1} + 1 \leq l \leq n$). Hence, using the notation

$$N_1 := n_1 \qquad \text{and} \qquad N_i := n_1 + \cdots + n_i, \; i = 2, \ldots, q,$$

which implies that $N_q = n$, the matrix $M_B^B(\varphi)$ of the linear map φ with respect to the basis B is the following:

$$\begin{pmatrix}
y_{11} & \cdots & y_{1,n_1} & y_{1,n_1+1} & \cdots & y_{1,N_2} & \cdots & y_{1,N_{q-1}+1} & \cdots & y_{1,N_q} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\
y_{n_1,1} & \cdots & y_{n_1,n_1} & y_{n_1,n_1+1} & \cdots & y_{n_1,N_2} & \cdots & y_{n_1,N_{q-1}+1} & \cdots & y_{n_1,N_q} \\
 & & & y_{n_1+1,n_1+1} & \cdots & y_{n_1+1,N_2} & \cdots & y_{n_1+1,N_{q-1}+1} & \cdots & y_{n_1+1,N_q} \\
 & & & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\
 & & & y_{N_2,n_1+1} & \cdots & y_{N_2,N_2} & \cdots & y_{N_2,N_{q-1}+1} & \cdots & y_{N_2,N_q} \\
 & & & & & & \ddots & \vdots & \vdots & \vdots \\
 & & & & & & & y_{N_{q-1}+1,N_{q-1}+1} & \cdots & y_{N_{q-1}+1,N_q} \\
 & & & & & & & \vdots & \ddots & \vdots \\
 & & & & & & & y_{N_q,N_{q-1}+1} & \cdots & y_{N_q,N_q}
\end{pmatrix}.$$

Consequently, $M_B^B(\varphi)$ is an element of the full subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \ldots, n_q)$. From the change-of-basis formula

$$M_B^B(\varphi) = M(id)_E^B \cdot M(\varphi)_E^E \cdot M(id)_B^E = M(id)_E^B \cdot Y \cdot M(id)_B^E,$$

where the change-of-basis matrix $M(id)_B^E$ is the matrix with vector v_j written in the j-th column, $j = 1, 2, \ldots, n$, and $M(id)_E^B = (M(id)_B^E)^{-1}$. Since Y was an arbitrary matrix of $\mathcal{A}$, we have proved that the algebra $X^{-1}\mathcal{A}X$ is in block triangular form, with $X = M(id)_B^E$.

In order to complete the proof, it remains to show that matrices from $X^{-1}IX$ have diagonal blocks only with zeros. Take an arbitrary matrix $Z \in I$. Let $\psi\colon V \to V$ be a linear map such that the (transformation) matrix of ψ with respect to the standard basis E for V is $M(\psi)_E^E = Z$. Since $I^q = 0$, it follows that $Z(I^{q-1}V) \subseteq I(I^{q-1}V) = 0$, and so, with $(v_1, v_2, \ldots, v_{n_1})$ being a basis for $I^{q-1}V$, we have, for $i = 1, 2, \ldots, n_1$,

$$\psi(v_i) = Zv_i = 0.$$

Next, $Z(I^{q-2}V) \subseteq I(I^{q-2}V) = I^{q-1}V$. Then for $j = n_1+1, n_1+2, \ldots, n_1+n_2$ $(= N_2)$ we can write

$$\psi(v_j) = Zv_j = z_{1j}v_1 + z_{2j}v_2 + \ldots + z_{n_1,j}v_{n_1}$$

for some scalars $z_{kj} \in K$ with $1 \leq k \leq n_1$ (and $n_1+1 \leq j \leq n_1+n_2$), because the basis $(v_1, v_2, \ldots, v_{n_1})$ for $I^{q-1}V$ was expanded to the basis $(v_1, \ldots, v_{n_1}, v_{n_1+1}, \ldots, v_{n_1+n_2})$ for $I^{q-2}V$. The pattern should now be clear from the arguments above, which leads us to concluding that the matrix $M_B^B(\psi)$ of the linear map ψ with respect to the basis B is the following:

$$\left(\begin{array}{ccccccccccccc}
0 & \dots & 0 & z_{1,n_1+1} & \cdots & z_{1,N_2} & z_{1,N_2+1} & \cdots & z_{1,N_3} & \cdots & z_{1,N_{q-1}+1} & \cdots & z_{1,N_q} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \ddots & \vdots \\
0 & \dots & 0 & z_{n_1,n_1+1} & \cdots & z_{n_1,N_2} & z_{n_1,N_2+1} & \cdots & z_{n_1,N_3} & \cdots & z_{n_1,N_{q-1}+1} & \cdots & z_{n_1,N_q} \\
 & & & 0 & \cdots & 0 & z_{n_1+1,N_2+1} & \cdots & z_{n_1+1,N_3} & \cdots & z_{n_1+1,N_{q-1}+1} & \cdots & z_{n_1+1,N_q} \\
 & & & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \ddots & \vdots \\
 & & & 0 & \dots & 0 & z_{N_2,N_2+1} & \cdots & z_{N_2,N_3} & \cdots & z_{N_2,N_{q-1}+1} & \cdots & z_{N_2,N_q} \\
 & & & & & & \ddots & & & \dots & \vdots & \vdots & \vdots \\
 & & & & & & & \ddots & & \dots & \vdots & \vdots & \vdots \\
 & & & & & & & & \ddots & \dots & \vdots & \vdots & \vdots \\
 & & & & & & & & & & z_{N_{q-2}+1,N_{q-1}+1} & \cdots & z_{N_{q-2}+1,N_q} \\
 & & & & & & & & & & \vdots & \ddots & \vdots \\
 & & & & & & & & & & z_{N_{q-1},N_{q-1}+1} & \cdots & z_{N_{q-1},N_q} \\
 & & & & & & & & & & 0 & \dots & 0 \\
 & & & & & & & & & & \vdots & \ddots & \vdots \\
 & & & & & & & & & & 0 & \dots & 0
\end{array}\right).$$

Hence, $M_B^B(\psi)$ is in the strictly upper block triangular part of the full subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \ldots, n_q)$. $\square$

Remark 2.2. The numbers $n_1, n_2, \ldots, n_q$ of Lemma 2.1, defined in the first line of the proof, are determined by algebra $\mathcal{A}$ and the dimensions of space V and subspaces I^iV for $i = 1, 2, \ldots, q-1$, where $V = K^n$.

Note that every finite dimensional K-algebra $\mathcal{A}$ can be identified with a subalgebra of $\mathrm{M}_n(K)$, for $n \leq \dim_K \mathcal{A}$. To do this we can use, for example, a regular representation. The Jacobson radical $J(\mathcal{A})$ of a finite dimensional algebra $\mathcal{A}$ is nilpotent (see [9, Theorem 4.12] for the broader class of Artinian rings), and so after such identification of $\mathcal{A}$ with a subalgebra of $\mathrm{M}_n(K)$ we can find an algebra in block triangular form (as in the above lemma) which is a conjugated of $\mathcal{A}$.

In the following example we start with the finite dimensional algebra $\mathcal{A} = \mathrm{M}_2(K[x]/(x^2))$. After identification with a subalgebra of matrices, the algebra $\mathcal{A}$ is conjugated with subalgebra of a block triangular matrices. We will describe the obtained blocks and see some "dependence" between them in the sense of the definition below formula (1.9). By $\overline{x}$ we will denote the image of $x \in K[x]$ in the natural homomorphism to the quotient algebra $K[x]/(x^2)$.

Example 2.3. Let $\mathcal{A}$ be the finite dimensional algebra $\mathrm{M}_2(K[x]/(x^2))$. Since, for any natural number n, the Jacobson radical satisfies $J(\mathrm{M}_n(\mathcal{A})) = \mathrm{M}_n(J(\mathcal{A}))$ (see [9, point (7), page 57]), we have $J(\mathcal{A}) = \mathrm{M}_2(J(K[x]/(x^2)) = \mathrm{M}_2(K\overline{x})$, which implies that $J(\mathcal{A})$ is a nonzero ideal with $(J(\mathcal{A}))^2 = 0$. Using the identification of an arbitrary element $a + b\overline{x}$ in $K[x]/(x^2)$ with the matrix $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in \mathrm{M}_2(K)$ we will treat (the 8-dimensional) K-algebra $\mathcal{A}$ as the subalgebra of $\mathrm{M}_4(K)$ comprising all matrices of the form

$$\begin{pmatrix} a_{11} & b_{11} & a_{12} & b_{12} \\ 0 & a_{11} & 0 & a_{12} \\ a_{21} & b_{21} & a_{22} & b_{22} \\ 0 & a_{21} & 0 & a_{22} \end{pmatrix}, \tag{2.2}$$

where $a_{ij}, b_{ij} \in K$ for $1 \leq i, j \leq 2$. With this identification, we have

$$J(\mathcal{A}) = \left\{ \begin{pmatrix} 0 & b_{11} & 0 & b_{12} \\ 0 & 0 & 0 & 0 \\ 0 & b_{21} & 0 & b_{22} \\ 0 & 0 & 0 & 0 \end{pmatrix} : \ b_{ij} \in K \right\}. \tag{2.3}$$

Now we are ready to use Lemma 2.1. With 2 being the nilpotency index q of $J(\mathcal{A})$, and with $V = K^4$, we have $J(\mathcal{A})V = \mathrm{span}(e_1, e_3)$, and so, following the notation in Lemma 2.1, we have $n_1 = \dim_K J(\mathcal{A})V = 2$ and $n_2 = \dim_K (J(\mathcal{A}))^0 V/J(\mathcal{A})V = \dim_K V/J(\mathcal{A})V = 2$. By Lemma 2.1, there exists an invertible matrix $X \in \mathrm{M}_4(K)$ such that $X^{-1}\mathcal{A}X$ is a subalgebra of the full subalgebra of $\mathrm{M}_4(K)$ of type $(2,2)$ which is $\begin{pmatrix} \mathrm{M}_2(K) & \mathrm{M}_2(K) \\ & \mathrm{M}_2(K) \end{pmatrix}$, and such that the ideal $X^{-1}J(\mathcal{A})X$ is contained in the strictly upper block triangular part $\begin{pmatrix} 0_2 & \mathrm{M}_2(K) \\ & 0_2 \end{pmatrix}$.

In order to find such an X we follow the proof of Lemma 2.1. We start the construction of a basis B for V by first finding basis vectors for $J(\mathcal{A})V$. As $J(\mathcal{A})V = \mathrm{span}(e_1, e_3)$, we take (e_1, e_3) as a basis for $J(\mathcal{A})V$. As $q = 2$, the second step is the last step, and in it we expand the basis (e_1, e_3) to a basis for V, by using e_2 and e_4, i.e., we take B as (e_1, e_3, e_2, e_4). An arbitrary matrix

$$\begin{pmatrix} a_{11} & b_{11} & a_{12} & b_{12} \\ 0 & a_{11} & 0 & a_{12} \\ a_{21} & b_{21} & a_{22} & b_{22} \\ 0 & a_{21} & 0 & a_{22} \end{pmatrix}$$

from algebra $\mathcal{A}$, treated as a linear map in the canonical basis (e_1, e_2, e_3, e_4), has

$$\begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & a_{21} & a_{22} \end{pmatrix}$$

as transformation matrix with respect to the basis B. It is obtained by conjugation with the matrix

$$X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

of vectors e_1, e_3, e_2, e_4 written in the first, second, third and fourth column, respectively. Consequently, every matrix $A \in X^{-1}\mathcal{A}X$ is in the block form $\begin{pmatrix} A_{11} & A_{12} \\ & A_{11} \end{pmatrix}$, where A_{11} and A_{12} are any matrices from $\mathrm{M}_2(K)$. Importantly, the two matrices in the diagonal blocks are equal (denoted here by A_{11}). Since every matrix in $J(\mathcal{A})$ has entries $a_{ij} = 0$ (see (2.2) and (2.3)), we have $X^{-1}J(\mathcal{A})X \subseteq \begin{pmatrix} 0_2 & \mathrm{M}_2(K) \\ & 0_2 \end{pmatrix}$.

Note that this example shows an interesting isomorphism, namely conjugation of the algebra $\mathrm{M}_2(\mathrm{U}_2^*(K))$ with the algebra $\mathrm{U}_2^*(\mathrm{M}_2(K))$.

For a D_q algebra $\mathcal{A}$, we denote the ideal of $\mathcal{A}$ generated by the set $\{[x, y] : x, y \in \mathcal{A}\}$ of commutators in $\mathcal{A}$ by $\mathcal{C}_\mathcal{A}$.

Proposition 2.4. *If $\mathcal{A}$ is a D_q algebra, then $\mathcal{C}_\mathcal{A}^q = 0$ and q is the nilpotency index of $\mathcal{C}_\mathcal{A}$. If, in addition, $\mathcal{A}$ is a subalgebra of $M_n(K)$, then $q \le n$.*

Proof. In order to show that the ideal $\mathcal{C}_\mathcal{A}$ is nilpotent with $\mathcal{C}_\mathcal{A}^q = 0$, take an element $x \in \mathcal{C}_\mathcal{A}$ of the following form:

$$x = r_1 \cdot [x_1, y_1] \cdot r_2 \cdot [x_2, y_2] \cdot \ldots \cdot r_q \cdot [x_q, y_q] \cdot r_{q+1}.$$

Since $\mathcal{C}_\mathcal{A}^q$ comprises (finite) sum of elements of this form, it suffices to show that $x = 0$.

For any $a, b, r \in \mathcal{A}$ we have the identity $[a, rb] = [a, r]b + r[a, b]$, and so

$$r[a, b] = [a, rb] - [a, r]b.$$

Applying the last equality to x_1, y_1, r_1, we have

$$\begin{aligned}
&r_1 \cdot [x_1, y_1] \cdot r_2 \cdot [x_2, y_2] \ldots \cdot r_q \cdot [x_q, y_q] \cdot r_{q+1} \\
&= ([x_1, r_1 y_1] - [x_1, r_1] y_1) \cdot r_2 \cdot [x_2, y_2] \cdot \ldots \cdot r_q \cdot [x_q, y_q] \cdot r_{q+1} \\
&= [x_1, r_1 y_1] \cdot r_2 \cdot [x_2, y_2] \cdot \ldots \cdot r_q \cdot [x_q, y_q] \cdot r_{q+1} \\
&- [x_1, r_1] \cdot y_1 r_2 \cdot [x_2, y_2] \cdot r_3 \cdot [x_3, y_3] \cdot \ldots \cdot r_q \cdot [x_q, y_q] \cdot r_{q+1}.
\end{aligned}$$

Next we can write

$$\begin{aligned}
&[x_1, r_1 y_1] \cdot r_2 \cdot [x_2, y_2] \cdot r_3 \cdot [x_3, y_3] \cdot r_4 \ldots \cdot r_q \cdot [x_q, y_q] \cdot r_{q+1} \\
&= [x_1, r_1 y_1] \cdot ([x_2, r_2 y_2] - [x_2, r_2] y_2) \cdot r_3 \cdot [x_3, y_3] \cdot r_4 \ldots \cdot r_q \cdot [x_q, y_q] \cdot r_{q+1} \\
&= [x_1, r_1 y_1][x_2, r_2 y_2] \cdot r_3 \cdot [x_3, y_3] \cdot r_4 \cdot [x_4, y_4] \cdot \ldots \cdot r_q \cdot [x_q, y_q] \cdot r_{q+1} \\
&- [x_1, r_1 y_1][x_2, r_2] \cdot y_2 r_3 \cdot [x_3, y_3] \cdot r_4 \cdot [x_4, y_4] \cdot \ldots \cdot r_q \cdot [x_q, y_q] \cdot r_{q+1}
\end{aligned}$$

and

$$\begin{aligned}
&[x_1, r_1] \cdot y_1 r_2 \cdot [x_2, y_2] \cdot r_3 \cdot [x_3, y_3] \cdot \ldots \cdot r_q \cdot [x_q, y_q] \cdot r_{q+1} \\
&= [x_1, r_1]([x_2, y_1 r_2 y_2] - [x_2, y_1 r_2] y_2) \cdot r_3 \cdot [x_3, y_3] \cdot \ldots \cdot r_q \cdot [x_q, y_q] \cdot r_{q+1} \\
&= [x_1, r_1][x_2, y_1 r_2 y_2] \cdot r_3 \cdot [x_3, y_3] \cdot r_4 \cdot [x_4, y_4] \cdot \ldots \cdot r_q \cdot [x_q, y_q] \cdot r_{q+1} \\
&- [x_1, r_1][x_2, y_1 r_2] \cdot y_2 r_3 \cdot [x_3, y_3] \cdot r_4 \cdot [x_4, y_4] \cdot \ldots \cdot r_q \cdot [x_q, y_q] \cdot r_{q+1}.
\end{aligned}$$

Continuing is this way, it is evident that x can be written as a sum of elements of the form

$$\pm([x'_1, y'_1][x'_2, y'_2] \ldots [x'_q, y'_q])r$$

for some $x'_1, y'_1, x'_2, y'_2, \ldots, x'_q, y'_q, r \in \mathcal{A}$. Such elements are all equal to zero, because $\mathcal{A}$ is a D_q algebra. Hence, $\mathcal{C}^q_{\mathcal{A}} = 0$.

Recall from the discussion preceding Theorem 1.1 that we always assume that $q > 1$ and that D_q algebra $\mathcal{A}$ is not a D_{q-1} algebra. So q is the nilpotency index of $\mathcal{C}_{\mathcal{A}}$.

If, in addition, $\mathcal{A}$ is a subalgebra of $\mathrm{M}_n(K)$, then it follows from Lemma 2.1 that $q \leq n$. □

Obviously, Proposition 2.4 implies the following fact:

Corollary 2.5. *For every positive integer n there are not D_q subalgebras of $M_n(K)$ for every $q > n$.*

3. Maximal D_q subalgebras of $\mathrm{M}_n(K)$ are conjugated with D_q subalgebras with max-comm db's

In this section we will characterize, up to conjugation, maximal D_q subalgebras of $\mathrm{M}_n(K)$, in particular these with maximum dimension.

We will show in Theorem 3.1 that every D_q subalgebra of $\mathrm{M}_n(K)$ is conjugated with a subalgebra of a full subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \dots, n_q)$, and it possesses some interesting additional properties.

Using this result we prove in Theorem 3.2 that every maximal D_q subalgebra of $\mathrm{M}_n(K)$ is conjugated with a D_q subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \dots, n_q)$ with max-comm db's.

We then conclude in Corollary 3.4 that every D_q subalgebra of $\mathrm{M}_n(K)$ with maximum dimension is conjugated with max-dim D_q subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \dots, n_q)$ with max-dim db's.

Theorem 3.1. *Let $\mathcal{A}$ be a D_q subalgebra of $M_n(K)$. Then there exist positive integers $n_1, n_2, \dots, n_q$, such that $\sum_{i=1}^{q} n_i = n$ and an invertible matrix $X \in \mathrm{M}_n(K)$, such that every matrix A' in the algebra $\mathcal{A}' = X^{-1}\mathcal{A}X$ is in block triangular form*

$$\begin{pmatrix} A'_{11} & A'_{12} & \dots & A'_{1q} \\ & A'_{22} & \dots & A'_{2q} \\ & & \ddots & \vdots \\ & & & A'_{qq} \end{pmatrix}, \tag{3.1}$$

where $A'_{ij} \in \mathrm{M}_{n_i \times n_j}(K)$ for all i and j such that $1 \le i \le j \le q$ (and other entries are zero) and $\overline{\mathcal{A}'}_{ii}$, defined in (1.8), is a commutative subalgebra of $M_{n_i}(K)$ for every i, $i = 1, 2 \dots, q$.

Proof. By Proposition 2.4, $\mathcal{C}_{\mathcal{A}}^q = 0$, where $\mathcal{C}_{\mathcal{A}}$ is the ideal of $\mathcal{A}$ generated by all commutators in $\mathcal{A}$, and q is the nilpotency index of $\mathcal{C}_{\mathcal{A}}$. Recall from the discussion preceding Theorem 1.1 that we always assume that $q > 1$. Thus, by Lemma 2.1, there exists an invertible matrix $X \in \mathrm{M}_n(K)$ such that every matrix A' in the algebra $\mathcal{A}' = X^{-1}\mathcal{A}X$ is in the block triangular form (3.1), and the ideal $X^{-1}\mathcal{C}_{\mathcal{A}}X$ of the algebra $\mathcal{A}'$ comprises zero matrices in the diagonal blocks.

It remains to show that, for $i = 1, 2, \dots, q$, the subalgebra $\overline{\mathcal{A}'}_{ii}$ of $\mathrm{M}_{n_i}(K)$ is commutative. Firstly, we will say more about the structure of the ideal $\mathcal{C}_{\mathcal{A}'}$ generated by all commutators $[z, w]$, $z, w \in \mathcal{A}'$. Since conjugation is an isomorphism of algebras, it follows readily that the ideal $X^{-1}\mathcal{C}_{\mathcal{A}}X$ is equal to $\mathcal{C}_{\mathcal{A}'}$. Therefore

$$\mathcal{C}_{\mathcal{A}'} \subseteq \begin{pmatrix} 0_{n_1} & \mathrm{M}_{n_1 \times n_2}(K) & \mathrm{M}_{n_1 \times n_3}(K) & \cdots & \mathrm{M}_{n_1 \times n_q}(K) \\ & 0_{n_2} & \mathrm{M}_{n_2 \times n_3}(K) & \cdots & \mathrm{M}_{n_2 \times n_q}(K) \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \mathrm{M}_{n_{q-1} \times n_q}(K) \\ & & & & 0_{n_q} \end{pmatrix}.$$

To complete the proof let X_{ii}, $Y_{ii} \in \overline{\mathcal{A}'}_{ii}$. Then, by the definition, there are block triangular matrices $X, Y \in \mathcal{A}'$ such that

$$X = \begin{pmatrix} X_{11} & \dots & X_{1i} & \dots & X_{1q} \\ & \ddots & \vdots & \ddots & \vdots \\ & & X_{ii} & \dots & X_{iq} \\ & & & \ddots & \vdots \\ & & & & X_{qq} \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} Y_{11} & \dots & Y_{1i} & \dots & Y_{1q} \\ & \ddots & \vdots & \ddots & \vdots \\ & & Y_{ii} & \dots & Y_{iq} \\ & & & \ddots & \vdots \\ & & & & Y_{qq} \end{pmatrix}.$$

The commutator $[X, Y]$ has the matrix $[X_{ii}, Y_{ii}]$ in the i-th diagonal block. Since we showed in the preceding paragraph that the diagonal blocks of the ideal generated by all commutators of $\mathcal{A}'$ are zero, we conclude that $[X_{ii}, Y_{ii}] = 0_{n_i}$, which completes the proof. $\square$

Next we show that if, in addition, $\mathcal{A}$ is a maximal D_q subalgebra of $\mathrm{M}_n(K)$, then the obtained algebras $\overline{\mathcal{A}'}_{ii}$ above are independent (see the definition below formula (1.9)). To be precise, we have the following characterization:

Theorem 3.2. *Let $\mathcal{A}$ be a maximal D_q subalgebra of $M_n(K)$. Then there exists an invertible matrix $X \in \mathrm{M}_n(K)$ such that $X^{-1}\mathcal{A}X$ is a D_q subalgebra of $\mathrm{M}_n(K)$ of some type $(n_1, n_2, \dots, n_q)$ with max-comm db's.*

Proof. By Theorem 3.1, there exists an invertible matrix X such that every matrix $A' \in \mathcal{A}' = X^{-1}\mathcal{A}X$ is in block triangular form

$$\begin{pmatrix} A'_{11} & A'_{12} & \dots & A'_{1q} \\ & A'_{22} & \dots & A'_{2q} \\ & & \ddots & \vdots \\ & & & A'_{qq} \end{pmatrix},$$

where $A'_{ij} \in \mathrm{M}_{n_i \times n_j}(K)$ for $1 \le i \le j \le q$ and each $\overline{\mathcal{A}'}_{ii}$ is a commutative subalgebra of $\mathrm{M}_{n_i}(K)$.

Let $\mathcal{B}$ be the subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \dots, n_q)$ equal to

$$\begin{pmatrix} \overline{\mathcal{A}'}_{11} & \mathrm{M}_{n_1 \times n_2}(K) & \mathrm{M}_{n_1 \times n_3}(K) & \cdots & \mathrm{M}_{n_1 \times n_q}(K) \\ & \overline{\mathcal{A}'}_{22} & \mathrm{M}_{n_2 \times n_3}(K) & \cdots & \mathrm{M}_{n_2 \times n_q}(K) \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \mathrm{M}_{n_{q-1} \times n_q}(K) \\ & & & & \overline{\mathcal{A}'}_{qq} \end{pmatrix}.$$

Note that by Remark 1.4 $\mathcal{B}$ is a D_q subalgebra of $\mathrm{M}_n(K)$. Since $\mathcal{A}$ is a maximal D_q subalgebra of $\mathrm{M}_n(K)$, it follows that $\mathcal{A}' = X^{-1}\mathcal{A}X$ is also a maximal D_q subalgebra of $\mathrm{M}_n(K)$. So from the inclusion $\mathcal{A}' \subseteq \mathcal{B}$ we obtain the equality $\mathcal{A}' = \mathcal{B}$.

In order to complete the proof we will show that each $\overline{\mathcal{A}}_{ii}$ is a maximal commutative subalgebra of $\mathrm{M}_{n_i}(K)$. Suppose, for the contrary, that, for some $j \in \{1, 2, \ldots, q\}$, the diagonal block $\overline{\mathcal{A}}_{jj}$ is properly contained in a commutative subalgebra C_{jj} of $\mathrm{M}_{n_j}(K)$. Then changing $\overline{\mathcal{A}}_{jj}$ to C_{jj} produces a D_q subalgebra of $\mathrm{M}_n(K)$ properly containing $\mathcal{A}'$, a contradiction. It completes the proof. □

Remark 3.3. Similar to Remark 2.2, the q-tuple $(n_1, n_2, \ldots, n_q)$ obtained in the proof of Theorem 3.2 is determined by the dimensions of the vector space V and the subspaces $\mathcal{C}^i_{\mathcal{A}}V$ for $i = 1, 2, \ldots, q$, where $V = K^n$ and $\mathcal{C}_{\mathcal{A}}$ is the ideal generated by all commutators of the maximal D_q subalgebra $\mathcal{A}$ of $\mathrm{M}_n(K)$.

We will show in Theorem 4.10 that two D_q subalgebras of types $(n_1, n_2, \ldots, n_q)$ and $(\ell_1, \ell_2, \ldots, \ell_q)$ with max-comm db's are conjugated if and only if $(n_1, n_2, \ldots, n_q) = (\ell_1, \ell_2, \ldots, \ell_q)$ (i.e., the q-tuple is uniquely determined) and the diagonal blocks of the D_q algebras are pairwise conjugated.

In summary, with an arbitrary maximal D_q subalgebra $\mathcal{A}$ of $\mathrm{M}_n(K)$ we can associate exactly one tuple $(n_1, n_2, \ldots, n_q)$ such that $\mathcal{A}$ is conjugated with a D_q subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \ldots, n_q)$ with max-comm db's.

If $\mathcal{A}$ is a D_q subalgebra of $\mathrm{M}_n(K)$ with maximum dimension, then by Theorem 3.2, $\mathcal{A}$ is conjugated with a D_q subalgebra $\mathcal{A}'$ of $\mathrm{M}_n(K)$ of some type $(n_1, n_2, \ldots, n_q)$ with max-comm db's. As in the proof of Theorem 3.2, if one of the diagonal blocks $\mathcal{A}'_{jj}$ of $\mathcal{A}'$ is not a commutative subalgebra of $\mathrm{M}_{n_j}(K)$ with maximum dimension, then we can change this block and obtain a D_q subalgebra with dimension greater than that of $\mathcal{A}'$. This contradiction yields to following result:

Corollary 3.4. *Let $\mathcal{A}$ be a D_q subalgebra of $M_n(K)$ with maximum dimension. Then there exists an invertible matrix X such that $X^{-1}\mathcal{A}X$ is a max-dim D_q subalgebra of $\mathrm{M}_n(K)$ of some type $(n_1, n_2, \ldots, n_q)$ with max-dim db's.*

4. Structure of D_q subalgebras with max-comm db's

In Section 3 (see Theorem 3.2) we showed that every maximal D_q subalgebra of $\mathrm{M}_n(K)$ is conjugated to a D_q subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \ldots, n_q)$ with max-comm db's. In the present section, in Theorem 4.3, we will prove that the converse is also true.

Next, we will further analyze conjugations of D_q subalgebras of $\mathrm{M}_n(K)$. In Proposition 4.8, we will establish that conjugation, which satisfies some additional properties, of a D_q subalgebra of $\mathrm{M}_n(K)$ of any type $(n_1, n_2, \ldots, n_q)$ with max-comm db's is also a D_q subalgebra of $\mathrm{M}_n(K)$ of the same type with max-comm db's. A consequence is Corollary 4.9, in which we will show that if $\mathcal{A}$ is a maximal D_q subalgebra of $\mathrm{M}_n(K)$

contained in $\mathrm{U}_n(K)$, then $\mathcal{A}$ is a D_q subalgebra of $\mathrm{M}_n(K)$ of some type $(n_1, n_2, \ldots, n_q)$ with max-comm db's. It leads us to a negative answer to Question 9 posed in [20]. Using Definition 1.5 and the paragraph immediately following it, we can rephrase the mentioned question as follows:

Question 4.1. [20, Question 9] For a field K, is there, for some n, a D_2 K-subalgebra of the upper triangular matrix algebra $\mathrm{U}_n(K)$ with maximum dimension $2 + \left\lfloor \frac{3n^2}{8} \right\rfloor$ which is not a max-dim D_2 subalgebra of $\mathrm{M}_n(K)$ of some type (n_1, n_2) with max-dim db's?

In the same paper (see [20, Theorem 15]), a block triangular structure as in max-dim D_q subalgebras of $\mathrm{M}_n(K)$ with max-dim db's was proven for D_2 subalgebras of $\mathrm{M}_n(K)$ with maximum dimension which are contained in $\mathrm{U}_n(K)$ and satisfy some additional conditions. Corollary 4.9 generalizes this result.

Moreover, from Proposition 4.8 we obtain Theorem 4.10, which says that any D_q subalgebras $\mathcal{A}$ and $\mathcal{B}$ of $\mathrm{M}_n(K)$ with max-comm db's $\mathcal{A}_{ii}$ and $\mathcal{B}_{ii}$, respectively, $i = 1, 2, \ldots, q$, are conjugated if and only if they are of the same type and for each i, $i = 1, 2, \ldots, q$, $\mathcal{A}_{ii}$ and $\mathcal{B}_{ii}$ are conjugates. It reduces the conjugation problem of maximal D_q subalgebras of $\mathrm{M}_n(K)$ to the conjugation problem of commutative subalgebras of $\mathrm{M}_\ell(K)$, for $\ell = 1, 2, \ldots, n-1$. We will discuss the obtained result in Section 5, restricting our attention to algebraically closed fields.

Recalling Proposition 2.4, the first result in the present section describes powers of the ideal $\mathcal{C}_{\mathcal{A}}$ generated by all commutators of a D_q subalgebra $\mathcal{A}$ of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \ldots, n_q)$ with max-comm db's.

Proposition 4.2. *If $\mathcal{A}$ is D_q subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \ldots, n_q)$ with max-comm db's, then, for $i = 1, 2, \ldots, q-1$,*

$$\mathcal{C}_{\mathcal{A}}^i = \begin{pmatrix} 0_{n_1} & \cdots & 0_{n_1 \times n_i} & \mathrm{M}_{n_1 \times n_{i+1}}(K) & \cdots & \mathrm{M}_{n_1 \times n_q}(K) \\ & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \ddots & \mathrm{M}_{n_{q-i} \times n_q}(K) \\ & & & \ddots & \ddots & 0_{n_{q-i+1} \times n_q} \\ & & & & \ddots & \vdots \\ & & & & & 0_{n_q} \end{pmatrix}.$$

Proof. We start with $i = 1$. Keeping in mind that the diagonal blocks of a D_q subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \ldots, n_q)$ with max-comm db's are commutative, the inclusion

$$\mathcal{C}_{\mathcal{A}} \subseteq \begin{pmatrix} 0_{n_1} & \mathrm{M}_{n_1 \times n_2}(K) & \mathrm{M}_{n_1 \times n_3}(K) & \cdots & \mathrm{M}_{n_1 \times n_q}(K) \\ & 0_{n_2} & \mathrm{M}_{n_2 \times n_3}(K) & \cdots & \mathrm{M}_{n_2 \times n_q}(K) \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \mathrm{M}_{n_{q-1} \times n_q}(K) \\ & & & & 0_{n_q} \end{pmatrix}$$

is immediate.

In order to show the converse inclusion (for $i = 1$), let j be any positive integer such that $j < q$, and take arbitrary matrices $X_{j,j+1} \in \mathrm{M}_{n_j \times n_{j+1}}(K)$, $X_{j,j+2} \in \mathrm{M}_{n_j \times n_{j+2}}(K)$, $\ldots$, $X_{jq} \in \mathrm{M}_{n_j \times n_q}(K)$. Then

$$\left(\begin{array}{c|c|c} & & \\ \hline & I_{n_j} & \\ \hline & & \end{array}\right) \quad \text{and} \quad \left(\begin{array}{c|c|ccc} & & & & \\ \hline & 0_{n_j} & X_{j,j+1} & \ldots & X_{jq} \\ \hline & & & & \end{array}\right)$$

are elements of every D_q subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \ldots, n_q)$ with max-comm db's and so, since

$$\left(\begin{array}{c|c|c} & & \\ \hline & I_{n_j} & \\ \hline & & \end{array}\right) \left(\begin{array}{c|c|ccc} & & & & \\ \hline & 0_{n_j} & X_{j,j+1} & \ldots & X_{jq} \\ \hline & & & & \end{array}\right) = \left(\begin{array}{c|c|ccc} & & & & \\ \hline & 0_{n_j} & X_{j,j+1} & \ldots & X_{jq} \\ \hline & & & & \end{array}\right),$$

$$\left(\begin{array}{c|c|ccc} & & & & \\ \hline & 0_{n_j} & X_{j,j+1} & \ldots & X_{jq} \\ \hline & & & & \end{array}\right) \left(\begin{array}{c|c|c} & & \\ \hline & I_{n_j} & \\ \hline & & \end{array}\right) = 0_n,$$

it follows that

$$\left(\begin{array}{c|c|ccc} & & & & \\ \hline & 0_{n_j} & X_{j,j+1} & \ldots & X_{jq} \\ \hline & & & & \end{array}\right) = \left[\left(\begin{array}{c|c|c} & & \\ \hline & I_{n_j} & \\ \hline & & \end{array}\right), \left(\begin{array}{c|c|ccc} & & & & \\ \hline & 0_{n_j} & X_{j,j+1} & \ldots & X_{jq} \\ \hline & & & & \end{array}\right)\right] \in \mathcal{C}_{\mathcal{A}}.$$

As j and the matrices $X_{j,j+1}$, $X_{j,j+2}$, $\ldots$, X_{jq} were arbitrary, the mentioned inclusion has been established.

The form of the ideal $\mathcal{C}^i_{\mathcal{A}}$, for $i = 2, \ldots, q-1$, is now evident. $\square$

With the help of Proposition 4.2 we are ready to prove the first of the main results of this section. Henceforth, e_{ij} denotes the matrix unit which has 1 in position (i, j) and zeroes elsewhere.

Theorem 4.3. *Let $\mathcal{A}$ be a D_q subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \ldots, n_q)$ with max-comm db's. Then $\mathcal{A}$ is a maximal D_q subalgebra of $M_n(K)$.*

Proof. Suppose, for the contrary, that a D_q subalgebra $\mathcal{A}$ of type $(n_1, n_2, \ldots, n_q)$ with max-comm db's $\mathcal{A}_{ii}$, $i = 1, 2, \ldots, q$ (using the notation (1.10)), is not a maximal D_q subalgebras of $\mathrm{M}_n(K)$. The block structure (1.10) of the algebra $\mathcal{A}$ will be essential to the proof.

Let $\mathcal{B}$ be a maximal D_q subalgebra of $\mathrm{M}_n(K)$ properly containing $\mathcal{A}$. So we can find a matrix $X \in \mathcal{B} \setminus \mathcal{A}$. Write it in the block form

$$\begin{pmatrix} X_{11} & \ldots & X_{1q} \\ \vdots & \ddots & \vdots \\ X_{q1} & \ldots & X_{qq} \end{pmatrix}, \tag{4.1}$$

where $X_{ij} \in \mathrm{M}_{n_i \times n_j}(K)$, $1 \leq i, j \leq q$. Note that the numbers n_i are the same as those in the definition of the type of subalgebra $\mathcal{A}$. Subalgebra $\mathcal{A}$ contains idempotents $E_1 = \sum_{i=1}^{n_1} e_{ii}$ and $E_j = \sum_{i=N_{j-1}+1}^{N_j} e_{ii}$ for $j = 2, 3, \ldots, q$, where $N_j = n_1 + n_2 + \cdots + n_j$. So they also belong to $\mathcal{B}$. It follows that for all indices $1 \leq i, j \leq q$, the matrices $E_i X E_j$ are in $\mathcal{B}$. These matrices in the form (4.1) have X_{ij} in their i-th row and j-th column, and 0 everywhere else. We conclude that there exists a matrix Z in the ideal $\mathcal{C}_{\mathcal{B}}$ generated by all the commutators of $\mathcal{B}$, such that, written in the block form analogous to (4.1), has exactly one nonzero block Z_{rs}, where r and s satisfy $1 \leq s \leq r \leq q$.

From the definition of $\mathcal{A}$ follows that $X \notin \mathcal{A}$ if and only if either there exists $i > j$ such that X_{ij} is a nonzero matrix or there exists $k \in \{1, 2, \ldots, q\}$ such that $X_{kk} \notin \mathcal{A}_{kk}$.

In the first case, for matrix Z we can take $E_{ii} X E_{jj}$. This matrix belongs to $\mathcal{C}_{\mathcal{B}}$, because $E_{ii} X E_{jj} = [E_{ii}, E_{ii} X E_{jj}]$. In the other case, $X_{kk} \notin \mathcal{A}_{kk}$. Since $\mathcal{A}_{kk}$ is a maximal commutative subalgebra, there exists a matrix $Y_{kk} \in \mathcal{A}_{kk}$ such that the commutator $[X_{kk}, Y_{kk}]$ is nonzero. Let Z be defined as follows:

$$Z = \left[\begin{pmatrix} \ddots & & & & \\ & 0_{n_{k-1}} & & & \\ & & X_{kk} & & \\ & & & 0_{n_{k+1}} & \\ & & & & \ddots \end{pmatrix}, \begin{pmatrix} \ddots & & & & \\ & 0_{n_{k-1}} & & & \\ & & Y_{kk} & & \\ & & & 0_{n_{k+1}} & \\ & & & & \ddots \end{pmatrix} \right].$$

Then $Z \in \mathcal{C}_{\mathcal{B}}$, with the only nonzero $Z_{rs} \in \mathrm{M}_{n_r \times n_s}(K), 1 \leq s \leq r \leq q$, in the form analogous to (4.1), exists. Let z_{ij}, $1 \leq i \leq n_r$, $1 \leq j \leq n_s$, be a nonzero entry of the matrix Z_{rs}. Note that in matrix Z it is entry $(N_{r-1}+i, N_{s-1}+j)$, where $N_{r-1} = n_1 + n_2 + \ldots + n_{r-1}$ and $N_{s-1} = n_1 + n_2 + \ldots + n_{s-1}$ (with $N_0 := 0$). Assume firstly that $r \notin \{1, q\}$. Then

$$e_{1,N_{r-1}+i} \cdot Z \cdot e_{N_{s-1}+j,n} = z_{ij} e_{1n} \neq 0.$$

By Proposition 4.2, $e_{1,N_{r-1}+i} \in (\mathcal{C}_{\mathcal{A}})^{r-1}$ and $e_{N_{s-1}+j,n} \in (\mathcal{C}_{\mathcal{A}})^{q-s}$, where $\mathcal{C}_{\mathcal{A}}$ is the ideal generated by all commutators $[x, y]$, $x, y \in \mathcal{A}$. Since $\mathcal{B}$ contains $\mathcal{A}$, we have that $\mathcal{C}_{\mathcal{B}}$ also contains $\mathcal{C}_{\mathcal{A}}$, and so

$$z_{ij} e_{ij} \in (\mathcal{C}_{\mathcal{B}})^{r-1} \cdot \mathcal{C}_{\mathcal{B}} \cdot (\mathcal{C}_{\mathcal{B}})^{q-s} = (\mathcal{C}_{\mathcal{B}})^{q+(r-s)} = \{0\}.$$

The above equality holds because $r - s \geq 0$ and by Proposition 2.4, $\mathcal{C}_{\mathcal{B}}^q = \{0\}$. It is a contradiction, since $z_{ij} \neq 0$.

When $r = 1$, then Z_{11} is the nonzero block of matrix Z. By Proposition 4.2, $e_{jn} \in \mathcal{C}_{\mathcal{A}}^{q-1} \subseteq \mathcal{C}_{\mathcal{B}}^{q-1}$, and so the product $Z \cdot e_{jn}$ in $\mathcal{C}_{\mathcal{B}} \cdot \mathcal{C}_{\mathcal{B}}^{q-1} = \mathcal{C}_{\mathcal{B}}^q$ is zero. However, $Z \cdot e_{jn}$ has as its n-th column the j-th column of Z, the latter column being nonzero, which is a contradiction.

Finally, if $r = q$, then Z_{qs} is the nonzero block of the matrix Z. In this case the first row of the matrix $e_{1,N_{q-1}+i} \cdot Z$, where $N_{q-1} = n_1 + n_2 + \cdots + n_{q-1}$, is nonzero. Similarly, it leads to a contradiction, which completes the proof. $\square$

Note that conjugation of a maximal D_q subalgebra of $\mathrm{M}_n(K)$ is still a maximal D_q subalgebra of $\mathrm{M}_n(K)$, and so, by Theorem 4.3 and Theorem 3.2, we have the following:

Corollary 4.4. *An algebra $\mathcal{A}$ is a maximal D_q subalgebra of $M_n(K)$ if and only if it is conjugated with a D_q subalgebra of $\mathrm{M}_n(K)$ of some type $(n_1, n_2, \ldots, n_q)$ with max-comm db's.*

In the rest of this section we will examine conjugations, which satisfy some additional properties, of a D_q subalgebra of $\mathrm{M}_n(K)$ of some type with max-comm db's. We need the following result involving some matrix equations:

Lemma 4.5. *Let r, s and t be positive integers, and let $Y \in \mathrm{M}_{r\times t}(K)$, $W \in \mathrm{M}_{s\times t}(K)$, with $W \neq 0_{s\times t}$. If $YZW = 0_{r\times t}$ for all $Z \in \mathrm{M}_{t\times s}(K)$, then $Y = 0_{r\times t}$.*

Proof. Write $Y = \begin{pmatrix} y_{11} & \cdots & y_{1t} \\ \vdots & \ddots & \vdots \\ y_{r1} & \cdots & y_{rt} \end{pmatrix}$, $W = \begin{pmatrix} w_{11} & \cdots & w_{1t} \\ \vdots & \ddots & \vdots \\ w_{s1} & \cdots & w_{st} \end{pmatrix}$, with (say) $w_{ij} \neq 0$ (for some indices i, j, with $1 \leq i \leq s$, $1 \leq j \leq t$). Consider the matrix unit $e_{ki} \in \mathrm{M}_{t\times s}(K)$ for any fixed k, $1 \leq k \leq t$.

By assumption and direct calculation, we have

$$0_{r\times t} = Ye_{ki}W = \begin{pmatrix} y_{1k}w_{i1} & \dots & y_{1k}w_{ij} & \dots & y_{1k}w_{it} \\ y_{2k}w_{i1} & \dots & y_{2k}w_{ij} & \dots & y_{2k}w_{it} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ y_{rk}w_{i1} & \dots & y_{rk}w_{ij} & \dots & y_{rk}w_{it} \end{pmatrix}, \tag{4.2}$$

and so, since $w_{ij} \neq 0$, we conclude that

$$y_{1k} = y_{2k} = \ldots = y_{rk} = 0,$$

i.e., the k-th column of Y is zero. As k was arbitrary, we conclude that $Y = 0_{r\times t}$. □

Since $\det(X) = \det(X_{11})\cdot\det(X_{22})$ if X is a block triangular matrix $\begin{pmatrix} X_{11} & X_{12} \\ 0_{n_2\times n_1} & X_{22} \end{pmatrix}$, where $X_{11} \in \mathrm{M}_{n_1}(K)$, $X_{22} \in \mathrm{M}_{n_2}(K)$, $X_{12} \in \mathrm{M}_{n_1\times n_2}(K)$, with n_1 and n_2 positive integers, it is evident that X_{11} and X_{22} are invertible if X is invertible, and direct matrix multiplication yields

$$X^{-1} = \begin{pmatrix} X_{11}^{-1} & -X_{11}^{-1}X_{12}X_{22}^{-1} \\ 0_{n_2\times n_1} & X_{22}^{-1} \end{pmatrix}. \tag{4.3}$$

Lemma 4.6. *Let $q, n_1, n_2, \ldots, n_q$ and n be positive integers such that $n_1+n_2+\cdots+n_q = n$, and let $\mathcal{A}$ be a subalgebra of $M_n(K)$. If*

$$\begin{pmatrix} 0_{n_1} & \mathrm{M}_{n_1\times n_2}(K) & \mathrm{M}_{n_1\times n_3}(K) & \cdots & \mathrm{M}_{n_1\times n_q}(K) \\ & 0_{n_2} & \mathrm{M}_{n_2\times n_3}(K) & \cdots & \mathrm{M}_{n_2\times n_q}(K) \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \mathrm{M}_{n_{q-1}\times n_q}(K) \\ & & & & 0_{n_q} \end{pmatrix} \subseteq \mathcal{A}$$

and $X \in \mathrm{M}_n(K)$ is an invertible matrix such that $X^{-1}\mathcal{A}X$ is contained in the full subalgebra of $M_n(K)$ of type $(n_1, n_2, \ldots, n_q)$ (see Definition 1.3), then matrix X also belongs to the full subalgebra of $M_n(K)$ of type $(n_1, n_2, \ldots, n_q)$.

Proof. The result that we want to prove is obvious if $q = 1$. Thus, building a proof using mathematical induction, we start with $q = 2$, and positive integers n_1, n_2 and n such that $n_1+n_2=n$. Write matrix X in block form $\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$, where $X_{ij} \in \mathrm{M}_{n_i \times n_j}(K)$, $1 \le i,j \le 2$, and suppose, for the contrary, that block $X_{21} \neq 0_{n_2 \times n_1}$. Write also the inverse matrix X^{-1} in block form $\begin{pmatrix} X'_{11} & X'_{12} \\ X'_{21} & X'_{22} \end{pmatrix}$, where $X'_{ij} \in \mathrm{M}_{n_i \times n_j}(K)$. Then, by assumption, the matrix $\begin{pmatrix} 0_{n_1} & Y \\ & 0_{n_2} \end{pmatrix}$ belongs to $\mathcal{A}$ for every $Y \in \mathrm{M}_{n_1 \times n_2}(K)$. Therefore,

$$X^{-1} \begin{pmatrix} 0_{n_1} & Y \\ & 0_{n_2} \end{pmatrix} X = \begin{pmatrix} X'_{11} & X'_{12} \\ X'_{21} & X'_{22} \end{pmatrix} \begin{pmatrix} 0_{n_1} & Y \\ & 0_{n_2} \end{pmatrix} \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$
$$= \begin{pmatrix} X'_{11}YX_{21} & X'_{11}YX_{22} \\ X'_{21}YX_{21} & X'_{21}YX_{22} \end{pmatrix} \in \begin{pmatrix} \mathrm{M}_{n_1}(K) & \mathrm{M}_{n_1 \times n_2}(K) \\ 0_{n_2 \times n_1} & \mathrm{M}_{n_2}(K) \end{pmatrix}.$$

Hence, $X'_{21}YX_{21} = 0_{n_2 \times n_1}$ for every $Y \in \mathrm{M}_{n_1 \times n_2}(K)$, and so Lemma 4.5 implies that $X'_{21} = 0_{n_2 \times n_1}$. We conclude from formula (4.3) that $X_{21} = 0_{n_2 \times n_1}$. This is a contradiction, which completes the desired result for $q = 2$.

Assume now inductively that the result holds for some $q \geq 2$, and let n and $n_1, n_2, \dots, n_{q+1}$ be positive integers such that $n_1 + n_2 + \cdots + n_{q+1} = n$. Write matrix X and its inverse in block form

$$X = \begin{pmatrix} X_{11} & \dots & X_{1,q+1} \\ \vdots & \ddots & \vdots \\ X_{q+1,1} & \dots & X_{q+1,q+1} \end{pmatrix}, \quad X^{-1} = \begin{pmatrix} X'_{11} & \dots & X'_{1,q+1} \\ \vdots & \ddots & \vdots \\ X'_{q+1,1} & \dots & X'_{q+1,q+1} \end{pmatrix},$$

with $X_{ij}, X'_{ij} \in \mathrm{M}_{n_i \times n_j}(K)$ for all $1 \le i,j \le q+1$. Firstly, we will show that $X_{j1} = 0_{n_j \times n_1}$ for $j = 2, 3, \dots, q+1$. Then we will use the inductive assumption. Suppose, for the contrary, that $X_{j1} \neq 0_{n_j \times n_1}$ for some j, $2 \le j \le q+1$. Let $Y_{1j} \in \mathrm{M}_{n_1 \times n_j}(K)$ be an arbitrary matrix. Then

$$\begin{pmatrix} \dots & 0_{n_1 \times n_{j-1}} & Y_{1j} & 0_{n_1 \times n_{j+1}} & \dots \\ \dots & 0_{n_2 \times n_{j-1}} & 0_{n_2 \times n_j} & 0_{n_2 \times n_{j+1}} & \dots \\ & \vdots & \vdots & \vdots & \end{pmatrix} \in \mathcal{A},$$

and direct calculation gives

$$X^{-1} \begin{pmatrix} \dots & 0_{n_1 \times n_{j-1}} & Y_{1j} & 0_{n_1 \times n_{j+1}} & \dots \\ \dots & 0_{n_2 \times n_{j-1}} & 0_{n_2 \times n_j} & 0_{n_2 \times n_{j+1}} & \dots \\ & \vdots & \vdots & \vdots & \end{pmatrix}$$
$$X = \begin{pmatrix} X'_{11}Y_{1j}X_{j1} & X'_{11}Y_{j1}X_{j2} & \dots \\ X'_{21}Y_{1j}X_{j1} & X'_{21}Y_{j1}X_{j2} & \dots \\ \vdots & \vdots & \end{pmatrix}.$$

Since the algebra $X^{-1}\mathcal{A}X$ is contained in the full subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \ldots, n_q)$, we obtain the equalities

$$X'_{21}Y_{1j}X_{j1} = 0_{n_2\times n_1}, \quad X'_{31}Y_{1j}X_{j1} = 0_{n_3\times n_1}, \quad \ldots \quad , X'_{q+1,1}Y_{1j}X_{j1} = 0_{n_{q+1}\times n_1}.$$

We now apply Lemma 4.5 to each of these equations and obtain that $X'_{21}, X'_{31}, \ldots, X'_{q+1,1}$ are zero matrices. Therefore we can write $X^{-1} = \begin{pmatrix} X'_{11} & \overline{X'}_{12} \\ 0_{N_2\times n_1} & \overline{X'}_{22} \end{pmatrix}$, where $N_2 = n_2 + n_3 + \cdots + n_{q+1}$ and

$$\overline{X'}_{12} = \begin{pmatrix} X'_{12} & X'_{13} & \ldots & X'_{1,q+1} \end{pmatrix}, \quad \overline{X'}_{22} = \begin{pmatrix} X'_{22} & \ldots & X'_{2,q+1} \\ \vdots & \ddots & \vdots \\ X'_{q+1,2} & \ldots & X'_{q+1,q+1} \end{pmatrix}.$$

From formula (4.3) on the inverse of a block triangular matrix it follows that

$$X_{21} = 0_{n_2\times n_1}, \quad X_{31} = 0_{n_3\times n_1}, \quad \ldots, \quad X_{q+1,1} = 0_{n_{q+1}\times n_1}.$$

This is a contradiction, since $X_{j1} \neq 0_{n_j\times n_1}$ for some $2 \leq j \leq q+1$. Hence, indeed $X_{21}, X_{31}, \ldots, X_{q+1,1}$ are zero matrices. Now we can use the inductive assumption to the subalgebra of $\mathrm{M}_{N_2}(K)$ obtained from the entries of $\mathcal{A}$ starting from row n_1+1 and column n_1+1. It implies that matrix X belongs to the full subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \ldots, n_q)$. □

Although conjugation of a block triangular D_q subalgebra $\mathcal{A}$ of $\mathrm{M}_n(K)$ of some type $(n_1, n_2, \ldots, n_q)$ with an invertible matrix $X \in \mathrm{M}_n(K)$ can result in the subalgebra $X^{-1}\mathcal{A}X$ of $\mathrm{M}_n(K)$ not being a block triangular subalgebra of $\mathrm{M}_n(K)$, as shown in the example below, we will prove in Proposition 4.8 that this does not happen if X is such $X^{-1}\mathcal{A}X$ is contained in the full subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \ldots, n_q)$.

Example 4.7. Let $\mathcal{A}$ be a D_4 subalgebra of $\mathrm{M}_n(K)$ of type (n_1, n_2, n_3, n_4), possibly with max-comm db's, and consider the invertible block matrix

$$X = \begin{pmatrix} 0_{n_1\times n_4} & 0_{n_1\times n_2} & 0_{n_1\times n_3} & I_{n_1} \\ 0_{n_2\times n_4} & I_{n_2} & 0_{n_2\times n_3} & 0_{n_2\times n_1} \\ 0_{n_3\times n_4} & 0_{n_3\times n_2} & I_{n_3} & 0_{n_3\times n_1} \\ I_{n_4} & 0_{n_4\times n_2} & 0_{n_4\times n_3} & 0_{n_4\times n_1} \end{pmatrix} \in \mathrm{M}_n(K),$$

which clearly is a "block" version of the permutation matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$X^{-1}=\begin{pmatrix} 0_{n_4\times n_1} & 0_{n_4\times n_2} & 0_{n_4\times n_3} & I_{n_4}\\ 0_{n_2\times n_1} & I_{n_2} & 0_{n_2\times n_3} & 0_{n_2\times n_4}\\ 0_{n_3\times n_1} & 0_{n_3\times n_2} & I_{n_3} & 0_{n_3\times n_4}\\ I_{n_1} & 0_{n_1\times n_2} & 0_{n_1\times n_3} & 0_{n_1\times n_4} \end{pmatrix}.$$

Writing the algebra $\mathcal{A}$ in block triangular form

$$\begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{A}_{13} & \mathcal{A}_{14}\\ & \mathcal{A}_{22} & \mathcal{A}_{23} & \mathcal{A}_{24}\\ & & \mathcal{A}_{33} & \mathcal{A}_{34}\\ & & & \mathcal{A}_{44} \end{pmatrix},$$

as in (1.10), direct verification yields

$$X^{-1}\mathcal{A}X=\begin{pmatrix} \mathcal{A}_{44} & 0_{n_4\times n_2} & 0_{n_4\times n_3} & 0_{n_4\times n_1}\\ \mathcal{A}_{24} & \mathcal{A}_{22} & \mathcal{A}_{23} & 0_{n_2\times n_1}\\ \mathcal{A}_{34} & 0_{n_3\times n_2} & \mathcal{A}_{33} & 0_{n_3\times n_1}\\ \mathcal{A}_{14} & \mathcal{A}_{12} & \mathcal{A}_{13} & \mathcal{A}_{11} \end{pmatrix},$$

implying that, by Definition 1.3, $X^{-1}\mathcal{A}X$ is not a subalgebra of $\mathrm{M}_n(K)$ of any type $(\ell_1, \ell_2, \ell_3, \ell_4)$.

Proposition 4.8. *Let $\mathcal{A}$ be a D_q subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \dots, n_q)$ with max-comm db's, and let $X \in \mathrm{M}_n(K)$ be an invertible matrix such that $X^{-1}\mathcal{A}X$ is contained in the full subalgebra of $M_n(K)$ of type $(n_1, n_2, \dots, n_q)$. Then $X^{-1}\mathcal{A}X$ is also a D_q subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \dots, n_q)$ with max-comm db's. Moreover, if we write $\mathcal{A}_{ii}$ and $\mathcal{A}'_{ii}$, $i = 1, 2, \dots, q$, for the diagonal blocks of the subalgebras $\mathcal{A}$ and $X^{-1}\mathcal{A}X$ (see (1.10)), respectively, then $\mathcal{A}_{ii}$ and $\mathcal{A}'_{ii}$ are conjugates.*

Proof. By Lemma 4.6, X is in the full subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \dots, n_q)$, and so

$$X=\begin{pmatrix} X_{11} & \dots & X_{1q}\\ & \ddots & \vdots\\ & & X_{qq} \end{pmatrix},$$

for some $X_{ij} \in \mathrm{M}_{n_i\times n_j}(K)$, $1 \le i \le j \le q$. Write the inverse X^{-1} in block form

$$\begin{pmatrix} X'_{11} & \dots & X'_{1q}\\ \vdots & \ddots & \vdots\\ X'_{q1} & \dots & X'_{qq} \end{pmatrix},$$

for $X'_{ij} \in \mathrm{M}_{n_i \times n_j}(K),\ 1 \le i, j \le q$. We will show that X^{-1} is also block triangular.

Divide X into four blocks of sizes $n_1 \times n_1$, $n_1 \times (n_2 + \cdots + n_q)$, $(n_2 + \cdots + n_q) \times n_1$ and $(n_2 + \cdots + n_q) \times (n_2 + \cdots + n_q)$, respectively. Formula (4.3) implies that $X'_{11} = X_{11}^{-1}$ and that the matrices $X'_{21}, X'_{31}, \ldots, X'_{q1}$ are all zero matrices. These facts lead us to finding the inverse of the block triangular matrix

$$\begin{pmatrix} X_{22} & \ldots & X_{2q} \\ & \ddots & \vdots \\ & & X_{qq} \end{pmatrix}.$$

Continuing in the above way, we finally get that X^{-1} is block triangular with $X'_{ii} = X_{ii}^{-1}$ for every $i,\ i = 1, 2, \ldots, q$.

Now write the D_q subalgebra $\mathcal{A}$ of $\mathrm{M}_n(K)$ with max-comm db's in the form

$$\begin{pmatrix} \mathcal{A}_{11} & \ldots & \mathcal{A}_{1q} \\ & \ddots & \vdots \\ & & \mathcal{A}_{qq} \end{pmatrix},$$

as in (1.10). Since the matrices X and X^{-1} are block triangular, it follows that $X^{-1}\mathcal{A}X$ is contained in the D_q subalgebra

$$\begin{pmatrix} X_{11}^{-1}\mathcal{A}_{11}X_{11} & \mathrm{M}_{n_1 \times n_2}(K) & \mathrm{M}_{n_1 \times n_3}(K) & \cdots & \mathrm{M}_{n_1 \times n_q}(K) \\ & X_{22}^{-1}\mathcal{A}_{22}X_{22} & \mathrm{M}_{n_2 \times n_3}(K) & \cdots & \mathrm{M}_{n_2 \times n_q}(K) \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \mathrm{M}_{n_{q-1} \times n_q}(K) \\ & & & & X_{qq}^{-1}\mathcal{A}_{qq}X_{qq} \end{pmatrix} \tag{4.4}$$

of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \ldots, n_q)$ with max-comm db's. We will prove that the reverse inclusion also holds.

We first show that

$$\begin{pmatrix} \cdots & 0_{n_{n_1 \times q-1}} & \mathrm{M}_{n_1 \times n_q}(K) \\ \cdots & 0_{n_2 \times n_{q-1}} & 0_{n_2 \times n_q} \\ & \vdots & \vdots \end{pmatrix} \subseteq X^{-1}\mathcal{A}X. \tag{4.5}$$

To this end, keep in mind that every $X_{ii},\ i = 1, 2, \ldots, q$, is invertible, and note that for an arbitrary matrix $Y_{1q} \in \mathrm{M}_{n_1 \times n_q}(K)$,

$$\begin{pmatrix} \cdots & 0_{n_1\times n_{q-1}} & Y_{1q} \\ \cdots & 0_{n_2\times n_{q-1}} & 0_{n_2\times n_q} \\ & \vdots & \vdots \end{pmatrix} = X^{-1}\begin{pmatrix} \cdots & 0_{n_1\times n_{q-1}} & X_{11}Y_{1q}X_{qq}^{-1} \\ \cdots & 0_{n_2\times n_{q-1}} & 0_{n_2\times n_q} \\ & \vdots & \vdots \end{pmatrix} X \in X^{-1}\mathcal{A}X,$$

which establishes (4.5). Next take an arbitrary matrix $Y_{1,q-1} \in \mathrm{M}_{n_1\times n_{q-1}}(K)$. Then

$$X^{-1}\begin{pmatrix} \cdots & 0_{n_1\times n_{q-2}} & X_{11}Y_{1,q-1}X_{q-1,q-1}^{-1} & 0_{n_1\times n_q} \\ \cdots & 0_{n_2\times n_{q-2}} & 0_{n_2\times n_{q-1}} & 0_{n_2\times n_q} \\ & \vdots & \vdots & \vdots \end{pmatrix} X =$$

$$= \begin{pmatrix} \cdots & 0_{n_1\times n_{q-2}} & Y_{1,q-1} & Z_{1,q} \\ \cdots & 0_{n_2\times n_{q-2}} & 0_{n_2\times n_{q-1}} & 0_{n_2\times n_q} \\ & \vdots & \vdots & \vdots \end{pmatrix} \in X^{-1}\mathcal{A}X$$

for some $Z_{1,q-1} \in \mathrm{M}_{n_1\times n_q}(K)$. Because of the inclusion in (4.5), we deduce that

$$\begin{pmatrix} \cdots & 0_{n_1\times n_{q-2}} & Y_{1,q-1} & 0_{n_1\times n_q} \\ \cdots & 0_{n_2\times n_{q-2}} & 0_{n_2\times n_{q-1}} & 0_{n_2\times n_q} \\ & \vdots & \vdots & \vdots \end{pmatrix} \in X^{-1}\mathcal{A}X.$$

Therefore, $X^{-1}\mathcal{A}X$ contains

$$\begin{pmatrix} \cdots & 0_{n_{n_1\times q-2}} & \mathrm{M}_{n_1\times n_{q-1}}(K) & 0_{n_1\times n_q} \\ \cdots & 0_{n_2\times n_{q-2}} & 0_{n_2\times n_{q-1}} & 0_{n_2\times n_q} \\ & \vdots & \vdots & \vdots \end{pmatrix},$$

Continuing in this way we obtain the inclusion

$$\begin{pmatrix} X_{11}^{-1}\mathcal{A}_{11}X_{11} & \mathrm{M}_{n_1\times n_2}(K) & \mathrm{M}_{n_1\times n_3}(K) & \dots & \mathrm{M}_{n_1\times n_q}(K) \\ & 0_{n_2} & 0_{n_2\times n_3} & \dots & 0_{n_1\times n_q} \\ & & 0_{n_3} & \dots & 0_{n_3\times n_q} \\ & & & \ddots & \vdots \\ & & & & 0_{n_q} \end{pmatrix} \subseteq X^{-1}\mathcal{A}X$$

Proceeding in the same manner we obtain that

$$\begin{pmatrix} 0_{n_1} & 0_{n_1\times n_2} & 0_{n_1\times n_3} & \dots & 0_{n_1\times n_q} \\ & X_{22}^{-1}\mathcal{A}_{22}X_{22} & \mathrm{M}_{n_2\times n_3}(K) & \dots & \mathrm{M}_{n_2\times n_q}(K) \\ & & 0_{n_3} & \dots & 0_{n_3\times n_q} \\ & & & \ddots & \vdots \\ & & & & 0_{n_q} \end{pmatrix} \subseteq X^{-1}\mathcal{A}X.$$

The pattern is now clear, and so finally we conclude that $X^{-1}\mathcal{A}X$ contains the entire subalgebra of $\mathrm{M}_n(K)$ in (4.4), which completes the proof. □

Corollary 4.9. *Let $\mathcal{A}$ be a subalgebra of $\mathrm{U}_n(K)$.*

(1) If $\mathcal{A}$ is a maximal D_q subalgebra of $\mathrm{M}_n(K)$, then $\mathcal{A}$ is a D_q subalgebra of $\mathrm{M}_n(K)$ of some type $(n_1, n_2, \ldots, n_q)$ with max-comm db's.
(2) If $\mathcal{A}$ is a D_q subalgebra of $\mathrm{M}_n(K)$ with maximum dimension, then $\mathcal{A}$ is a max-dim D_q subalgebra of $\mathrm{M}_n(K)$ of some type $(n_1, n_2, \ldots, n_q)$ with max-dim db's.

Proof. (1) By Theorem 3.2, there is an invertible matrix X such that $X^{-1}\mathcal{A}X$ is a D_q subalgebra of $\mathrm{M}_n(K)$ of some type $(n_1, n_2, \ldots, n_q)$ with max-comm db's. The desired result now follows immediately from Proposition 4.8, since $\mathcal{A} = X(X^{-1}\mathcal{A}X)X^{-1}$.

(2) By (1), $\mathcal{A}$ is a D_q subalgebra of $\mathrm{M}_n(K)$ of some type $(n_1, n_2, \ldots, n_q)$ with max-comm db's. Had any of the commutative algebras in the diagonal blocks of $\mathcal{A}$ not been of maximum dimension, we would have been able to replace it by a commutative algebra with larger dimension, thereby obtaining a D_q subalgebra of $\mathrm{M}_n(K)$ with dimension larger than that of $\mathcal{A}$; a contradiction. □

The construction of a D_q subalgebra of $\mathrm{M}_n(K)$ with maximum dimension described in Theorem 1.2, combined with the examples in (1.2) and in (1.3) of commutative subalgebras of $\mathrm{M}_n(K)$ with maximum dimension, gives an example of a D_q subalgebra of $\mathrm{M}_n(K)$ with maximum dimension contained in $\mathrm{U}_n(K)$. So if $\mathcal{A}$ is a D_q subalgebra of $\mathrm{U}_n(K)$ with maximum dimension, then $\mathcal{A}$ is a D_q subalgebra of $\mathrm{M}_n(K)$ with maximum dimension. Consequently, Corollary 4.9(2) confirms the "underlying conjecture" embodied in Question 4.1 by answering the question in the negative.

We conclude the section with a characterization of when D_q subalgebras of $\mathrm{M}_n(K)$ with max-comm db's are conjugated.

Theorem 4.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be D_q subalgebras of $\mathrm{M}_n(K)$ of types $(n_1, n_2, \ldots, n_q)$ and $(\ell_1, \ell_2, \ldots, \ell_q)$, respectively, with max-comm db's. Write $\mathcal{A}_{ii}$ and $\mathcal{B}_{ii}$, $i = 1, 2, \ldots, q$, for the diagonal blocks of the subalgebras $\mathcal{A}$ and $\mathcal{B}$, respectively (see (1.10)). Then $\mathcal{A}$ and $\mathcal{B}$ are conjugates if and only if the q-tuples $(n_1, n_2, \ldots, n_q)$ and $(\ell_1, \ell_2, \ldots, \ell_q)$ are equal and, for every i, $i = 1, 2, \ldots, q$, $\mathcal{A}_{ii}$ and $\mathcal{B}_{ii}$ are conjugates.*

Proof. Firstly, assume that $\mathcal{A}$ and $\mathcal{B}$ are D_q subalgebras of $\mathrm{M}_n(K)$ of the same type $(n_1, n_2, \ldots, n_q)$ with max-comm db's such that for every j, $j = 1, 2, \ldots, q$, the diagonal blocks $\mathcal{A}_{jj}$ and $\mathcal{B}_{jj}$ are conjugates. Then there are invertible matrices $X_{jj} \in \mathrm{M}_{n_j}(K)$ such that $X_{jj}^{-1}\mathcal{A}_{jj}X_{jj} = \mathcal{B}_{jj}$. We will show that

$$\begin{pmatrix} X_{11}^{-1} & & & \\ & X_{22}^{-1} & & \\ & & \ddots & \\ & & & X_{qq}^{-1} \end{pmatrix} \mathcal{A} \begin{pmatrix} X_{11} & & & \\ & X_{22} & & \\ & & \ddots & \\ & & & X_{qq} \end{pmatrix} = \mathcal{B}. \tag{4.6}$$

Note that, by Proposition 4.8, the subalgebra on the left hand side in (4.6) is a D_q subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \ldots, n_q)$ with max-comm db's. It is easy to check that this subalgebra has $X_{jj}^{-1}\mathcal{A}_{jj}X_{jj}$ as its diagonal blocks, which establishes the equality in (4.6).

Conversely, assume that the algebras $\mathcal{A}$ and $\mathcal{B}$ are conjugated. If we can show that tuples the q-tuples $(n_1, n_2, \ldots, n_q)$ and $(\ell_1, \ell_2, \ldots, \ell_q)$ are equal, then the result follows directly from Proposition 4.8. To this end, first note that, by Proposition 4.2,

$$n_1 + n_2 + \cdots + n_i = \dim_K \mathcal{C}_{\mathcal{A}}^{q-i} V$$

for $i = 1, 2, \ldots, q-1$, where $V = K^n$ and $\mathcal{C}_{\mathcal{A}}$ is the ideal of $\mathcal{A}$ generated by all $[x, y],\ x, y \in \mathcal{A}$. Since $\sum_{j=1}^{q} n_j = n$, we can write

$$n_i = \begin{cases} \dim_K \mathcal{C}_{\mathcal{A}}^{q-i} V - \dim_K \mathcal{C}_{\mathcal{A}}^{q-i+1} V, & \text{if } i = 1, 2, \ldots, q-1 \\ n - \dim_K \mathcal{C}_{\mathcal{A}} V, & \text{if } i = q. \end{cases}$$

Hence the type of the subalgebra $\mathcal{A}$ of $\mathrm{M}_n(K)$ is determined by $\dim_K \mathcal{C}_{\mathcal{A}}^i V$ for $i = 1, 2, \ldots, q-1$. Similarly, the type of the subalgebra $\mathcal{B}$ is determined by $\dim_K \mathcal{C}_{\mathcal{B}}^i V$, where $\mathcal{C}_{\mathcal{B}}$ is the ideal of $\mathcal{B}$ generated by all $[x, y],\ x, y \in \mathcal{B}$.

By assumption, there exists an invertible matrix $X \in \mathrm{M}_n(K)$ such that $X^{-1}\mathcal{A}X = \mathcal{B}$, which implies that $\mathcal{C}_{\mathcal{B}}^i = (X^{-1}\mathcal{C}_{\mathcal{A}}X)^i = X^{-1}\mathcal{C}_{\mathcal{A}}^i X$ for all $i = 1, 2 \ldots, q-1$. To complete the proof we will show that $\dim_K \mathcal{C}_{\mathcal{A}}^i V = \dim_K \mathcal{C}_{\mathcal{B}}^i V$.

If vectors $C_1v_1, C_2v_2, \ldots, C_kv_k$ constitute a basis of $\mathcal{C}_{\mathcal{A}}^i V$ for some matrices $C_j \in \mathcal{C}_{\mathcal{A}}^i$ and some vectors $v_j \in V$, then it can be shown directly that the vectors $X^{-1}C_jv_j = (X^{-1}C_jX)X^{-1}v_j,\ j = 1, 2, \ldots, k$, of the vector space $(X^{-1}\mathcal{C}_{\mathcal{A}}^i X)V$ are linearly independent. Therefore, $\dim_K \mathcal{C}_{\mathcal{A}}^i V \le \dim_K (X^{-1}\mathcal{C}_{\mathcal{A}}^i X)V = \dim_K \mathcal{C}_{\mathcal{B}}^i V$. Similarly, we can show that $\dim_K \mathcal{C}_{\mathcal{B}}^i V = \dim_K (X^{-1}\mathcal{C}_{\mathcal{A}}^i X)V \le \dim_K \mathcal{C}_{\mathcal{A}}^i V$ by taking basis vectors of the vector space $(X^{-1}\mathcal{C}_{\mathcal{A}}^i X)V$ and producing linearly independent vectors in the vector space $\mathcal{C}_{\mathcal{A}}^i V$. □

5. Remarks on a result by Jacobson

In this section, we clarify the structure of commutative subalgebras of $\mathrm{M}_n(K)$, with K an algebraically closed field, as discussed in [7].

Throughout this section K is an algebraically closed field, and $\mathcal{A}$ is a commutative subalgebra of $\mathrm{M}_n(K)$ with maximum dimension. We focus in particular on the structure of $\mathcal{A}$ for $n \in \{2, 3\}$, which, in the light of the footnote in [7, page 436], does not seem to be so readily obtained after all. These considerations will be used in the subsequent sections.

By [7], for $n > 3$ we have the following:

- If n is an even integer ($n = 2\ell$), then $\mathcal{A}$ is conjugated to

$$C^1_{2\ell}(K) := KI_n + \begin{pmatrix} 0_\ell & \mathrm{M}_\ell(K) \\ & 0_\ell \end{pmatrix}. \tag{5.1}$$

- If n is an odd integer ($n = 2\ell + 1$), then $\mathcal{A}$ is conjugated to

$$C^1_{2\ell+1}(K) := KI_n + \begin{pmatrix} 0_\ell & \mathrm{M}_{\ell\times(\ell+1)}(K) \\ & 0_{\ell+1} \end{pmatrix} \tag{5.2}$$

  or

$$C^2_{2\ell+1}(K) := KI_n + \begin{pmatrix} 0_{\ell+1} & \mathrm{M}_{(\ell+1)\times\ell}(K) \\ & 0_\ell \end{pmatrix}. \tag{5.3}$$

It is possible to show (see Corollary 5.3) that the algebras $C^1_{2\ell+1}(K)$ and $C^2_{2\ell+1}(K)$ are not conjugated. However, they are isomorphic. Indeed, it is easily verified that the map from $C^1_{2\ell+1}(K)$ to $C^2_{2\ell+1}(K)$ which rotates the rectangular block $\mathrm{M}_{\ell\times(\ell+1)}(K)$ counterclockwise through $90°$ is an isomorphism.

Next, let $n \leq 3$. Obviously, for $n = 1$ we have

$$\mathcal{A} = K =: C^1_1(K). \tag{5.4}$$

We now carefully study the two remaining cases, i.e., when $n \in \{2, 3\}$. Either $\mathcal{A}$ is isomorphic to a nontrivial product $\mathcal{A}_1 \times \mathcal{A}_2$ of algebras, or it is not isomorphic to such a product. Suppose that we have the latter situation. Since K is algebraically closed and $\mathcal{A}$ is a finite dimensional commutative algebra, it follows from [9, (3.5) Wedderburn-Artin Theorem] that

$$\mathcal{A}/J(\mathcal{A}) \cong K_1 \times K_2 \times \ldots \times K_t$$

for some $t \geq 1$, with $K_i = K$ for every i. Since $J(\mathcal{A})$ is nilpotent (see, for example, [9, (4.12) Theorem]), it follows from [9, (21.28) Theorem] that if $t > 1$, then, lifting the idempotent $(1, 0, \ldots, 0)$ of $\mathcal{A}/J(\mathcal{A})$, we get a nontrivial (i.e., $e \notin \{0, 1\}$) idempotent $e \in \mathcal{A}$. Therefore,

$$\mathcal{A} \cong e\mathcal{A}e \times (1 - e)\mathcal{A}(1 - e),$$

which contradicts our assumption. Consequently, $\mathcal{A}$ is local, and so by [18, Proposition 10], $\mathcal{A}$ is conjugated to some subalgebra C of $\mathrm{U}_n^*(K)$. For C we have the following candidates:

- For $n = 2$ and the (commutative) algebra $\mathrm{U}_2^*(K)$ we have only one possibility, namely

$$C_2^1(K) = \mathrm{U}_2^*(K) = KI_2 + \begin{pmatrix} 0 & K \\ & 0 \end{pmatrix}. \tag{5.5}$$

- For $n = 3$ and the algebra $\mathrm{U}_3^*(K)$, taking an arbitrary $x \in K$, the matrix $\begin{pmatrix} 0 & 0 & x \\ & 0 & 0 \\ & & 0 \end{pmatrix}$ commutes with every $X \in \mathrm{U}_3^*(K)$. So we can see that $\begin{pmatrix} 0 & 0 & x \\ & 0 & 0 \\ & & 0 \end{pmatrix} \in C$, and consequently, $\begin{pmatrix} 0 & 0 & K \\ & 0 & 0 \\ & & 0 \end{pmatrix} \subseteq C$. Since the maximal possible dimension for a commutative subalgebra of $\mathrm{M}_3(K)$ is $1 + \left\lfloor \frac{3^2}{4} \right\rfloor = 3$, we deduce that there exist $\alpha, \beta \in K$, not both equal to zero, such that $C = \left\{ \begin{pmatrix} a & \alpha b & c \\ & a & \beta b \\ & & a \end{pmatrix} : a, b, c \in K \right\}$.

  For $\alpha \neq 0$ and $\beta = 0$ we get

$$C_3^1(K) = KI_3 + \begin{pmatrix} 0 & K & K \\ & 0 & 0 \\ & & 0 \end{pmatrix}. \tag{5.6}$$

  For $\alpha = 0$ and $\beta \neq 0$ we get

$$C_3^2(K) = KI_3 + \begin{pmatrix} 0 & 0 & K \\ & 0 & K \\ & & 0 \end{pmatrix}. \tag{5.7}$$

  Finally, for $\alpha \neq 0$ and $\beta \neq 0$, conjugation by $\begin{pmatrix} \alpha & 0 & 0 \\ & 1 & 0 \\ & & \beta^{-1} \end{pmatrix}$ gives

$$\begin{pmatrix} \alpha^{-1} & 0 & 0 \\ & 1 & 0 \\ & & \beta \end{pmatrix} \begin{pmatrix} a & \alpha b & c \\ & a & \beta b \\ & & a \end{pmatrix} \begin{pmatrix} \alpha & 0 & 0 \\ & 1 & 0 \\ & & \beta^{-1} \end{pmatrix} = \begin{pmatrix} a & b & \alpha^{-1}\beta^{-1}c \\ & a & b \\ & & a \end{pmatrix},$$

  and so

$$C_3^3(K) = KI_3 + K \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & K \\ & 0 & 0 \\ & & 0 \end{pmatrix}. \tag{5.8}$$

Now we consider the case where $\mathcal{A}$ is isomorphic to a proper product of algebras $\mathcal{A} \cong \mathcal{A}_1 \times \mathcal{A}_2$. In this situation, $1 = e_1 + e_2$ for some nontrivial idempotents e_1, e_2. Let $\epsilon_1 \colon K^n \to K^n$ be the linear map such that e_1 is its matrix in the standard basis. Then as $\epsilon_1^2 = \epsilon_1$ we get $K^n = \mathrm{im}(\epsilon_1) \oplus \ker(\epsilon_1)$. If $v_1, v_2, \ldots, v_k$ form a basis for $\mathrm{im}(\epsilon_1)$, and $v_{k+1}, v_{k+2}, \ldots, v_n$ form a basis for $\ker(\epsilon_1)$ then in the basis $B = (v_1, v_2, \ldots, v_n)$ for K^n we have $M(\epsilon_1)_B^B = \left(\begin{array}{c|c} I_k & 0_{k\times(n-k)} \\ \hline & 0_{n-k} \end{array}\right)$. In the same basis B, for the map ϵ_2 related to $e_2 = 1 - e_1$, we have $M(\epsilon_2)_B^B = I_n - M(\epsilon_1)_B^B = \left(\begin{array}{c|c} 0_k & 0_{k\times(n-k)} \\ \hline & I_{n-k} \end{array}\right)$.

Since the idempotents e_1 and e_2 are orthogonal and the algebra $\mathcal{A}$ is commutative, we have $a = 1 \cdot a \cdot 1 = (e_1 + e_2)a(e_1 + e_2) = e_1 a e_1 + e_2 a e_2$ for every $a \in \mathcal{A}$. Therefore, as a vector space, $\mathcal{A} = e_1 \mathcal{A} e_1 \oplus e_2 \mathcal{A} e_2$. Considering conjugation of $\mathcal{A}$ with the change-of-basis matrix from the standard basis for K^n to B, we get

$$\begin{aligned} \mathcal{A}' = {} & \left(\begin{array}{c|c} I_k & 0_{k\times(n-k)} \\ \hline & 0_{n-k} \end{array}\right) \mathcal{A}' \left(\begin{array}{c|c} I_k & 0_{k\times(n-k)} \\ \hline & 0_{n-k} \end{array}\right) \\ & \oplus \left(\begin{array}{c|c} 0_k & 0_{k\times(n-k)} \\ \hline & I_{n-k} \end{array}\right) \mathcal{A}' \left(\begin{array}{c|c} 0_k & 0_{k\times(n-k)} \\ \hline & I_{n-k} \end{array}\right) = \\ & = \left(\begin{array}{c|c} \mathcal{A}_1' & 0_{k\times(n-k)} \\ \hline & \mathcal{A}_2' \end{array}\right), \end{aligned}$$

where $\mathcal{A}'$ is the conjugation of $\mathcal{A}$, and $\mathcal{A}_1' \subseteq \mathrm{M}_k(K)$ and $\mathcal{A}_2' \subseteq \mathrm{M}_{n-k}(K)$ are commutative subalgebras of $\mathrm{M}_n(K)$ with maximum dimensions isomorphic to $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively.

- For $n = 2$ we have only one possibility, namely, $\mathcal{A}$ is conjugated with

$$C_2^2(K) = \begin{pmatrix} K & 0 \\ & K \end{pmatrix}. \tag{5.9}$$

- For $n = 3$, either both algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are indecomposable or exactly one is indecomposable (recall that the maximal possible dimension of a commutative subalgebra of $\mathrm{M}_3(K)$ is 3). In the first case, at first glance, $\mathcal{A}$ is conjugated either with

$$\Lambda_1 = \left\{ \begin{pmatrix} a & b & 0 \\ & a & 0 \\ & & c \end{pmatrix} : a, b, c \in K \right\} \text{ or } \Lambda_2 = \left\{ \begin{pmatrix} a & 0 & 0 \\ & b & c \\ & & b \end{pmatrix} : a, b, c \in K \right\}.$$

Note, however, that considering the invertible matrix $Z = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ we have

$$Z^{-1}\Lambda_1 Z = \Lambda_2.$$

Thus $\mathcal{A}$ in this case is conjugated with

$$C_3^4(K) = \Lambda_1 = \left\{ \begin{pmatrix} a & b & 0 \\ & a & 0 \\ & & c \end{pmatrix} : a, b, c \in K \right\}. \tag{5.10}$$

In the case where exactly one of $\mathcal{A}_1$ and $\mathcal{A}_2$ is indecomposable, we get that $\mathcal{A}$ is conjugated with

$$C_3^5(K) = \begin{pmatrix} K & 0 & 0 \\ & K & 0 \\ & & K \end{pmatrix}. \tag{5.11}$$

Amongst the algebras $C_1^1(K), C_2^1(K), C_2^2(K), C_3^1(K), C_3^2(K), C_3^3(K), C_3^4(K)$ and $C_3^5(K)$, only $C_3^1(K)$ and $C_3^2(K)$ are isomorphic. However, by Corollary 5.3 these isomorphic subalgebras are not conjugated.

We summarize the above considerations (in this section) as follows:

Remark 5.1. Every commutative subalgebra of $\mathrm{M}_n(K)$ with maximum dimension is conjugated with precisely one of these presented in (5.1) – (5.11). Amongst these algebras, $C_{2\ell+1}^1(K)$ and $C_{2\ell+1}^2(K)$ are isomorphic for every $\ell \geq 1$.

By [7] and the presentation above for algebraically closed fields K, we have determined, up to conjugation, all commutative subalgebras of $\mathrm{M}_n(K)$ with maximum dimension. In fact, in [7] for $n > 3$, there is even a weaker assumption on the field K (not imperfect of characteristic 2), but in the above characterization of commutative subalgebras of $\mathrm{M}_2(K)$ and $\mathrm{M}_3(K)$ with maximum dimension, the assumption that K is algebraically closed is used. For example, in the case of the field $\mathbb{R}$ of real numbers, $\left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$ is a 2-dimensional subalgebra of $\mathrm{M}_2(\mathbb{R})$ (isomorphic to the field of complex numbers) which is not conjugated with $C_2^1(K)$ nor $C_2^2(K)$.

We conclude the section by showing that, for some of the commutative subalgebras $\mathcal{A}$ of $\mathrm{M}_n(K)$ with maximum dimension, namely those $\mathcal{A}$'s presented in (5.1) – (5.7), the only possible conjugation of $\mathcal{A}$ which is contained in $\mathrm{U}_n(K)$ is equal to $\mathcal{A}$ itself. We will use this result to give an exact description of D_q subalgebras of $\mathrm{U}_n(K)$ with maximum dimension in Corollary 6.8.

Proposition 5.2. *Let $\mathcal{A}$ be a commutative subalgebra of $M_n(K)$ with maximum dimension presented in one of (5.1) – (5.7). If $X \in \mathrm{M}_n(K)$ is invertible and $X^{-1}\mathcal{A}X \subseteq \mathrm{U}_n(K)$, then $X^{-1}\mathcal{A}X = \mathcal{A}$.*

Proof. If $n = 1$, then $\mathcal{A} = K$ and there is nothing to prove. Assume now that $n \geq 2$, with $\mathcal{A}$ equal to one of the algebras in (5.1) – (5.3) or (5.5) – (5.7). Then there exist positive integers r and s such that $r + s = n$ and

$$\mathcal{A} = KI_n + \begin{pmatrix} 0_r & \mathrm{M}_{r\times s}(K) \\ 0_{s\times r} & 0_s \end{pmatrix},$$

and so we have the inclusions

$$\begin{pmatrix} 0_r & \mathrm{M}_{r\times s}(K) \\ 0_{s\times r} & 0_s \end{pmatrix} \subseteq \mathcal{A} \quad \text{and} \quad X^{-1}\mathcal{A}X \subseteq \mathrm{U}_n(K) \subseteq \begin{pmatrix} \mathrm{M}_r(K) & \mathrm{M}_{r\times s}(K) \\ 0_{s\times r} & \mathrm{M}_s(K) \end{pmatrix}.$$

Invoking Lemma 4.6, with $q = 2$, we deduce that there exist matrices $X_{11} \in \mathrm{M}_r(K)$, $X_{12} \in \mathrm{M}_{r\times s}(K)$, and $X_{22} \in \mathrm{M}_s(K)$ such that

$$X = \begin{pmatrix} X_{11} & X_{12} \\ 0_{s\times r} & X_{22} \end{pmatrix}.$$

Therefore, by (4.3),

$$X^{-1} = \begin{pmatrix} X_{11}^{-1} & X_{12}' \\ 0_{s\times r} & X_{22}^{-1} \end{pmatrix},$$

where $X_{12}' = -X_{11}^{-1}X_{12}X_{22}^{-1}$. Let $\begin{pmatrix} aI_r & A_{12} \\ & aI_s \end{pmatrix}$ be an arbitrary element of $\mathcal{A}$, with $a \in K$, $A_{12} \in \mathrm{M}_{r\times s}(K)$. Then

$$X^{-1}\begin{pmatrix} aI_r & A_{12} \\ & aI_s \end{pmatrix}X = \begin{pmatrix} X_{11}^{-1} & X_{12}' \\ 0_{s\times r} & X_{22}^{-1} \end{pmatrix}\left(aI_n + \begin{pmatrix} 0_r & A_{12} \\ & 0_s \end{pmatrix}\right)\begin{pmatrix} X_{11} & X_{12} \\ 0_{s\times r} & X_{22} \end{pmatrix} =$$
$$= aI_n + \begin{pmatrix} 0_r & X_{11}^{-1}A_{12}X_{22} \\ & 0_s \end{pmatrix},$$

and so $X^{-1}\mathcal{A}X \subseteq \mathcal{A}$. Since we can take any matrix of $\mathrm{M}_{r\times s}(K)$ for A_{12}, it is not hard to show that the opposite inclusion also holds. This completes the proof. □

For an odd integer $n \geq 3$, the algebras $C_n^1(K)$ and $C_n^2(K)$ are distinct subalgebras of $\mathrm{U}_n(K)$. Moreover, we have the following:

Corollary 5.3. *Let n be an odd integer greater than or equal to 3. Then the commutative subalgebras $C_n^1(K)$ and $C_n^2(K)$ of $M_n(K)$ are not conjugated.*

Note that Proposition 5.2 cannot be extended to the subalgebras presented in (5.8)−(5.11):

Example 5.4. Conjugation of the subalgebra $C_2^2(K) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in K \right\}$ of $\mathrm{M}_2(K)$ with the invertible matrix $X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ gives

$$X^{-1} C_2^2(K) X = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} C_2^2(K) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \left\{ \begin{pmatrix} a & a-b \\ 0 & b \end{pmatrix} : a, b \in K \right\},$$

which is a subalgebra of $\mathrm{U}_2(K)$ different from $C_2^2(K)$.

Similar examples can also be found in case of the subalgebras $C_3^3(K)$, $C_3^4(K)$ and $C_3^5(K)$ of $\mathrm{M}_3(K)$. Two of them can be already constructed from the previous analysis, while defining $C_3^3(K)$ by formula (5.8) and $C_3^4(K)$ by formula (5.10).

In the following sections we will use the algebras presented in (5.1)−(5.11) for arbitrary fields. If we need the assumption that K is an algebraically closed field, then we will stress this assumption explicitly.

6. Isomorphic D_q subalgebras of $\mathrm{M}_n(K)$ with max-dim db's are of the same type

In this section, we will treat the isomorphism problem of D_q subalgebras of $\mathrm{M}_n(K)$ with max-dim db's. Before we state the main results we will give a characterization of the subalgebras we are dealing with.

Note that by Remark 3.3, for any maximal D_q subalgebra of $\mathrm{M}_n(K)$ there exists exactly one q-tuple $(n_1, n_2, \ldots, n_q)$, which indicates the type of conjugated D_q subalgebra of $\mathrm{M}_n(K)$ with max-comm db's. So we can look at the class of all maximal D_q subalgebras of $\mathrm{M}_n(K)$ associated with the fixed tuple $(n_1, n_2, \ldots, n_q)$. In general, we are not able to say much about this class, even if we treat algebras up to conjugation. However, if we restrict it to these algebras with maximum possible dimension for the considered class, then it consists of subalgebras conjugated with a D_q subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \ldots, n_q)$ with max-dim db's, which is more accessible. We want to stress that only for specific tuples, which will be described in Section 7, the restricted class consists of D_q subalgebras of $\mathrm{M}_n(K)$ with maximum dimension.

In Theorem 6.2 we prove that if two D_q subalgebras of $\mathrm{M}_n(K)$ with max-dim db's are isomorphic, then they are of the same type and the algebras in their diagonal blocks are pairwise isomorphic. Next, we will show that this theorem cannot be inverted, which indicates what should be modified to completely solve the isomorphism problem. Finally, for an algebraically closed field K, we will prove that two D_q subalgebras of $\mathrm{M}_n(K)$ with max-dim db's are isomorphic if and only if they are of the same type and their diagonal blocks are pairwise conjugated. In contrast with the examples of isomorphic but not conjugated commutative subalgebras of $\mathrm{M}_n(K)$ with maximum dimension, it implies that isomorphic D_q subalgebras of $\mathrm{M}_n(K)$ with maximum dimension are conjugated.

Recall that if we consider a D_q subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \dots, n_q)$ with max-dim db's, then we always use the notation related to (1.10). In other words, for D_q subalgebras $\mathcal{A}$ and $\mathcal{B}$ of $\mathrm{M}_n(K)$ of types $(n_1, n_2, \dots, n_q)$ and $(\ell_1, \ell_2, \dots, \ell_q)$, respectively, with max-dim db's, we will write

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \dots & \mathcal{A}_{1q} \\ & \mathcal{A}_{22} & \dots & \mathcal{A}_{2q} \\ & & \ddots & \vdots \\ & & & \mathcal{A}_{qq} \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} & \dots & \mathcal{B}_{1q} \\ & \mathcal{B}_{22} & \dots & \mathcal{B}_{2q} \\ & & \ddots & \vdots \\ & & & \mathcal{B}_{qq} \end{pmatrix}, \tag{6.1}$$

where $\mathcal{A}_{ii}$ (respectively, $\mathcal{B}_{ii}$) is a commutative subalgebra of $\mathrm{M}_{n_i}(K)$ (respectively, $\mathrm{M}_{\ell_i}(K)$) with maximum dimension for every i, $i = 1, 2 \dots, q$, and $\mathcal{A}_{ij} = \mathrm{M}_{n_i \times n_j}(K)$ and $\mathcal{B}_{ij} = \mathrm{M}_{\ell_i \times \ell_j}(K)$ for all i and j such that $1 \le i < j \le q$. Following the notation in Proposition 4.2, we henceforth denote the ideal of $\mathcal{A}$ (respectively, $\mathcal{B}$) generated by all commutators $[x, y]$, with $x, y \in \mathcal{A}$ (respectively, $x, y \in \mathcal{B}$) by $\mathcal{C}_{\mathcal{A}}$ (respectively, $\mathcal{C}_{\mathcal{B}}$).

In order to build a proof of Theorem 6.2, we will need a technical lemma based on Proposition 4.2.

Lemma 6.1. *Let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be an isomorphism of D_q subalgebras $\mathcal{A}$ and $\mathcal{B}$ of $M_n(K)$ of types $(n_1, n_2, \dots, n_q)$ and $(\ell_1, \ell_2, \dots \ell_q)$, respectively, with max-dim db's. Then, using the notation in (6.1),*

$$\varphi\left(\left(\begin{array}{c|c|c} & & \\ \hline & \mathcal{A}_{ii} & \\ \hline & & \end{array}\right)\right) \subseteq \left(\begin{array}{c|ccc} & \mathcal{B}_{1i} & \dots & \mathcal{B}_{1q} \\ & \vdots & \ddots & \vdots \\ & \mathcal{B}_{ii} & \dots & \mathcal{B}_{iq} \\ \hline & & & \end{array}\right)$$

for every i, $i = 1, 2, \dots, q$.

Proof. It follows from $\varphi(\mathcal{C}_{\mathcal{A}}) = \mathcal{C}_{\mathcal{B}}$ that $\varphi(\mathcal{C}^i_{\mathcal{A}}) = \mathcal{C}^i_{\mathcal{B}}$ for every i, $i = 1, 2, \dots, q$, and so, by Proposition 4.2,

$$\mathcal{C}^i_{\mathcal{A}} \cdot \left(\begin{array}{c|c|c} & & \\ \hline & \mathcal{A}_{ii} & \\ \hline & & \end{array}\right) = \{0_n\},$$

implying that

$$\mathcal{C}_{\mathcal{B}}^{i} \cdot \varphi\left(\left(\begin{array}{c|c|c} & & \\ \hline & \mathcal{A}_{ii} & \\ \hline & & \end{array}\right)\right) = \{0_n\}. \tag{6.2}$$

Let $1 < i < q$. We first show that the equality in (6.2) implies that

$$\varphi\left(\left(\begin{array}{c|c|c} & & \\ \hline & \mathcal{A}_{ii} & \\ \hline & & \end{array}\right)\right) \subseteq \begin{pmatrix} \mathcal{B}_{11} & \dots & \mathcal{B}_{1i} & \mathcal{B}_{1,i+1} & \dots & \mathcal{B}_{1q} \\ & \ddots & \vdots & \vdots & \ddots & \vdots \\ & & \mathcal{B}_{ii} & \mathcal{B}_{i,i+1} & \dots & \mathcal{B}_{iq} \\ & & & 0_{n_{i+1}} & \dots & 0_{n_{i+1},n_q} \\ & & & & \ddots & \vdots \\ & & & & & 0_{n_q} \end{pmatrix}. \tag{6.3}$$

To this end, let $Y = (y_{ij})$ be an arbitrary matrix in $\varphi\left(\left(\begin{array}{c|c|c} & & \\ \hline & \mathcal{A}_{ii} & \\ \hline & & \end{array}\right)\right)$, and let j be any integer such that $\ell_1 + \ell_2 + \cdots + \ell_i < j \leq n$ (recall that $\ell_1 + \ell_2 + \cdots + \ell_q = n$). Then it follows again from Proposition 4.2 that $e_{1j} \in \mathcal{C}_{\mathcal{B}}^{i}$, and so by (6.2),

$$0_n = e_{1j}Y = \begin{pmatrix} y_{j1} & y_{j2} & \dots & y_{jn} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

Hence, rows $\ell_1+\ell_2+\cdots+\ell_i+1, \ell_1+\ell_2+\cdots+\ell_i+2, \dots, n$ of Y are zero, which establishes (6.3).

Similar arguments, starting from the equality

$$\left(\begin{array}{c|c|c} & & \\ \hline & \mathcal{A}_{ii} & \\ \hline & & \end{array}\right) \cdot \mathcal{C}_{\mathcal{A}}^{q-i+1} = \{0_n\},$$

can be used to show that columns $1, 2, \dots, \ell_1 + \ell_2 + \cdots + \ell_{i-1}$ columns of an arbitrary matrix in the image $\varphi\left(\left(\begin{array}{c|c|c} & & \\ \hline & \mathcal{A}_{ii} & \\ \hline & & \end{array}\right)\right)$ are zero.

As far as the cases $i = 1$ and $i = q$ are concerned, it is evident that the gist of the above arguments also shows that rows $\ell_1 + 1, \ell_1 + 2, \dots, n$ of every matrix in

$\varphi\left(\left(\begin{array}{c|c|c} \mathcal{A}_{11} & & \\ \hline & & \\ \hline & & \end{array}\right)\right)$ are zero, and that columns 1,2,..., $\ell_1 + \ell_2 + \cdots + \ell_{q-1}$ of every matrix in $\varphi\left(\left(\begin{array}{c|c|c} & & \\ \hline & & \\ \hline & & \mathcal{A}_{qq} \end{array}\right)\right)$ are zero, which completes the proof. □

Theorem 6.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be D_q subalgebras of $\mathrm{M}_n(K)$ of types $(n_1, n_2, \ldots, n_q)$ and $(\ell_1, \ell_2, \ldots, \ell_q)$, respectively, with max-dim db's. If $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, then the q-tuples $(n_1, n_2, \ldots, n_q)$ and $(\ell_1, \ell_2, \ldots, \ell_q)$ are equal and, using the notation in (6.1), the algebras $\mathcal{A}_{ii}$ and $\mathcal{B}_{ii}$ are isomorphic for every i, $i = 1, 2, \ldots, q$.*

Proof. We will use the notation in (6.1). Let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be an isomorphism of the algebras $\mathcal{A}$ and $\mathcal{B}$. As in Lemma 6.1, we have $\varphi(\mathcal{C}_{\mathcal{A}}) = \mathcal{C}_{\mathcal{B}}$, which implies the induced isomorphism $\overline{\varphi}\colon \mathcal{A}/\mathcal{C}_{\mathcal{A}} \to \mathcal{B}/\mathcal{C}_{\mathcal{B}}$. By Proposition 4.2, we identify the quotient algebras $\mathcal{A}/\mathcal{C}_{\mathcal{A}}$ and $\mathcal{B}/\mathcal{C}_{\mathcal{B}}$ with the direct products $\mathcal{A}_{11} \times \mathcal{A}_{22} \times \cdots \times \mathcal{A}_{qq}$ and $\mathcal{B}_{11} \times \mathcal{B}_{22} \times \cdots \times \mathcal{B}_{qq}$ of the algebras $\mathcal{A}_{ii}$ and $\mathcal{B}_{ii}$, $i = 1, 2, \ldots, q$, respectively. As $\mathcal{C}_{\mathcal{B}}$ comprises all matrices with zero entries in the diagonal blocks, the inclusion

$$\begin{aligned}&\overline{\varphi}(0_{n_1} \times \ldots \times 0_{n_{i-1}} \times \mathcal{A}_{ii} \times 0_{n_{i+1}} \times \ldots \times 0_{n_q})\\ &\quad \subseteq 0_{\ell_1} \times \ldots \times 0_{\ell_{i-1}} \times \mathcal{B}_{ii} \times 0_{\ell_{i+1}} \times \ldots \times 0_{\ell_q},\end{aligned}$$

follows from Lemma 6.1. Similarly, Lemma 6.1 applied to the inverse φ^{-1} yields

$$\begin{aligned}&(\overline{\varphi})^{-1}(0_{\ell_1} \times \ldots \times 0_{\ell_{i-1}} \times \mathcal{B}_{ii} \times 0_{\ell_{i+1}} \times \ldots \times 0_{\ell_q})\\ &\quad \subseteq 0_{n_1} \times \ldots \times 0_{n_{i-1}} \times \mathcal{A}_{ii} \times 0_{n_{i+1}} \times \ldots \times 0_{n_q}.\end{aligned}$$

These two inclusions imply the equality

$$\overline{\varphi}(0_{n_1} \times \ldots \times 0_{n_{i-1}} \times \mathcal{A}_{ii} \times 0_{n_{i+1}} \times \ldots \times 0_{n_q}) = 0_{\ell_1} \times \ldots \times 0_{\ell_{i-1}} \times \mathcal{B}_{ii} \times 0_{\ell_{i+1}} \times \ldots \times 0_{\ell_q},$$

and so the algebras $\mathcal{A}_{ii}$ and $\mathcal{B}_{ii}$ are isomorphic.

Next, since $\mathcal{A}_{ii}$ and $\mathcal{B}_{ii}$ are commutative subalgebras of $\mathrm{M}_{n_i}(K)$ and $\mathrm{M}_{\ell_i}(K)$ (respectively) with maximum dimension, it follows from Schur's Theorem that

$$\dim_K \mathcal{A}_{ii} = 1 + \left\lfloor \frac{n_i^2}{4} \right\rfloor = 1 + \left\lfloor \frac{\ell_i^2}{4} \right\rfloor = \dim_K \mathcal{B}_{ii}, \tag{6.4}$$

which implies the equality $n_i = \ell_i$ for any $i = 1, 2, \ldots, q$. □

Note that in the equality (6.4) the assumption that $\mathcal{A}_{ii}$ and $\mathcal{B}_{ii}$ are commutative subalgebras of matrices with maximum dimension is essential. By Theorem 4.10 a conclusion related to that presented in Theorem 6.2 holds for D_q subalgebras $\mathcal{A}$ and $\mathcal{B}$ of

$\mathrm{M}_n(K)$ of types $(n_1, n_2, \ldots, n_q)$ and $(\ell_1, \ell_2, \ldots, \ell_q)$, respectively with max-comm db's, if we assume that $\mathcal{A}$ and $\mathcal{B}$ are conjugated. In this regard, we pose the following two questions:

Question 6.3. Do there exist isomorphic D_q subalgebras $\mathcal{A}$ and $\mathcal{B}$ of $\mathrm{M}_n(K)$ of types $(n_1, n_2, \ldots, n_q)$ and $(\ell_1, \ell_2, \ldots, \ell_q)$, respectively, with max-comm db's, which are not conjugated?

Question 6.4. Do there exist isomorphic D_q subalgebras $\mathcal{A}$ and $\mathcal{B}$ of $\mathrm{M}_n(K)$ of types $(n_1, n_2, \ldots, n_q)$ and $(\ell_1, \ell_2, \ldots, \ell_q)$, respectively, with max-comm db's, such that $n_i \neq \ell_i$ for at least one i?

Note that the paragraph preceding the two questions above implies that a positive answer to Question 6.4 would also answer Question 6.3 in the positive.

The next part of this section will lead us to a full characterization of isomorphisms between D_q subalgebras of $\mathrm{M}_n(K)$ with max-dim db's if the field K is algebraically closed. Firstly, without this assumption on K, we will show that there exist non-isomorphic D_q subalgebras of $\mathrm{M}_n(K)$ of the same type with max-dim db's, and so the converse of Theorem 6.2 does not hold.

Lemma 6.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be D_q subalgebras of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \ldots, n_q)$ with max-dim db's, such that, for some j, n_j is odd, $n_j \geq 3$ and the j-th diagonal blocks of algebras $\mathcal{A}$ and $\mathcal{B}$ are $C^1_{n_j}(K)$ and $C^2_{n_j}(K)$, respectively. Then $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic.*

Proof. Let $n_{j1} = \lfloor \frac{n_j}{2} \rfloor$, $n_{j2} = \lfloor \frac{n_j}{2} \rfloor + 1$. Then $n_{j1} \neq n_{j2}$ and by (5.2), (5.3), (5.6) and (5.7),

$$C^1_{n_j}(K) = KI_{n_j} + \begin{pmatrix} 0_{n_{j1}} & \mathrm{M}_{n_{j1}\times n_{j2}}(K) \\ 0_{n_{j2}\times n_{j1}} & 0_{n_{j2}} \end{pmatrix}$$

and

$$C^2_{n_j}(K) = KI_{n_j} + \begin{pmatrix} 0_{n_{j2}} & \mathrm{M}_{n_{j2}\times n_{j1}}(K) \\ 0_{n_{j1}\times n_{j2}} & 0_{n_{j1}} \end{pmatrix}.$$

We first consider the case $q = 2$ and $j = 1$. Then the Jacobson radical $J(\mathcal{A})$ of $\mathcal{A}$ satisfies

$$\left(\begin{array}{cc|c} 0_{n_{11}} & \mathrm{M}_{n_{11}\times n_{12}}(K) & \mathrm{M}_{n_1\times n_2}(K) \\ & 0_{n_{12}} & \\ \hline & & 0_{n_2} \end{array}\right) \subseteq J(\mathcal{A}) \subseteq \left(\begin{array}{cc|c} 0_{n_{11}} & \mathrm{M}_{n_{11}\times n_{12}}(K) & \mathrm{M}_{n_1\times n_2}(K) \\ & 0_{n_{12}} & \\ \hline & & \mathrm{M}_{n_2}(K) \end{array}\right),$$

and by Proposition 4.2,

$$\mathcal{C}_{\mathcal{A}} = \left(\begin{array}{c|c} 0_{n_1} & \mathrm{M}_{n_1\times n_2}(K) \\ \hline & 0_{n_2} \end{array}\right),$$

implying that

$$\dim_K(J(\mathcal{A})\mathcal{C}_{\mathcal{A}}) = \dim_K\left(\begin{array}{c|c} 0_{n_{11}\times n_1} & \mathrm{M}_{n_{11}\times n_2}(K) \\ \hline & 0_{(n_{12}+n_2)\times n_2} \end{array}\right) = n_{11}n_2.$$

Similarly, $\dim_K(J(\mathcal{B})\mathcal{C}_{\mathcal{B}}) = n_{12}n_2$. If the algebras $\mathcal{A}$ and $\mathcal{B}$ were isomorphic, then the images of the ideals $J(\mathcal{A})$ and $\mathcal{C}_{\mathcal{A}}$ of $\mathcal{A}$ under any algebra isomorphism from $\mathcal{A}$ to $\mathcal{B}$ would be the ideals $J(\mathcal{B})$ and $\mathcal{C}_{\mathcal{B}}$ of $\mathcal{B}$, respectively, and since the respective dimensions would be equal, we would have that

$$n_{11}n_2 = \dim_K(J(\mathcal{A})\mathcal{C}_{\mathcal{A}}) = \dim_K(J(\mathcal{B})\mathcal{C}_{\mathcal{B}}) = n_{12}n_2,$$

i.e., $n_{11} = n_{12}$; a contradiction. Therefore, $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic.

The case $q = 2$ and $j = 2$ is very similar to the above one. Instead of the dimensions of $J(\mathcal{A})\mathcal{C}_{\mathcal{A}}$ and $J(\mathcal{B})\mathcal{C}_{\mathcal{B}}$ we have to compare the dimensions of $\mathcal{C}_{\mathcal{A}}J(\mathcal{A})$ and $\mathcal{C}_{\mathcal{B}}J(\mathcal{B})$.

Now we assume that $q > 2$. Suppose (for the contrary) that $\varphi\colon \mathcal{B} \to \mathcal{A}$ is an isomorphism, and let $\overline{\varphi}\colon \mathcal{B}/\mathcal{C}_{\mathcal{B}}^2 \to \mathcal{A}/\mathcal{C}_{\mathcal{A}}^2$ be the induced isomorphism. (A similar strategy was followed in the proof of Theorem 6.2.) By $\overline{x}$ and $\overline{y}$ we denote the images of elements $x \in \mathcal{A}$ and $y \in \mathcal{B}$ in the quotient algebras $\mathcal{A}/\mathcal{C}_{\mathcal{A}}^2$ and $\mathcal{B}/\mathcal{C}_{\mathcal{B}}^2$, respectively. Consider the subspace

$$V_j = \overline{\left(\begin{array}{c|c|c} & & \\ \hline & J(\mathcal{B}_{jj}) & \\ \hline & & \end{array}\right)}\,\overline{\mathcal{C}_{\mathcal{B}}}$$

of the quotient algebra $\mathcal{B}/\mathcal{C}_{\mathcal{B}}^2$. Since, by assumption, $\mathcal{B}_{jj} = C_{n_j}^2(K)$, we have that

$$J(\mathcal{B}_{jj}) = \begin{pmatrix} 0_{n_{j2}} & \mathrm{M}_{n_{j2}\times n_{j1}}(K) \\ 0_{n_{j1}\times n_{j2}} & 0_{n_{j1}} \end{pmatrix}, \tag{6.5}$$

and so by Proposition 4.2,

$$V_j = \overline{\left(\begin{array}{c|c|c} & & \\ \hline & J(\mathcal{B}_{jj}) & \\ \hline & & \end{array}\right)}\,\overline{\mathcal{C}_{\mathcal{B}}} = \overline{\left(\begin{array}{c|c|c} & & \\ \hline & J(\mathcal{B}_{jj}) & \\ \hline & & \end{array}\right)} \cdot \overline{\begin{pmatrix} 0_{n_1} & \mathcal{B}_{12} & 0_{n_1\times n_3} & \cdots & 0_{n_1\times n_q} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & 0_{n_{q-2}\times n_q} \\ & & & \ddots & \mathcal{B}_{q-1,q} \\ & & & & 0_{n_q} \end{pmatrix}}. \tag{6.6}$$

We consider two possibilities, namely $j < q$ or $j = q$, as we did above with the case $q = 2$.

Firstly, let $j < q$. Then, by (6.6),

$$\begin{aligned}\dim_K V_j &= \dim_K(J(\mathcal{B}_{jj})\mathcal{B}_{j,j+1}) \\ &= \dim_K\left(\begin{pmatrix} 0_{n_{j2}} & \mathrm{M}_{n_{j2}\times n_{j1}}(K) \\ 0_{n_{j1}\times n_{j2}} & 0_{n_{j1}} \end{pmatrix}\cdot \mathrm{M}_{n_j\times n_{j+1}}(K)\right) = n_{j2}n_{j+1}. \end{aligned} \tag{6.7}$$

Under an isomorphism of algebras, nilpotent elements are mapped to nilpotent elements, and so, invoking Lemma 6.1, we have the inclusion

$$\overline{\varphi}(V_j) = \overline{\varphi\left(\left(\begin{array}{c|c|c} & & \\ \hline & J(\mathcal{B}_{jj}) & \\ \hline & & \end{array}\right)\right)\overline{\mathcal{C}_{\mathcal{A}}}} \subseteq \overline{\left(\begin{array}{|cccc} \mathcal{A}_{1j} & \mathcal{A}_{1,j+1} & \dots & \mathcal{A}_{1q} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{A}_{j-1,j} & \mathcal{A}_{j-1,j+1} & \dots & \mathcal{A}_{j-1,q} \\ J(\mathcal{A}_{jj}) & \mathcal{A}_{j,j+1} & \dots & \mathcal{A}_{jq} \\ \hline & & & \end{array}\right)}\,\overline{\mathcal{C}_A}, \tag{6.8}$$

where

$$J(\mathcal{A}_{jj}) = J(C^1_{n_j}(K)) = \begin{pmatrix} 0_{n_{j1}} & \mathrm{M}_{n_{j1}\times n_{j2}}(K) \\ 0_{n_{j2}\times n_{j1}} & 0_{n_{j2}} \end{pmatrix}. \tag{6.9}$$

Since the diagonal blocks of $\mathcal{C}_{\mathcal{A}}$ are zero and the product of such elements in the quotient algebra $\mathcal{A}/\mathcal{C}^2_{\mathcal{A}}$ is zero, it follows from (6.9) that

$$\begin{aligned}\dim_K\left(\overline{\left(\begin{array}{|cccc} \mathcal{A}_{1j} & \mathcal{A}_{1,j+1} & \dots & \mathcal{A}_{1q} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{A}_{j-1,j} & \mathcal{A}_{j-1,j+1} & \dots & \mathcal{A}_{j-1,q} \\ J(\mathcal{A}_{jj}) & \mathcal{A}_{j,j+1} & \dots & \mathcal{A}_{jq} \\ \hline & & & \end{array}\right)}\,\overline{\mathcal{C}_A}\right) &= \dim_K(J(\mathcal{A}_{jj})\mathcal{A}_{j,j+1}) \\ = \dim_K\left(\begin{pmatrix} 0_{n_{j1}} & \mathrm{M}_{n_{j1}\times n_{j2}}(K) \\ 0_{n_{j2}\times n_{j1}} & 0_{n_{j2}} \end{pmatrix}\cdot \mathrm{M}_{n_j\times n_{j+1}}(K)\right) &= n_{j1}n_{j+1}. \end{aligned} \tag{6.10}$$

We have thus found (see (6.7)) that $\dim_K V_j = n_{j2}n_{j+1}$, and by (6.8) and (6.10), $\dim_K\overline{\varphi}(V_j) \le n_{j1}n_{j+1}$. However, these dimensions are equal, implying that $n_{j2} \le n_{j1}$. This is a contradiction, since $n_{j2} = n_{j1} + 1$.

Lastly, let $j = q$. Now we need another argument, because in this case, by (6.6), V_j is the zero space (and so $\dim_K V_j = 0 = \dim_K \overline{\varphi}(V_j)$). Instead, we will compare the dimensions of the spaces $\mathcal{C}^{q-1}_{\mathcal{A}}J(\mathcal{A})$ and $\mathcal{C}^{q-1}_{\mathcal{B}}J(\mathcal{B})$. Note that

$$\begin{pmatrix} 0_{n_1} & \mathcal{A}_{12} & \dots & \mathcal{A}_{1q} \\ & \ddots & \ddots & \vdots \\ & & 0_{q-1} & \mathcal{A}_{q-1,q} \\ & & & J(\mathcal{A}_{qq}) \end{pmatrix} \subseteq J(\mathcal{A}) \subseteq \begin{pmatrix} \mathcal{A}_{11} & \dots & \mathcal{A}_{1,q-1} & \mathcal{A}_{1q} \\ & \ddots & \vdots & \vdots \\ & & \mathcal{A}_{q-1,q-1} & \mathcal{A}_{q-1,q} \\ & & & J(\mathcal{A}_{qq}) \end{pmatrix}$$

and

$$\begin{pmatrix} 0_{n_1} & \mathcal{B}_{12} & \dots & \mathcal{B}_{1q} \\ & \ddots & \ddots & \vdots \\ & & 0_{q-1} & \mathcal{B}_{q-1,q} \\ & & & J(\mathcal{B}_{qq}) \end{pmatrix} \subseteq J(\mathcal{B}) \subseteq \begin{pmatrix} \mathcal{B}_{11} & \dots & \mathcal{B}_{1,q-1} & \mathcal{B}_{1q} \\ & \ddots & \vdots & \vdots \\ & & \mathcal{B}_{q-1,q-1} & \mathcal{B}_{q-1,q} \\ & & & J(\mathcal{B}_{qq}) \end{pmatrix},$$

where $J(\mathcal{A}_{qq})$ and $J(\mathcal{B}_{qq})$ are as in (6.9) and (6.5), respectively, with $j = q$. Then, by Proposition 4.2, it is not hard to show that $\dim_K(\mathcal{C}_{\mathcal{A}}^{q-1} J(\mathcal{A})) = n_1 n_{q2}$ and $\dim_K(\mathcal{C}_{\mathcal{B}}^{q-1} J(\mathcal{B})) = n_1 n_{q1}$. We conclude, as before, that $n_{q2} = n_{q1}$. This contradiction completes the proof. □

Theorem 6.6. *Let K be an algebraically closed field, and let $\mathcal{A}$ and $\mathcal{B}$ be D_q subalgebras of $\mathrm{M}_n(K)$ of types $(n_1, n_2, \dots, n_q)$ and $(\ell_1, \ell_2, \dots, \ell_q)$, respectively, with max-dim db's. Then $\mathcal{A}$ and $\mathcal{B}$ are isomorphic if and only if the q-tuples $(n_1, n_2, \dots, n_q)$ and $(\ell_1, \ell_2, \dots, \ell_q)$ are equal and the diagonal blocks $\mathcal{A}_{jj}$ and $\mathcal{B}_{jj}$ are pairwise conjugated for $j = 1, 2, \dots, q$. Moreover, for every j, $j = 1, 2, \dots, q$, there is exactly one k_j such that $\mathcal{A}_{jj}$ and $\mathcal{B}_{jj}$ are conjugated with $C_{n_j}^{k_j}(K)$.*

Proof. Firstly, assume that the q-tuples $(n_1, n_2, \dots, n_q)$ and $(\ell_1, \ell_2, \dots, \ell_q)$ are equal and that the diagonal blocks $\mathcal{A}_{jj}$ and $\mathcal{B}_{jj}$ are pairwise conjugated for $j = 1, 2, \dots, q$. It follows from Theorem 4.10 that $\mathcal{A}$ and $\mathcal{B}$ are conjugated, and hence isomorphic.

Conversely, assume that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic. By Remark 5.1, each diagonal block $\mathcal{A}_{jj}$ (respectively, $\mathcal{B}_{jj}$) is conjugated by a matrix $X_{jj} \in \mathrm{M}_{n_j}(K)$ (respectively, $Y_{jj} \in \mathrm{M}_{\ell_j}(K)$) with some subalgebra $C_{n_j}^{s_j}(K)$ (respectively, $C_{\ell_j}^{t_j}(K)$). We stress that, for the algebra $C_{n_j}^{s_j}(K)$, the number n_j is exactly the number appearing in the sequence $(n_1, n_2, \dots, n_q)$ which is the type of $\mathcal{A}$; similarly for the algebra $C_{\ell_j}^{t_j}(K)$. So, $X_{jj}^{-1}\mathcal{A}_{jj}X_{jj} = C_{n_j}^{s_j}(K)$ and $Y_{jj}^{-1}\mathcal{B}_{jj}Y_{jj} = C_{\ell_j}^{t_j}(K)$. Define the following block diagonal matrices:

$$X = \begin{pmatrix} X_{11} & & & \\ & X_{22} & & \\ & & \ddots & \\ & & & X_{qq} \end{pmatrix}, \quad Y = \begin{pmatrix} Y_{11} & & & \\ & Y_{22} & & \\ & & \ddots & \\ & & & Y_{qq} \end{pmatrix}.$$

It follows from Proposition 4.8 that $\mathcal{A}' = X^{-1}\mathcal{A}X$ and $\mathcal{B}' = Y^{-1}\mathcal{B}Y$ are D_q subalgebras of $\mathrm{M}_n(K)$ of types $(n_1, n_2, \dots, n_q)$ and $(\ell_1, \ell_2, \dots, \ell_q)$, respectively, with max-dim db's.

It is straightforward to check that the diagonal blocks of the algebras $\mathcal{A}'$ and $\mathcal{B}'$ are equal to $C_{n_j}^{s_j}(K)$ and $C_{\ell_j}^{t_j}(K)$, respectively. Since $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, so are $\mathcal{A}'$ and $\mathcal{B}'$. Therefore, by Theorem 6.2, the q-tuples $(n_1, n_2, \ldots, n_q)$ and $(\ell_1, \ell_2, \ldots, \ell_q)$ are equal and the diagonal blocks $C_{n_j}^{s_j}(K)$ and $C_{\ell_j}^{t_j}(K)$ are isomorphic for $j = 1, 2, \ldots, q$. By Remark 5.1 and the arguments preceding it, an isomorphism is possible only for equal blocks, or for pairs $C_{n_j}^1(K)$ and $C_{\ell_j}^2(K)$, or for pairs $C_{n_j}^2(K)$ and $C_{\ell_j}^1(K)$, with odd integer $n_j = \ell_j \geq 3$. However, in the case of at least one pair of distinct blocks, Lemma 6.5 can be applied, and so $\mathcal{A}'$ and $\mathcal{B}'$ are not isomorphic; a contradiction. Consequently, $C_{n_j}^{s_j}(K) = C_{\ell_j}^{t_j}(K)$ for $j = 1, 2, \ldots, q$, and so

$$X_{jj}^{-1}\mathcal{A}_{jj}X_{jj} = Y_{jj}^{-1}\mathcal{B}_{jj}Y_{jj},$$

from which we get $(X_{jj}Y_{jj}^{-1})^{-1}\mathcal{A}_{jj}X_{jj}Y_{jj}^{-1} = \mathcal{B}_{jj}$. Hence the diagonal blocks $\mathcal{A}_{jj}$ and $\mathcal{B}_{jj}$ are conjugated for $j = 1, 2, \ldots, q$, which completes the proof. □

Note that from Theorem 6.6 and Theorem 4.10 it follows that D_q subalgebras of $\mathrm{M}_n(K)$ of some types with max-dim db's over an algebraically closed field K are isomorphic if and only if these subalgebras are conjugated.

Remark 6.7. For algebraically closed fields, we have another tool which can help to even better describe a D_q subalgebra $\mathcal{A}$ of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \ldots, n_q)$ with max-dim db's, namely we can say that $\mathcal{A}$ is of type $\big((n_1, k_1), (n_2, k_2), \ldots, (n_q, k_q)\big)$ where for every j, $j = 1, 2, \ldots, q$, the number k_j appears as the superscript in $C_{n_j}^{k_j}(K)$. It should be clear that k_j depends on n_j, as follows:

$$k_j = \begin{cases} 1, & \text{if } n_j = 1; \\ 1 \text{ or } 2, & \text{if } n_j = 2; \\ 1 \text{ or } 2 \text{ or } 3 \text{ or } 4 \text{ or } 5, & \text{if } n_j = 3; \\ 1, & \text{if } n_j \geq 4 \text{ and } n_j \text{ is even}; \\ 1 \text{ or } 2, & \text{if } n_j \geq 5 \text{ and } n_j \text{ is odd}. \end{cases}$$

Moreover, the q-tuples of ordered pairs $\big((n_1, k_1), (n_2, k_2), \ldots, (n_q, k_q)\big)$ determine all D_q subalgebras of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \ldots, n_q)$ with max-dim db's up to conjugation (and isomorphism).

We have already shown that every D_q subalgebra of $\mathrm{U}_n(K)$ with maximum dimension is a max-dim D_q subalgebra of $\mathrm{M}_n(K)$ of some type $(n_1, n_2, \ldots, n_q)$ with max-dim db's (see Corollary 4.9). With the help of Remark 6.7 we can almost precisely say what the diagonal blocks of this subalgebra look like when the field K is algebraically closed.

Corollary 6.8. *Let K be an algebraically closed field, and let $\mathcal{A}$ be a D_q subalgebra of $U_n(K)$ with maximum dimension. Then $\mathcal{A}$ is a max-dim* D_q *subalgebra of* $\mathrm{M}_n(K)$ *of* *some type $(n_1, n_2, \ldots, n_q)$ with max-dim db's, such that in form (1.10) each diagonal block $\mathcal{A}_{ii}$, $i = 1, 2, \ldots, q$, satisfies one of the following conditions:*

(1) $\mathcal{A}_{ii}$ is equal to $C^1_{n_i}(K)$, with integer n_i greater than or equal to 1,
(2) $\mathcal{A}_{ii}$ is equal to $C^2_{n_i}(K)$, with odd integer n_i greater than or equal to 3,
(3) $\mathcal{A}_{ii}$ is conjugated with $C^2_2(K)$, $C^3_3(K)$, $C^4_3(K)$ or $C^5_3(K)$.

Proof. By Corollary 3.4 and Remark 6.7, there exists an invertible matrix $Y \in \mathrm{M}_n(K)$ such that $Y^{-1}\mathcal{A}Y$ is a max-dim D_q subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \ldots, n_q)$ with the $C^{t_i}_{n_i}(K)$'s as max-dim db's. By hypothesis, $\mathcal{A}$ is contained in $\mathrm{U}_n(K)$, and we have that $\mathcal{A} = Y(Y^{-1}\mathcal{A}Y)Y^{-1}$. By Proposition 4.8, $\mathcal{A}$ is a max-dim D_q subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \ldots, n_q)$ with max-dim db's conjugated with the $C^{t_i}_{n_i}(K)$'s. A conjugation of $C^{t_i}_{n_i}(K)$ is contained in $\mathrm{U}_{n_i}(K)$, because $\mathcal{A} \subseteq \mathrm{U}_n(K)$, and so the conditions in the statement of the corollary follow from Proposition 5.2. □

7. max-dim $\mathbf{D}_q$ subalgebras of $\mathbf{M}_n(K)$ with max-dim db's

In this section, we will describe the q-tuples $(n_1, n_2, \ldots, n_q)$ such that $\mathcal{A}$ is a max-dim D_q subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \ldots, n_q)$ with max-dim db's, and we will provide examples illustrating our study.

All the results in this section are based on the description in [19, pages 251-253], including [19, Lemma 12 and Lemma 13], where it was shown that if n_1, n_2, …, n_q are positive integers such that $n_1 + n_2 + \cdots + n_q = n$, then a D_q subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \ldots, n_q)$ with max-dim db's has maximum dimension if and only if, for all i and j,

$$|n_i - n_j| = \begin{cases} 0 \text{ or } 2, & \text{if both } n_i \text{ and } n_j \text{ are even;} \\ 0 \text{ or } 1, & \text{otherwise.} \end{cases} \tag{7.1}$$

We recall and reformulate slightly the mentioned results in [19] by starting in Proposition 7.1 with a description of max-dim D_2 subalgebras of $\mathrm{M}_n(K)$ with max-dim db's, which follows directly from (7.1).

Proposition 7.1. *Let n_1 and n_2 be positive integers such that $n_1 + n_2 = n$, and consider a D_2 subalgebra of $M_n(K)$ of type (n_1, n_2) with max-dim db's. Then $\mathcal{A}$ is a D_2 subalgebra of $M_n(K)$ of maximum dimension if and only if one of the following possibilities occurs:*

a) *n is odd, and $(n_1, n_2) = (\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1)$ or $(\lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor)$.*
b) *$4|n$ and $(n_1, n_2) = (\frac{n}{2}, \frac{n}{2})$.*
c) *$n = 2$ and $(n_1, n_2) = (1, 1)$.*
d) *$n \geq 6$, $n \equiv 2 \pmod 4$ and $(n_1, n_2) = (\frac{n}{2}, \frac{n}{2})$ or $(\frac{n}{2} - 1, \frac{n}{2} + 1)$ or $(\frac{n}{2} + 1, \frac{n}{2} - 1)$.*

Invoking Proposition 7.1 and the examples of commutative subalgebras of $\mathrm{M}_n(K)$ with maximum dimensions given in Section 5 (see (5.1) – (5.11)), we obtain the following examples of D_2 subalgebras of $\mathrm{M}_5(K)$ and $\mathrm{M}_6(K)$ with maximum dimension.

Example 7.2. The max-dim D_2 subalgebras of $\mathrm{M}_5(K)$ with max-dim db's are of types $(2,3)$ or $(3,2)$, for example,

$$K(e_{11}+e_{22})+K(e_{33}+e_{44}+e_{55})+\left(\begin{array}{cc|ccc} 0 & K & K & K & K \\ & 0 & K & K & K \\ \hline & & 0 & K & K \\ & & & 0 & 0 \\ & & & & 0 \end{array}\right),$$

$$K(e_{11}+e_{22}+e_{33})+K(e_{44}+e_{55})+\left(\begin{array}{ccc|cc} 0 & K & K & K & K \\ & 0 & 0 & K & K \\ & & 0 & K & K \\ \hline & & & 0 & K \\ & & & & 0 \end{array}\right).$$

In such a way we can construct 20 different max-dim D_2 subalgebras of $\mathrm{M}_5(K)$ of types $(2,3)$ or $(3,2)$ with max-dim db's equal to $C_2^i(K)$ or $C_3^j(K)$, with $i \in \{1,2\}$ and $j \in \{1,2,3,4,5\}$. By Theorem 6.2 and Lemma 6.5, these 20 different max-dim D_2 subalgebras of $\mathrm{M}_5(K)$ of types $(2,3)$ or $(3,2)$ with max-dim db's are pairwise non-isomorphic. If K is algebraically closed, then by Corollary 3.4 and Theorem 6.6, any D_2 subalgebra of $\mathrm{M}_5(K)$ with maximum dimension is conjugated with exactly one of them.

Similarly, the max-dim D_2 subalgebras of $\mathrm{M}_6(K)$ with max-dim db's are of types $(2,4)$, $(3,3)$ or $(4,2)$. There are 29 such pairwise non-isomorphic subalgebras with diagonal blocks equal to $C_2^i(K)$ or $C_3^j(K)$, with $i \in \{1,2\}$ and $j \in \{1,2,3,4,5\}$, or $C_4^1(K)$.

The procedure of determining q-tuples $(n_1,n_2,\ldots,n_q)$, such that $n_i \le n_j$ for all $i<j$ and such that a D_q subalgebra of $\mathrm{M}_n(K)$ of type $(n_1,n_2,\ldots,n_q)$ with max-dim db's is a D_q subalgebra of $\mathrm{M}_n(K)$ with maximum dimension was discussed in Remark 15 of [19]. It starts with determining numbers satisfying $|n_i-n_j| \le 1$ (for $s=0$ and $t=0$ in the theorem below). Next, in some situations we can add 1 to some of the numbers in the q-tuple $(n_1,n_2,\ldots,n_q)$ and simultaneously subtract 1 from some of the others numbers in such a way that condition (7.1) is satisfied.

The introductory paragraphs in Section 1 show that a max-dim D_q subalgebra of $\mathrm{M}_n(K)$ of some type $(n_1,n_2,\ldots,n_q)$ with max-dim db's has dimension equal to the expression in (1.6). It follows that a permutation which changes the (ordered) q-tuple $(n_1,n_2,\ldots,n_q)$ give rise to a max-dim D_q subalgebra of $\mathrm{M}_n(K)$ of another type with max-dim db's. Combining this observation with [19, Remark 15] we get the following:

Theorem 7.3. *Let $\mathcal{A}$ be a D_q subalgebra of $\mathrm{M}_n(K)$ of type $(n_1, n_2, \ldots, n_q)$ with max-dim db's. Write $n = q\left\lfloor \frac{n}{q} \right\rfloor + r$, where r is the non-negative integer less than q in the Division Algorithm. Then $\mathcal{A}$ is a D_q subalgebra of $M_n(K)$ with maximum dimension if and only if there exists a permutation $\sigma \in S_q$ such that $n_{\sigma(i)} \leq n_{\sigma(j)}$ for all $i < j$ and one of two possibilities occurs:*

a) $\left\lfloor \frac{n}{q} \right\rfloor$ *is even and there exists a non-negative integer s, with $s \leq \left\lfloor \frac{r}{2} \right\rfloor$, such that*

$$n_{\sigma(i)} = \begin{cases} \left\lfloor \frac{n}{q} \right\rfloor, & \text{if } 1 \leq i \leq q - r + s; \\ \left\lfloor \frac{n}{q} \right\rfloor + 1, & \text{if } q - r + s < i \leq q - s; \\ \left\lfloor \frac{n}{q} \right\rfloor + 2, & \text{if } q - s < i \leq q. \end{cases}$$

b) $\left\lfloor \frac{n}{q} \right\rfloor$ *is odd and there exists a non-negative integer t, with $t \leq \left\lfloor \frac{q-r}{2} \right\rfloor$, such that*

$$n_{\sigma(i)} = \begin{cases} \left\lfloor \frac{n}{q} \right\rfloor - 1, & \text{if } 1 \leq i \leq t; \\ \left\lfloor \frac{n}{q} \right\rfloor, & \text{if } t < i \leq q - r - t; \\ \left\lfloor \frac{n}{q} \right\rfloor + 1, & \text{if } q - r - t < i \leq q. \end{cases}$$

We conclude with two examples illustration the two parts of Theorem 7.3.

Example 7.4. We will find all 5-tuples $(n_1, n_2, n_3, n_4, n_5)$ with $n_1 \leq n_2 \leq n_3 \leq n_4 \leq n_5$ resulting in a max-dim D_5 subalgebra of $\mathrm{M}_{14}(K)$ of type $(n_1, n_2, n_3, n_4, n_5)$ with max-dim db's.

Using the notation in Theorem 7.3, we have $n = 14$, $q = 5$, $\left\lfloor \frac{n}{q} \right\rfloor = 2$ and $r = 4$. Then, for $s = 0, 1, 2$, the 5-tuples $(n_1, n_2, n_3, n_4, n_5)$ are respectively equal to

$$(2, 3, 3, 3, 3), \quad (2, 2, 3, 3, 4) \quad \text{and} \quad (2, 2, 2, 4, 4).$$

If we do not assume that the n_i's appear in increasing order, then we can permute them, which leads to 4, $\frac{5!}{2! \cdot 2!} - 1 = 29$ and $\frac{5!}{3! \cdot 2!} - 1 = 9$ more possibilities, respectively.

Similarly, we will find all the 7-tuples $(n_1, n_2, \ldots, n_7)$ with $n_1 \leq n_2 \leq \ldots \leq n_7$ resulting in a max-dim D_7 subalgebra of $\mathrm{M}_{22}(K)$ of type $(n_1, n_2, \ldots, n_7)$ with max-dim db's. Again, using the notation from in Theorem 7.3, we have $n = 22$, $q = 7$, $\left\lfloor \frac{n}{q} \right\rfloor = 3$, $r = 1$. So, for $t = 0, 1, 2, 3$, the 7-tuples $(n_1, n_2, n_3, n_4, n_5, n_6, n_7)$ are respectively equal to

$$(3, 3, 3, 3, 3, 3, 4),\ (2, 3, 3, 3, 3, 4, 4),\ (2, 2, 3, 3, 4, 4, 4) \text{ and } (2, 2, 2, 4, 4, 4, 4).$$

In this case we have 6, $\frac{7!}{4!\cdot 2!} - 1 = 104$, $\frac{7!}{2!\cdot 2!\cdot 3!} - 1 = 209$ and $\frac{7!}{3!\cdot 4!} - 1 = 34$, respectively, more sequences if we don't assume that the n_i's appear in increasing order.

Data availability

No data was used for the research described in the article.

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