Journal of Algebra 564 (2020) 247–275

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Peirce decompositions, idempotents and rings

Pham N. Ánh [a], Gary F. Birkenmeier [b,*], Leon van Wyk [c]

[a] Rényi Institute of Mathematics, Hungarian Academy of Sciences, 1364 Budapest, Pf. 127, Hungary
[b] Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA 70504-1010, USA
[c] Department of Mathematical Sciences, Stellenbosch University, P/Bag X1, Matieland 7602, Stellenbosch, South Africa

ARTICLE INFO

Article history:
Received 26 April 2018
Available online 11 August 2020
Communicated by Louis Rowen

MSC:
16S50
15A33
16D20
16D70

Keywords:
Idempotent
Peirce decomposition
Peirce trivial
n-Peirce ring
Generalized matrix ring
Morita context
J-trivial
B-trivial

ABSTRACT

Idempotents dominate the structure theory of rings. The Peirce decomposition induced by an idempotent provides a natural environment for defining and classifying new types of rings. This point of view offers a way to unify and to expand the classical theory of semiperfect rings and idempotents to much larger classes of rings. Examples and applications are included.

* Corresponding author.
E-mail addresses: anh@renyi.hu (P.N. Ánh), gfb1127@louisiana.edu (G.F. Birkenmeier), LvW@sun.ac.za (L. van Wyk).

https://doi.org/10.1016/j.jalgebra.2020.08.003
0021-8693/

1. Introduction

Since the coordinatization of projective and continuous geometries (see [18]), it is well-known that idempotents induce direct sum decompositions of regular representations which determine a structure of rings, provided that the rings have enough idempotents. This wealth of idempotents can be ensured if rings are proper matrix rings, i.e., they are n-by-n for $n > 1$. An idempotent $e = e^2$ in a ring R not necessarily with unity induces the (two-sided) Peirce decomposition

$$R = eRe \oplus eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e),$$

or more transparently, e induces on R the generalized matrix ring structure

$$R = \begin{bmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{bmatrix},$$

with the obvious matrix addition and multiplication. Here eRe $(= \{ere \mid r \in R\})$, $eR(1-e)$, $(1-e)Re$ and $(1-e)R(1-e)$ are abelian subgroups of R, where the abbreviated notation $eR(1-e)$ stands formally for the set $\{e(r-re) = er-ere \mid r \in R\}$; and similarly, $(1-e)Re = \{re-ere \mid r \in R\}$, $(1-e)R(1-e) = \{r-er-re+ere \mid r \in R\}$. Henceforth, there are generally two ways to treat idempotents concerning their structural influence. The first is an internal way given by the classical Peirce decompositions; the second way is an external one provided by generalized (or formal) matrix rings. It is well known (e.g., see [1]) that with each Peirce decomposition, we can associate a generalized matrix ring; and with each generalized matrix ring, we can associate a Peirce decomposition. Observe that a Morita context is a 2-by-2 generalized matrix ring. Recall that a Morita context is a quadruple $(A, B, {}_AM_B, {}_BN_A)$ of rings A and B and bimodules ${}_AM_B$ and ${}_BN_A$, together with (A, A)- and (B, B)-bimodule homomorphisms

$$(-,-) : M \underset{B}{\otimes} N \longrightarrow {}_A A_A, \qquad [-,-] : N \underset{A}{\otimes} M \longrightarrow {}_B B_B,$$

satisfying the conditions

$$(m,n)m_1 = m[n,m_1] \qquad \text{and} \qquad [n,m]n_1 = n(m,n_1)$$

of associativity for all $m, m_1 \in M$ and all $n, n_1 \in N$. It is not necessary to require A and B to be unital rings and M and N to be unitary bimodules. Consequently, every Morita context provides a generalized matrix ring

$$R = \begin{bmatrix} A & M \\ N & B \end{bmatrix},$$

endowed with the usual matrix addition and multiplication by using bimodule homomorphisms $(-,-)$ and $[-,-]$; and vice versa in the case when at least one of A and B is unital, by putting $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ or $e = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, one obtains a Peirce decomposition.

Generalized matrix rings, in particular Morita contexts, provide an efficient way to obtain rings with prescribed idempotents of a certain type. Then, using the prescribed idempotents to obtain Peirce decompositions, one can obtain further information about the rings. For example, a ring with unity is a 2-by-2 generalized upper triangular matrix ring if and only if it has a left semicentral idempotent which is neither 0 nor 1. Moreover, the Peirce decomposition may provide a means to unify a class of generalized matrix rings. For example, renumbering pairwise orthogonal idempotents leads to formally different generalized matrix rings which can be transformed from one to another by appropriate interchanging of rows and columns, respectively. However, the associated Peirce decomposition is the same, because addition is commutative.

The associativity condition imposed on Morita contexts is satisfied trivially if the considered bilinear products are trivial, i.e., zero. This naturally suggests the notion of Peirce idempotents. An idempotent $e = e^2 \in R$ is called *Peirce trivial* if $eR(1-e)Re = 0 = (1-e)ReR(1-e)$ (see [1]). By defining the class of rings which are indecomposable relative to the Peirce trivial concept (i.e., rings in which 0 and 1 are the only Peirce trivial idempotents) one obtains building blocks for a new decomposition theory (see Definition 2.1). We refer to Peirce's original paper [14] for decompositions induced by idempotents. Other aspects and related properties of matrix and generalized matrix rings can be found also in [1], [7], [8], [11], [12] and [17].

In this paper we devote our attention to the investigation of n-Peirce rings. In contrast to our other work in [1], in this article we give a coordinatization-free treatment, i.e., we look for results which are independent of particular generalized matrix ring representations. In Section 2, the main result shows that one can develop a structure theory of Peirce rings similar to that of Bass for semiperfect rings. Thorough discussions on conditions weakening Peirce trivial idempotents can be found in [1]. In Section 3, following the program suggested by Jacobson's classic (see [10]), we define so-called trivial idempotents relative to certain radicals, like **J**-trivial and **B**-trivial idempotents; and we sketch the process of how to lift results on semisimple factors to the rings. This is closely related to the classical theory of lifting idempotents modulo radicals. Applications of our theory are developed in the last section. In particular, we show that a variety of well known and useful conditions produce an n-Peirce ring with a generalized matrix representation whose diagonal rings are 1-Peirce rings which satisfy the respective condition. Moreover, we provide many well known classes of rings for which an n-Peirce ring has a generalized matrix representation in which each diagonal ring is 1-Peirce and in the respective class.

A word about notation and convention: in the rest of this paper all rings are unital and all modules are unitary. When a ring R with an idempotent $e^2 = e \in R$ is viewed as a generalized matrix ring $R = \begin{bmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{bmatrix}$, then the identity element of the rings $A = eRe$ and $B = (1-e)R(1-e)$ is 1 by convention, not e or $1-e$, respectively. This convention will simplify and make routine calculations transparent. We consider an element $r = 1 \cdot r \cdot 1 = [e + (1-e)]r[e + (1-e)]$ both as a sum $r = ere + er(1-e) + (1-e)re + (1-e)r(1-e)$ and as a formal matrix $r = \begin{bmatrix} ere & er(1-e) \\ (1-e)re & (1-e)r(1-e) \end{bmatrix}$.

2. General structure theory

For the sake of self-containment we provide the following definition (see [1]).

Definition 2.1. An idempotent $e = e^2$ in a ring R is called *inner Peirce trivial* if $eR(1-e)Re = 0$. Dually, e is *outer Peirce trivial* if $1-e$ is an inner Peirce trivial. An idempotent e is *Peirce trivial* if it is both inner and outer Peirce trivial. The set of Peirce trivial idempotents of R is denoted by $\mathfrak{P}_{\mathrm{t}}(R)$. A ring R is a 0-*Peirce ring* if it has only one element $1 = 0$, and R is called a *Peirce ring*, or more precisely, a 1-*Peirce ring* if $\mathfrak{P}_{\mathrm{t}}(R) = \{0, 1\}$, with $1 \neq 0$. Inductively, for a natural number $n > 1$, a ring R is called an *n-Peirce ring* if there is an $e \in \mathfrak{P}_{\mathrm{t}}(R)$ such that eRe is an m-Peirce ring for some m, $1 \leq m < n$, and $(1-e)R(1-e)$ is an $(n-m)$-Peirce ring. An idempotent $e \in R$ is called an *n-Peirce idempotent* if eRe is an n-Peirce ring. In particular, $e = e^2 \in R$ is called a *1-Peirce idempotent* if eRe is a 1-Peirce ring. Henceforth, for every natural number n, we denote the class of n-Peirce rings by $\mathbf{P}_n$.

Recall from [1] that $\mathbf{P}_n$ is a proper subclass of $\mathcal{T}_n$. Thus when $R \in \mathbf{P}_n$ is represented in generalized matrix form and $[a_{ij}], [b_{ij}] \in R$ with $[c_{ij}] = [a_{ij}][b_{ij}]$, then $c_{ii} = a_{ii}b_{ii}$, for all i and j (i.e., the diagonal entries of the product of two matrices equals the product of the corresponding diagonal entries of the factor matrices).

Remark 2.2. Since all central idempotents are Peirce trivial, every $R \in \mathbf{P}_1$ is indecomposable as a ring. In particular, if a ring R is semiprime or Abelian then both inner and outer Peirce trivial idempotents in R are central; and such a ring R is in $\mathbf{P}_1$ if and only if R is indecomposable as a ring. Recall that a ring is *Abelian* if its idempotents are central. Peirce trivial idempotents generalize the notion of semicentral idempotents which occur naturally in the structure of 2-by-2 generalized triangular matrix rings. For a natural number n, n-Peirce rings are generalizations of n-strongly triangular matrix rings (see [3]), or in another terminology, rings with a complete set of triangulating idempotents (see [4]). For a thorough and subtle analysis of inner and outer Peirce idempotents, see [1]. It is also worth noting that for an idempotent $e^2 = e \in R$ the set $e + eR(1-e)$ is characterized in [18, Part II, Chapter II, Lemma 2.7] as the set of idempotents $f^2 = f \in R$ such that e and f generate the same right ideal.

The following characterization (which is related to [1, Corollary 3.6]) of Peirce trivial idempotents is obvious in view of Definition 2.1.

Proposition 2.3. *Let $e = e^2 \in R$ and $I = eR(1-e) + (1-e)Re$. Then $e \in \mathfrak{P}_t(R)$ if and only if I is an ideal of R.*

Direct matrix computations (see [2] and [3]) yield the following:

Proposition 2.4. *Let $e \in \mathfrak{P}_t(R)$, and put $A = eRe,\ B = (1-e)R(1-e),\ M = eR(1-e)$ and $N = (1-e)Re$. For arbitrary elements $m \in M,\ n \in N$ the element $f = \begin{bmatrix} 1 & m \\ n & 0 \end{bmatrix}$ is an idempotent, the rings A and $fRf = \left\{ \begin{bmatrix} a & am \\ na & 0 \end{bmatrix} : a \in A \right\}$ are isomorphic under the map φ, sending $a \in A$ to $\varphi(a) = \begin{bmatrix} a & am \\ na & 0 \end{bmatrix}$, and B and $(1-f)R(1-f) = \left\{ \begin{bmatrix} 0 & -mb \\ -bn & 1 \end{bmatrix} : b \in B \right\}$ are isomorphic under the map ϱ, sending $b \in B$ to $\varrho(b) = \begin{bmatrix} 0 & -mb \\ -bn & b \end{bmatrix}$. Also, the modules ${}_RRe$ and ${}_RRf$ are isomorphic by sending $e \mapsto ef$ and $f \mapsto fe$. Moreover, $M = fR(1-f),\ N = (1-f)Rf$ and the identity maps on M and N are (φ, ϱ)-bimodule isomorphisms, i.e., for any $a \in eRe,\ b \in (1-e)R(1-e),\ x \in M$ and $y \in N$ one has $axb = \varphi(a)x\varrho(b)$ and $bya = \varrho(b)y\varphi(a)$. Consequently, f is also in $\mathfrak{P}_t(R)$.*

Simple formal calculations with matrices also show the following result.

Lemma 2.5. *If $e \in \mathfrak{P}_t(R)$ and $g \in \mathfrak{P}_t(eRe)$, then for any $m \in eR(1-e)$ and $n \in (1-e)Re$:*

(1) the element $h = \begin{bmatrix} g & gm \\ ng & 0 \end{bmatrix}$ is an inner Peirce trivial idempotent in R;
(2) the rings gRg and hRh are isomorphic;
(3) the modules ${}_RRg$ and ${}_RRh$ are isomorphic.

Remark 2.6. It can be seen from [1, Example 3.9] that if $e \in \mathfrak{P}_t(R)$ and $g \in \mathfrak{P}_t(eRe)$, then g need not be in $\mathfrak{P}_t(R)$; but g is inner Peirce trivial in R (see [1, Lemma 3.8(1)]). Observe that in general a product of two Peirce trivial idempotents is not even an idempotent; to wit, let R be the 2-by-2 upper triangular matrix ring over a ring A. Then $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ is a product of Peirce trivial idempotents which is not an idempotent.

The following result provides basic properties of Peirce trivial idempotents.

Proposition 2.7. *Let R be a ring, and let $e \in \mathfrak{P}_{\mathrm{t}}(R)$. Then any idempotent $f \in R = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$, where $A = eRe$, $B = (1-e)R(1-e)$, $M = eR(1-e)$ and $N = (1-e)Re$, can be written as a sum of two orthogonal idempotents $\alpha = \begin{bmatrix} g & gm \\ ng & 0 \end{bmatrix}$ and $\beta = \begin{bmatrix} 0 & mh \\ hn & h \end{bmatrix}$ ($f = \alpha + \beta$ and $\alpha\beta = \beta\alpha = 0$) for appropriate $g^2 = g \in A$, $h^2 = h \in B$, $m \in M$ and $n \in N$. Furthermore,*

(1) $\alpha, \beta \in \mathfrak{P}_{\mathrm{t}}(fRf)$;
(2) the modules ${}_RRf$ and ${}_RRf_e$, where $f_e = g + h$, are isomorphic;
(3) $f \in \mathfrak{P}_{\mathrm{t}}(R)$ if and only if $f_e \in \mathfrak{P}_{\mathrm{t}}(R)$.

Moreover, if $f \in \mathfrak{P}_{\mathrm{t}}(R)$, then $g \in \mathfrak{P}_{\mathrm{t}}(A)$ and $h \in \mathfrak{P}_{\mathrm{t}}(B)$. They are inner Peirce trivial idempotents of R, but not necessarily outer Peirce trivial idempotents of R. The same is true for both α and β, i.e., they are inner Peirce trivial idempotents of R.

Proof. Since f can be written uniquely as the generalized matrix $f = \begin{bmatrix} g & m \\ n & h \end{bmatrix}$ for uniquely determined elements $g \in A$, $h \in B$, $m \in M$ and $n \in N$, the equality $f^2 = f$ implies that

$$g^2 = g, \quad h^2 = h, \quad m = gm + mh \ \text{ and } \ n = ng + hn,$$

which in turn implies that

$$gmh = 0 \ \text{ and } \ hng = 0.$$

Let

$$\alpha = \begin{bmatrix} g & gm \\ ng & 0 \end{bmatrix} \ \text{ and } \ \beta = \begin{bmatrix} 0 & mh \\ hn & h \end{bmatrix}.$$

Then one can verify directly that $\alpha = \alpha^2$, $\beta = \beta^2$, $f = \alpha + \beta$ and $\alpha\beta = \beta\alpha = 0$.

To see that $\alpha, \beta \in \mathfrak{P}_{\mathrm{t}}(fRf)$, one has to verify that $\alpha fRf\beta fRf\alpha = \alpha R\beta R\alpha = 0 = \beta fRf\alpha fRf\beta = \beta R\alpha R\beta$, which is obvious by observing the inclusions $\alpha R\beta \subseteq M$ and $\beta R\alpha \subseteq N$. The modules ${}_RRf$ and ${}_RRf_e$ are isomorphic by the equalities $f = ff_ef$ and $f_e = f_eff_e$. Since $f = f_e + (gm + hn) + (mh + ng)$, $gm + hn \in f_eR(1 - f_e)$, $mh + ng \in (1-f_e)Rf_e$, $f_e = f-(gm+hn)-(mh+ng)$, $gm+hn \in fR(1-f)$ and $mh+ng \in (1-f)Rf$, it follows immediately in view of Proposition 2.4 that $f \in \mathfrak{P}_{\mathrm{t}}(R)$ if and only if $f_e \in \mathfrak{P}_{\mathrm{t}}(R)$.

Assume now in addition that $f \in \mathfrak{P}_{\mathrm{t}}(R)$. The idempotent e is now the identity 1_A of A, i.e., $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and similarly $1 - e = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity 1_B of B. The equality

$0 = fR(1-f)Rf = (\alpha+\beta)R(1-f)R(\alpha+\beta)$ implies that $0 = \alpha eRe(1-f)eRef = gA(e-g)Afe = aA(e-g)Ag$, whence g is an inner Peirce trivial idempotent of $A = eRe$. On the other hand, the equality $0 = (1-f)RfR(1-f)$ shows that $0 = (1-f)eRefeRe(1-f) = (1-f)eAgAe(1-f)$, from which $0 = e(1-f)eAgAe(1-f)e = (e-g)AgA(e-g)$ follows. Therefore, g is also an outer Peirce trivial idempotent of A. Consequently, $g \in \mathfrak{P}_t(A)$. By symmetry, $h \in \mathfrak{P}_t(B)$, with $B = (1-e)R(1-e)$. The remaining assertions are now simply consequences of Lemma 2.5. $\square$

As an obvious consequence of Proposition 2.7 and Remark 2.6, routine matrix multiplication shows that:

Corollary 2.8. *In the notation of Proposition 2.7, the products efe and $(1-e)f(1-e)$ of an $e \in \mathfrak{P}_t(R)$ and an idempotent $f \in R$ are the idempotents g and h of R, respectively. Moreover, in the case of a Peirce trivial idempotent f, the idempotents g and h are inner Peirce trivial, but not necessarily outer Peirce trivial idempotents of R.*

To justify Definition 2.1, one has to show that n is an invariant of an n-Peirce ring, i.e., n does not depend on the choice of elements of $\mathfrak{P}_t(R)$. This fact is shown in the following result.

Theorem 2.9. *Let $R \in \mathbf{P}_n$, and let $f \in \mathfrak{P}_t(R)$. Then $fRf \in \mathbf{P}_k$ for some $k \leq n$, and $(1-f)R(1-f) \in \mathbf{P}_{n-k}$.*

Proof. We use induction. The case $n = 1$ is obvious from Definition 2.1. Assume now that $n > 1$ and that the theorem is true for all $m < n$. Consider an $R \in \mathbf{P}_n$ defined by an $e \in \mathfrak{P}_t(R)$ such that $eRe \in \mathbf{P}_m$ $(1 \leq m < n)$ and $(1-e)R(1-e) \in \mathbf{P}_{n-m}$. For simplifying calculations put $A = eRe,\ M = eR(1-e),\ N = (1-e)Re$ and $B = (1-e)R(1-e)$, and write the elements of R as generalized matrices $r = \begin{bmatrix} a & m \\ n & b \end{bmatrix}$. Therefore, if f is an arbitrary element in $\mathfrak{P}_t(R)$, then in view of Proposition 2.7, for the unique generalized matrix representation $f = \begin{bmatrix} g & m \\ n & h \end{bmatrix}$, with uniquely determined elements $g \in A,\ h \in B,\ m \in M$ and $n \in N$, by putting $\alpha = \begin{bmatrix} g & gm \\ ng & 0 \end{bmatrix}$ and $\beta = \begin{bmatrix} 0 & mh \\ hn & h \end{bmatrix}$, one has that $f = \alpha + \beta,\ \alpha\beta = \beta\alpha = 0,\ \alpha, \beta \in \mathfrak{P}_t(fRf),\ g \in \mathfrak{P}_t(A)$ and $h \in \mathfrak{P}_t(B)$. Without loss of generality, we may assume that $f \neq 0, 1$.

By the induction hypothesis applied to both A and B, we have that $gAg \in \mathbf{P}_p$ and $hBh \in \mathbf{P}_q$ for some $p,\ 0 \leq p \leq m$, and some $q,\ 0 \leq q \leq n-m$, such that at least one of the two inequalities is proper by the extra assumption on f. For the sake of simplicity, by putting $t = gm = gt,\ u = ng = ug,\ v = mh = vh$ and $w = hn = hw$, one has

$$\alpha = \begin{bmatrix} g & t \\ u & 0 \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} 0 & v \\ w & g \end{bmatrix}.$$

Routine matrix calculations show that

$$\alpha f R f \alpha = \alpha R \alpha = \left\{ \begin{bmatrix} gag & gat \\ uag & 0 \end{bmatrix} : \ gag \in gAg \right\}$$

and

$$(f-\alpha) f R f (f-\alpha) = \beta R \beta = \left\{ \begin{bmatrix} 0 & vbh \\ hbw & hbh \end{bmatrix} : \ hbh \in hBh \right\},$$

whence $\alpha R \alpha$ and $\beta R \beta$ are isomorphic to gAg and hBh via the maps

$$\begin{bmatrix} gag & gat \\ uag & 0 \end{bmatrix} \longmapsto gag \in gAg$$

and

$$\begin{bmatrix} 0 & vbh \\ hbw & hbh \end{bmatrix} \longmapsto hbh \in hBh,$$

respectively. Consequently $fRf \in \mathbf{P}_{p+q}$. Moreover, the left R-modules ${}_RRg$ and ${}_RR\alpha$ are isomorphic, as are ${}_RRh$ and ${}_RR\beta$. By the same manner and by the induction hypothesis we have also that $(1-f)R(1-f) \in \mathbf{P}_{n-p-q}$, completing the proof. □

Since an idempotent is either Peirce trivial or not Peirce trivial, Theorem 2.9 suggests the following dichotomy.

Definition 2.10. A ring R has *Peirce dimension* 0 if it has only one element $1 = 0$, and R has *Peirce dimension* n $(n > 0)$ if $R \in \mathbf{P}_n$. All other rings are said to have *infinite Peirce dimension.*

As an obvious consequence of Definition 2.10 and Theorem 2.9 we have:

Corollary 2.11. *The Peirce dimension is additive, i.e., the Peirce dimension of a finite direct sum of rings is the sum of the Peirce dimensions of the direct summands. In particular, if* $e \in \mathfrak{P}_{\mathrm{t}}(R)$*, then the Peirce dimension of* R *is the sum of the Peirce dimensions of* eRe *and* $(1-e)R(1-e)$*.*

The following consequence deals with rings of infinite Peirce dimension.

Corollary 2.12. *A ring R has infinite Peirce dimension if and only if there is an infinite set of nonzero pairwise distinct idempotents $e_0 = 1, e_1, \dots, e_n, \dots$ such that e_{i+1} is a 1-Peirce idempotent of e_iRe_i for every $i = 0, 1, 2, \dots$.*

The class $\mathbf{P}_1$ (of 1-Peirce rings) covers rings with a variety of different properties. It contains, for example, all prime rings and rings with only the two idempotents 1 and 0. Furthermore, all matrix rings over local rings are also in $\mathbf{P}_1$, as is easily verified. The problem of characterizing or describing the class $\mathbf{P}_1$ seems to be quite interesting. Since $\mathbf{P}_n$ is even larger, one needs additional invariants to get a closer look at them.

The following invariant is an obvious consequence of Definition 2.1.

Definition 2.13. Let I be a finite nonempty set. A *partition of I* is a finite set of nonempty pairwise disjoint subsets whose union is I. A *dyadic partition* of I is a partition into two disjoint subsets. A partition λ is called a *dyadic refinement* of a partition γ if all elements of γ, with one exception, are elements of λ, and the exceptional element is a union of two elements of λ. A set $\Lambda = \{\lambda_0 = \{I\}, \lambda_1, \lambda_2, \cdots, \lambda_k\}$ of partitions λ_i is called a *complete dyadic set of partitions* if λ_{i+1} is a dyadic refinement of λ_i for all $i = 0, \dots, k-1$ and all elements of λ_k are singletons. Therefore $k = n - 1$ if I has n elements.

For a subset I of $\{1, 2, \dots, n\}$ and a set $\{e_1, e_2, \dots, e_n\}$ of idempotents in a ring R the sum $\sum_{i\in I} e_i$ is denoted by e_I.

The following important characterization of n-Peirce rings is an easy consequence of Definitions 2.1 and 2.13.

Corollary 2.14. *A ring R is in $\mathbf{P}_n$ if and only if there are n pairwise orthogonal 1-Peirce idempotents $e_1, \dots, e_n$ whose sum is 1, and a complete dyadic set $\Lambda = \Big\{\lambda_0 = \big\{\{1, 2, \dots, n\}\big\}, \lambda_1, \lambda_2, \cdots, \lambda_k\Big\}$ of partitions of $\{1, 2, \dots, n\}$ such that for an exceptional element I of λ_i, $i = 0, \dots k-1$, which is a union of two elements J and L of λ_{i+1}, $e_J \in \mathfrak{P}_{\mathrm{t}}(e_IRe_I)$.*

Proof. The sufficiency is obvious. For the necessity we use induction on n. The claim is obvious for $n = 2, 3$. Let $n > 3$ and assume that the claim is true for all $m < n$. By Definition 2.1 there is an $E \in \mathfrak{P}_{\mathrm{t}}(R)$ such that $ERE \in \mathbf{P}_{n_1}$ and $FRF \in \mathbf{P}_{n_2}$, with $F = 1 - E$ and $n_1 + n_2 = n$, $n_1n_2 \neq 0$. Therefore by the induction hypothesis there are pairwise orthogonal 1-Peirce idempotents $e_1, \dots, e_{n_1}$ in ERE and $f_1, \dots, f_{n_2}$ in FRF together with complete dyadic sets $\Lambda_1 = \{\alpha_0, \dots, \alpha_{n_1-1}\}$ and $\Lambda_2 = \{\delta_0, \dots, \delta_{n_2-1}\}$ of partitions of $\{1, \dots, n_1\}$ and $\{1, \dots, n_2\}$, resulting in $ERE \in \mathbf{P}_{n_1}$ and $FRF \in \mathbf{P}_{n_2}$. Now, putting $e_i = f_{i-n_1}$ for all i, $n_i < i < n + 1$, we obtain a set of n pairwise orthogonal idempotents $\{e_1, \dots, e_n\}$ with sum 1. Partitions of $\{1, \dots, n_2\}$ in the set Λ_2 define partitions of the set $\{n_1 + 1, \dots, n\}$ by sending i, $0 < i < n_2 + 1$, to $n_1 + i$, whence the set Λ_2 of partitions of $\{1, \dots, n_2\}$ defines the set $\Lambda_3 = \{\beta_0, \dots, \beta_{n_2-1}\}$ of partitions of the set $\{n_1 + 1, \dots, n\}$. Putting $\lambda_0 = \{1, \dots, n\}$ and $\lambda_{i+1} = \alpha_i \cup \beta_0$ for all $i,\ 0 \le i < n_1$, and $\lambda_{n_1+i-1} = \alpha_{n_1-1} \cup \beta_i$ for all $i,\ 0 < i < n_1$, one obtains a required complete dyadic set $\Lambda = \{\lambda_0, \ldots, \lambda_{n-1}\}$ of partitions of $\{1, \ldots, n\}$, which completes the proof. □

We emphasize the "advantage" of Peirce trivial idempotents over inner or outer Peirce trivial idempotents in that the symmetry of the Peirce trivial idempotents and the dyadic (refinement) partitioning provide the mechanism for decomposing a ring in $\mathbf{P}_n$ into n rings in $\mathbf{P}_1$.

With the notation of Corollary 2.14, for an $R \in \mathbf{P}_n$ together with n pairwise orthogonal 1-Peirce idempotents e_i with sum 1 and a complete dyadic set Λ of partitions, the subset $\mathfrak{D}(R)^- = \sum\limits_{i \neq j} e_i R e_j$ (see [1]) is a nilpotent ideal of nilpotency index at most n.

This simple assertion is based on the next observation. If I is an exceptional subset in the partition λ_{k-1} which is a disjoint union of two subsets J, K in λ_k, put $\mathfrak{D}_{\lambda_i}(R)^- = e_J R e_K + e_K R e_J$. Then in view of Proposition 2.3, $\mathfrak{D}_{\lambda_i}(R)^-$ is an ideal of $E_I R E_I$ with square 0. Since $\mathfrak{D}(R)^- = \sum\limits_{\lambda_i} \mathfrak{D}_{\lambda_i}(R)^-$, the claim is proved.

We will see later that $\mathfrak{D}(R)^-$ is independent of the choice of a set of idempotents e_i.

It is worth stating separately a result which is similar to the classical Wedderburn Principal Theorem:

Corollary 2.15. *Under the above notation, an n-Peirce ring R is a direct sum of $\mathfrak{D}(R)^-$ and a subring which is a direct sum of n 1-Peirce rings.*

The converse of this simple result is, in general, not true. It is quite interesting to find sufficient conditions such that a ring R is an n-Peirce ring if it has a direct decomposition $R = S \oplus D$ of a subring S, which is a direct product of n 1-Peirce rings, and a nilpotent ideal D of nilpotency index (at most) n.

The proof of Corollary 2.14 shows that there are several complete dyadic sets of partitions that define the same n-Peirce ring within a given set of pairwise orthogonal 1-Peirce idempotents whose sum is 1, and there is freedom and room in numbering the idempotents under consideration.

Now we give an obvious way to construct one such set of idempotents with a possible numbering (indexing) and a complete dyadic set of partitions. Since Definition 2.1 is deductive, first the identity $E_0 = 1$ is an orthogonal sum of two proper Peirce trivial idempotents $E_0 = E_{00} + E_{01}$ of R. Second, each of E_{00} and E_{01} is again an orthogonal sum of two proper Peirce trivial idempotents in the associated rings $E_{00} R E_{00}$ and $E_{01} R E_{01}$, respectively, except the case when they are 1-Peirce idempotents. In this exceptional case, they are elements of a required set of pairwise orthogonal 1-Peirce idempotents.

Continuing in this way, after finitely many steps one obtains, for an $R \in \mathbf{P}_n$, a sequence $e_1, \ldots, e_n$ of pairwise orthogonal 1-Peirce idempotents with sum 1 and a complete dyadic set of partitions $\lambda_0 = \{I\} \subseteq \lambda_1 \subseteq \lambda_2 \subseteq \cdots \subseteq \lambda_k$ such that for an exceptional element I of $\lambda_i,\ i = 0, \ldots k-1$, which is the union of two elements J and L of $\lambda_{i+1},\ e_J \in$

$\mathfrak{P}_{\mathrm{t}}(e_I Re_I)$. Furthermore, one can index them such that for each index $i < n$ there is an index j_i, $i < j_i \leq n$, maximal with respect to the property that $e_i \in \mathfrak{P}_{\mathrm{t}}(E_i RE_i)$, where $E_i = e_i + e_{i+1} + \cdots + e_{j_i}$.

However, a sequence $e_1, \ldots, e_n$ of pairwise orthogonal 1-Peirce idempotents with sum 1 such that for each index $i < n$ there is an index j_i, $i < j_i \leq n$, maximal with respect to the property that $e_i \in \mathfrak{P}_{\mathrm{t}}(E_i RE_i)$, where $E_i = e_i + e_{i+1} + \cdots + e_{j_i}$, is not sufficient to ensure that a ring is in $\mathbf{P}_n$. The reason is that such a sequence is far from ensuring that there exists a subsum of the e_i's which is Peirce trivial in the ring.

Furthermore, if $f^2 = f \in R$ is an arbitrary 1-Peirce idempotent of R, then according to Proposition 2.7 together with its notation, in the expression $f = \alpha + \beta$ of f as a sum of two orthogonal idempotents α and β in $\mathfrak{P}_{\mathrm{t}}(fRf)$, one of α and β must be 0, say, $\beta = 0$. Then g is a 1-Peirce idempotent in a subring eRe in $\mathbf{P}_m$, with $m < n$. Therefore, after finitely many steps one finds an idempotent e_i, uniquely determined by f, such that there is a 1-Peirce idempotent $g \in e_i Re_i$, with $f = \begin{bmatrix} g & gm \\ ng & 0 \end{bmatrix}$ or $f = \begin{bmatrix} 0 & mg \\ gn & g \end{bmatrix}$, where m and n are appropriate elements of $e_i R(1 - e_i)$ and $(1 - e_i)Re_i$, respectively. Note that g is, in general, not equal to e_i, the identity of the ring $e_i Re_i$, as one can see easily in the case of a matrix ring over a division ring.

These arguments lead to:

Proposition 2.16. *Under the hypothesis and notation of Corollary 2.14, any 1-Peirce idempotent f in a ring $R \in \mathbf{P}_n$ determines uniquely an $i \in \{1, 2, \ldots, n\}$ and a 1-Peirce idempotent $g \in e_i Re_i$ such that*

(1) ${}_R Rf$ and ${}_R Rg$ are isomorphic;
(2) $f = \begin{bmatrix} g & gm \\ ng & 0 \end{bmatrix}$ or $f = \begin{bmatrix} 0 & mg \\ gn & g \end{bmatrix}$ for appropriate $m \in e_i R(1 - e_i)$ and $n \in (1 - e_i)Re_i$. If $f \in \mathfrak{P}_{\mathrm{t}}(R)$, then $g = e_i$.

Remark 2.17. If f is an m-Peirce idempotent in a ring $R \in \mathbf{P}_n$, then f is an orthogonal sum of m pairwise orthogonal 1-Peirce idempotents f_j, and in view of Proposition 2.16, there are uniquely determined indices i_j associated to j and a 1-Peirce idempotent $g_j \in e_{i_j} Re_{i_j}$ such that

(1) ${}_R Rf_j$ and ${}_R Rg_j$ are isomorphic, and
(2) $f_j = \begin{bmatrix} g_j & g_j m_j \\ n_j g_j & 0 \end{bmatrix}$ or $f_j = \begin{bmatrix} 0 & m_j g_j \\ g_j n_j & g_j \end{bmatrix}$ for appropriate $m_j \in e_{i_j} R(1 - e_{i_j})$ and $n_j \in (1 - e_{i_j})Re_{i_j}$.

However, it is possible that the indices i_j are the same for different indices j as in the following example. Let

$$R = \begin{bmatrix} \mathbb{Z}/8\mathbb{Z} & 4\mathbb{Z}/8\mathbb{Z} & 2\mathbb{Z}/8\mathbb{Z} \\ 2\mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/8\mathbb{Z} & 2\mathbb{Z}/8\mathbb{Z} \\ 2\mathbb{Z}/8\mathbb{Z} & 2\mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/8\mathbb{Z} \end{bmatrix}.$$

Then one can check that $R \in \mathbf{P}_1$ and $fRf \in \mathbf{P}_2$, where $f = f_1 + f_2$ and

$$f = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ f_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } f_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

whence the corresponding indices i_1 and i_2 coincide. Although R seems to be quite simple, it has 2^{20} elements! One can verify that $R \in \mathbf{P}_1$ in the same way as in [1, Example 5.9].

Note that [1, Example 5.9] also provides an example of a ring in $\mathbf{P}_1$ with 3 pairwise orthogonal 1-Peirce idempotents whose sum is 1. There is another handy short way to check this assertion without computation, as follows. Observing that R is a semiperfect ring, in fact, a finite ring, all complete sets of pairwise orthogonal primitive idempotents of R are conjugate, i.e., any such set can be transformed into another one by an inner automorphism (see Lemma 2.18 below). Hence it suffices to check for the complete set $\{f_1, f_2, 1-(f_1+f_2)\}$ of pairwise orthogonal primitive idempotents of R, which is obvious. In this way one can construct quite a large class of semiperfect rings in $\mathbf{P}_1$.

This example shows also that a ring in $\mathbf{P}_n$ can contain a proper l-Peirce idempotent e, i.e., $e \neq 1, 0$, with $l > n$. For example, let $A = \mathbb{Z}/2^n\mathbb{Z}$ $(n > 2)$ and let $X = 2A$, $Y = 2^{n-1}A$. The above method implies immediately that the finite generalized $n \times n$ matrix ring

$$R = \begin{pmatrix} A & Y & Y & \cdots & Y & X \\ Y & A & Y & \cdots & Y & X^2 \\ Y & Y & A & \cdots & Y & X^3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ Y & Y & Y & \cdots & A & X^{n-1} \\ X^{n-1} & X^{n-2} & X^{n-3} & \cdots & X & A \end{pmatrix}$$

is in $\mathbf{P}_1$ together with an idempotent $f^2 = f \in R$ such $fRf \in \mathbf{P}_{n-1}$. Consequently, a ring R in $\mathbf{P}_1$ can contain an idempotent f such that the subring fRf has an arbitrary finite Peirce dimension.

In order to describe a relation between two sets of pairwise orthogonal 1-Peirce idempotents showing that a ring is in $\mathbf{P}_n$, let us recall the following more general, but folklore, result.

Lemma 2.18. *If $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_n\}$ are two sets of pairwise orthogonal idempotents in a ring R whose sums are 1, such that the modules ${}_RRe_i$ and ${}_RRf_i$ are isomorphic* *for all* $i = 1, 2, \ldots, n$, *then there is an invertible element* $s \in R$ *such that* $se_is^{-1} = f_i$ *for all* $i = 1, 2, \ldots, n$.

Proof. By assumption, for each $i = 1, 2, \ldots, n$ there are elements $s_i, t_i \in R$ such that the equalities $e_is_if_i = s_i,\ f_it_ie_i = t_i,\ s_it_i = e_i$ and $t_is_i = f_i$ hold. Put $s = s_1 + \cdots + s_n$ and $t = t_1 + \cdots + t_n$. Simple calculations show that $st = ts = 1$ and $sf_it = sf_is^{-1} = e_i$ for all $i = 1, 2, \ldots, n$. □

In spite of Proposition 2.16, we are now in a position to give some partial positive results showing some similarity to the theory of semiperfect rings.

Theorem 2.19. *Let* $R \in \mathbf{P}_n$ *be defined by* n *pairwise orthogonal 1-Peirce idempotents* $e_1, \ldots, e_n$ *with sum 1 and a complete dyadic set of partitions* $\{\lambda_0 = \{I\}, \lambda_1, \lambda_2, \ldots, \lambda_k\}$ *of* $\{1, 2, \ldots, n\}$ *such that for an exceptional element* I *of* $\lambda_i,\ i = 0, \ldots, k-1$, *which is the union of two elements* J *and* L *of* $\lambda_{i+1},\ e_J \in \mathfrak{P}_{\mathrm{t}}(e_IRe_I)$.

(1) If $\{f_1, \ldots, f_m\}$, *with* $m \le n$, *is any set of pairwise orthogonal 1-Peirce idempotents with sum 1, then* $m = n$ *and there is an invertible element* $s \in R$ *and a permutation* σ *of the set* $\{1, \ldots, n\}$ *such that* $f_{\sigma(i)} = se_is^{-1}$ *for all* $i = 1, \ldots, n$.

(2) If $f^2 = f \in R$ *is a* k*-Peirce idempotent, then for each index* j *in a representation of* $f = \sum_{j=1}^{k} f_j$ *as a sum of* k *pairwise orthogonal 1-Peirce idempotents, there exists an index* i *and a* 1*-Peirce idempotent* $\epsilon_{i_j} \in e_iRe_i$ *such that* ${}_RRf_j$ *is isomorphic to* ${}_RR\epsilon_{i_j}$, *where the* ϵ_{i_j}*'s are pairwise orthogonal appropriate* 1*-Peirce idempotents contained in* e_iRe_i. *Consequently,* ${}_RRf$ *is isomorphic to* ${}_RR\epsilon$, $\epsilon = \sum_{j=1}^{k} \epsilon_{i_j}$.

(3) Moreover, if $f \in \mathfrak{P}_{\mathrm{t}}(R)$, *then* f *is a* k*-Peirce idempotent for some* $k \le n$, *and in the above representation of* $f = \sum_{j=1}^{k} f_j$ *as a sum of* k *pairwise orthogonal 1-Peirce idempotents* f_j, *for each index* j *one has* $\epsilon_{i_j} = e_i$, *i.e., the correspondence* $j \mapsto i_j$ *is injective.*

Proof. (1) By Proposition 2.16, for each f_j there is a uniquely determined e_{i_j} with a 1-Peirce idempotent $g_{i_j} \in e_{i_j}Re_{i_j}$ such that f_j and g_{i_j} are equal modulo $\mathfrak{D}(R)^-$. Since the factor of R by $\mathfrak{D}(R)^-$ is a direct product of n rings e_iRe_i, and $\sum f_j$ maps to 1 in this factor ring, together with $m \le n$, one gets that all the g_{i_j} are distinct and each g_{i_j} is the identity e_{i_j} of the ring $e_{i_j}Re_{i_j}$. This shows that $m = n$, and that ${}_RRf_j$ and ${}_RRe_{i_j}$ are isomorphic R-modules, whence the existence of an inner automorphism of R sending the e_i onto the f_i follows in view of Lemma 2.18.

(2) We use the notation in the proof of Theorem 2.9. Let $e \in \mathfrak{P}_{\mathrm{t}}(R)$, ensuring that $R \in \mathbf{P}_n$, and put $A = eRe,\ M = eR(1-e),\ N = (1-e)Re$ and $B = (1-e)R(1-e)$, where $A \in \mathbf{P}_m$ and $B \in \mathbf{P}_{n-m}$ for some $m,\ 1 \le m < n$. For a k-Peirce idempotent

$f = \begin{bmatrix} g & m \\ n & h \end{bmatrix}$ with uniquely determined elements $g \in A,\ h \in B,\ m \in M$ and $n \in N$, let $\alpha = \begin{bmatrix} g & gm \\ ng & 0 \end{bmatrix}$ and $\beta = \begin{bmatrix} 0 & mh \\ hn & h \end{bmatrix}$. One has that $f = \alpha + \beta,\ \alpha\beta = \beta\alpha = 0$, and $\alpha, \beta \in \mathfrak{P}_{\mathrm{t}}(fRf)$, whence they are again l_1- and l_2-Peirce idempotents with $l_1, l_2 \leq k$ of R, respectively, in view of Theorem 2.9 and Proposition 2.7. Simple formal matrix calculation shows that gRg and hRh are isomorphic to $\alpha R\alpha$ and $\beta R\beta$, respectively. Consequently, g and h are l_1- and l_2-Peirce idempotents of R, respectively. Then the obvious induction finishes the proof of the first part of (2).

(3) If $f \in \mathfrak{P}_{\mathrm{t}}(R)$, then again by Theorem 2.9 f is a k-Peirce idempotent for some $k \leq n$. In a representation of f as a sum $\sum_{j=1}^{k} f_j$ of k pairwise orthogonal 1-Peirce idempotents above, Proposition 2.7 shows that, for each index j, the corresponding idempotent ϵ_{i_j} associated with f_j is in $\mathfrak{P}_{\mathrm{t}}(e_iRe_i)$, whence $\epsilon_{i_j} = e_i$, as required. It is worth noting that, in view of Remark 2.17, $k > n$ can happen for Peirce idempotents f which are not Peirce trivial. □

One can see assertion (1) of Theorem 2.19 by using [1, Theorem 5.7(2)]. By this result, $R \in \mathbf{P}_k$ for some $k,\ k \leq m \leq n$, whence $k = m = n$ by Theorem 2.9.

As a consequence of the above proof we obtain immediately that

$$\sum_{i \neq j} f_iRf_j \subseteq \sum_{i \neq j} e_iRe_j.$$

Since the role of e_i and of f_j are now quite symmetric, in view of Proposition 2.7 and Theorem 2.19, by interchanging the role of f_i and of e_i in the above inclusion, we obtain the equality

$$\sum_{i \neq j} f_iRf_j = \sum_{i \neq j} e_iRe_j,$$

showing that:

Corollary 2.20. *The ideal $\mathfrak{D}(R)^-$ of a ring $R \in \mathbf{P}_n$ is independent of the choice of the set $\{e_1, e_2, \ldots, e_n\}$ of n pairwise orthogonal 1-Peirce idempotents with sum 1. In particular, a ring-theoretical direct sum $\sum_i e_iRe_i$ of n subrings $e_iRe_i \in \mathbf{P}_1$ is uniquely determined by R up to isomorphisms, i.e., independent of the choice of the corresponding n pairwise orthogonal 1-Peirce idempotents whose sum is 1. Consequently, the subrings $e_iRe_i,\ i = 1, 2, \ldots, n$, are also invariants of R and the bimodules ${}_{e_iRe_i}e_iRe_j{}_{e_jRe_j}$ are uniquely determined up to bimodule isomorphisms, too.*

Unfortunately, the converse of this result is not true. However, in view of [1, Theorem 5.7(2)] every ring with a complete set $\{e_1, e_2, \ldots, e_n\}$ of pairwise orthogonal 1-Peirce idempotents e_i is in $\mathbf{P}_k$ for some $k \leq n$, whence R admits a Wedderburn-like principal decomposition described in Corollaries 2.15 and 2.20 with another complete set of k pairwise orthogonal 1-Peirce idempotents. Moreover, [1, Theorem 5.7(2)] together with the above consequences provides a very handy tool to determine if certain rings are in $\mathbf{P}_1$. To state the criterion, we define an auxiliary notion. A subset X of a ring R is said to be *nilpotent of index n* if its *ring closure*, i.e., the smallest additive group of R containing X and closed under multiplication is a nilpotent ring (without identity) of index n.

Corollary 2.21. *Let R be a ring with a complete set $\{e_1, e_2, \ldots, e_n\}$ of pairwise orthogonal 1-Peirce idempotents e_i, i.e., $\sum e_i = 1, e_i e_j = \delta_{ij} e_i$. Then $R \in \mathbf{P}_1$ if for every subset $I \subseteq \{1, 2, \ldots, n\}$ of at least two elements, the nilpotency index of $\mathfrak{D}_I{}^- = \sum_{i,j \in I,\ i \neq j} e_i R e_j$ is larger than the cardinality of I.*

Note that $\mathfrak{D}_I{}^-$ may not be nilpotent. Furthermore, it should be noted separately that [1, Theorem 5.7(2)] is an efficient tool for constructing certain rings in $\mathbf{P}_n$ with prescribed properties. In view of Corollary 2.14 one can refine the definition of rings in $\mathbf{P}_n$ by including a complete dyadic set of partitions of $\{1, 2, \ldots, n\}$ as an additional invariant.

Definition 2.22. A ring R is called an n-Peirce ring associated with a complete dyadic set of partitions $\lambda_0 = \{I\} \subseteq \lambda_1 \subseteq \lambda_2 \subseteq \cdots \subseteq \lambda_k$ of $\{1, 2, \ldots, n\}$ if there are n pairwise orthogonal 1-Peirce idempotents $e_1, \ldots, e_n$ with sum 1 such that for an exceptional element I of $\lambda_i,\ i = 0, \ldots, k-1$, which is the union of two elements J and L of λ_{i+1}, $e_J \in \mathfrak{P}_{\mathrm{t}}(e_I R e_I)$.

It is worth noting that a ring in $\mathbf{P}_n$ in the sense of Definition 2.1 can admit different complete dyadic sets of partitions of $\{1, 2, \ldots, n\}$. It is quite an interesting combinatorial question to determine all complete dyadic set of partitions of $\{1, 2, \ldots, n\}$ for a ring in $\mathbf{P}_n$. This freedom would provide room for a combinatorial description of certain automorphisms of rings in $\mathbf{P}_n$.

To justify Definition 2.22 we give an example of a ring in $\mathbf{P}_4$ associated with the complete dyadic set $\Big\{\lambda_0 = \big\{\{1,2,3,4\}\big\}, \lambda_1 = \big\{\{1,2\},\{3,4\}\big\}, \lambda_2 = \big\{\{1\},\{2\},\{3,4\}\big\}, \lambda_3 = \big\{\{1\},\{2\},\{3\},\{4\}\big\}\Big\}$ of partitions of $\{1,2,3,4\}$ which does not contain a Peirce trivial 3-Peirce idempotent. Consider the field $K = \mathbb{Z}_2$ of 2 elements, together with the trivial bilinear forms $[-,-] = (-,-) : \mathbb{Z}_2 \otimes_{\mathbb{Z}_2} \mathbb{Z}_2 \to \mathbb{Z}_2$, and let $A = B = \begin{bmatrix} K & K \\ K & K \end{bmatrix}$ be the generalized matrix ring induced by these trivial bilinear forms. Let $M = N = A = B$, considering ${}_A M_B$ and ${}_B N_A$ as bimodules equipped with the trivial bilinear form $(-,-)_B : M \otimes_B N \to A$ and $[-,-]_A : N \otimes_A M \to B$. Now the generalized matrix ring $R = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$ induced by these bilinear forms is the required example, as is easily verified by using the method described in Remark 2.17.

Since a complete dyadic set associated with a 2-Peirce or a 3-Peirce ring is unique, or equivalently, a Peirce trivial idempotent that defines a 2-Peirce or 3-Peirce ring, can be chosen to be a 1-Peirce idempotent, we can make the definition of a complete dyadic set essentially simpler as follows.

Definition 2.23. A set $\Lambda = \{\lambda_0 = \{I\}, \lambda_1, \lambda_2, \cdots, \lambda_k\}$ of partitions λ_i of a finite nonempty set I is called a *reduced dyadic set of partitions* if λ_{i+1} is a dyadic refinement of λ_i for all $i = 0, \ldots, k-1$ and all elements of λ_k are sets having at most three elements.

Definition 2.23 simplifies Corollary 2.14 as

Corollary 2.24. *A ring R is in $\mathbf{P}_n$ if and only if there are n pairwise orthogonal 1-Peirce idempotents $e_1, \ldots, e_n$ with sum 1 and a reduced dyadic set $\Lambda = \Big\{\lambda_0 = \big\{\{1, 2, \ldots, n\}\big\}, \lambda_1, \lambda_2, \cdots, \lambda_k\Big\}$ of partitions of $\{1, 2, \ldots, n\}$ such that for an exceptional element I of λ_i, $i = 0, \ldots, k-1$, which is the union of two elements J and L of λ_{i+1}, $e_J \in \mathfrak{P}_{\mathrm{t}}(e_I R e_I)$ and for all subsets J in λ_k, $e_J R e_J \in \mathbf{P}_{|J|}$, where $|J| \in \{1, 2, 3\}$ denotes the cardinality of J.*

In the last part of this section we describe automorphisms of certain n-Peirce rings associated with a complete dyadic set of partitions, generalizing the notion of strongly generalized triangular matrix rings. First we need the following definition.

Definition 2.25. An idempotent $e = e^2$ in a ring R is called a *strict 1-Peirce* idempotent if $e \in \mathfrak{P}_{\mathrm{t}}(R)$ and $eRe \in \mathbf{P}_1$. Also, e is called a *strict n-Peirce* idempotent if $e \in \mathfrak{P}_{\mathrm{t}}(R)$ and $eRe \in \mathbf{P}_n$. A ring R is called inductively a *strict n-Peirce* ring if there is a strict 1-Peirce $e \in R$ such that $(1-e)R(1-e)$ is a strict $(n-1)$-Peirce ring. *Strict 1-Peirce* rings are precisely 1-Peirce rings, whence strict 2- and strict 3-Peirce rings coincide also with 2- and 3-Peirce rings, respectively.

Observe that if $\{e_i\}_{i=1}^n$ is a complete set of orthogonal 1-Peirce idempotents contained in $\mathfrak{P}_{\mathrm{t}}(R)$, then R is a strict n-Peirce ring. In particular, if $\{e_i\}_{i=1}^n$ is a complete set of orthogonal primitive idempotents contained in $\mathfrak{P}_{\mathrm{t}}(R)$, then R is a strict n-Peirce ring. To construct such rings, see [1, Example 4.5(3)].

Strict n-Peirce rings are precisely n-Peirce rings associated with a complete dyadic set $\{\lambda_1, \ldots, \lambda_n\}$ of partitions of $\{1, 2, \ldots, n\}$ given by $\lambda_k = \big\{\{1\}, \{2\}, \ldots, \{k-1\}, I_k\big\}$, $k = 1, \ldots, n$, where $I_k = \{k, \ldots, n\}$. Therefore strict n-Peirce rings are natural extensions of strongly generalized triangular matrix rings (see [3]), or in alternative terminology, rings with a complete set of triangulating idempotents (see [4]).

Remark 2.26. It is worth emphasizing the subtle difference between 1-Peirce idempotents and strict 1-Peirce idempotents. The former are not necessarily Peirce trivial while the latter are such idempotents. For example, all proper idempotents in a matrix ring over a division ring are 1-Peirce idempotents, but they are never strict 1-Peirce idempotents!

In order to obtain a description of isomorphisms between strict n-Peirce rings, one needs some technical preparation.

Let A be a strict m-Peirce ring defined by an ordered sequence $e_1, \dots, e_m$ of pairwise orthogonal Peirce idempotents with sum 1 such that every $e_i \in \mathfrak{P}_{\mathrm{t}}(A_i),\ i = 1, \dots, m-1$, where $A_i = (e_i + \cdots + e_m)A(e_i + \cdots + e_m)$. Then $A_1 = A$. Letting $R_i = e_iAe_i$, we have $A_m = R_m$.

Next, let B be another strict n-Peirce ring defined by an ordered sequence $f_1, \dots, f_n$ of pairwise orthogonal Peirce idempotents with sum 1 such that every $f_i \in \mathfrak{P}_{\mathrm{t}}(B_i),\ i = 1, \dots, n-1$, where $B_i = (f_i + \cdots + f_n)B(f_i + \cdots + f_n)$. Then $B_1 = B$. Letting $S_i = f_iBf_i$, we have $B_n = S_n$.

If σ is any permutation of $\{1, \dots, n\}$, put $f_1^\sigma := f_{\sigma(1)}, \dots, f_n^\sigma := f_{\sigma(n)}$. According to this notation, if we write $g_i = f_i^\sigma$, then one can identify the above convention as follows:

$$S_i^\sigma := S_{\sigma(i)} = g_iBg_i, \quad B_i^\sigma := B_{\sigma(i)} = (g_i + \cdots + g_n)B(g_i + \cdots + g_n).$$

We are now in a position to describe isomorphisms between strict n-Peirce rings (see [3, Theorem]).

Theorem 2.27. *Let A and B be strict m- and strict n-Peirce rings defined by ordered sequences $e_1, \dots, e_m$ and $f_1, \dots, f_n$ of pairwise orthogonal 1-Peirce idempotents (with sum 1 in both cases) associated with the complete dyadic sets $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$ and $\{\lambda_1', \lambda_2', \dots, \lambda_n'\}$ of partitions of $\{1, 2, \dots, m\}$ and $\{1, 2, \dots, n\}$, respectively, where $\lambda_k = \{\{1\}, \{2\}, .., \{k-1\}, I_k\},\ k = 1, \dots, m$, and $\lambda'_{k'} = \{\{1\}, \{2\}, .., \{k'-1\}, I_{k'}\},\ k' = 1, \dots, n$, with $I_k = \{k, \dots, m\}$ and $I_{k'} = \{k', \dots, n\}$. Then A and B are isomorphic via an isomorphism $\varphi : A \to B$ iff $m = n$ and there is a permutation σ of $\{1, \dots, m\}$ together with ring isomorphisms $\rho_i : R_i = e_iAe_i \to S_i^\sigma = S_{\sigma(i)} = f_i^\sigma B f_i^\sigma = f_{\sigma(i)}Bf_{\sigma(i)},\ i = 1, \dots, m = n$, and for $i = 1, \dots, m-1$ there are elements $m_i \in M_i^\sigma = f_i^\sigma B_i^\sigma(1 - f_i^\sigma)$ and $n_i \in N_i^\sigma = (1 - f_i^\sigma)B_i^\sigma f_i^\sigma$, ring isomorphisms $\varphi_{i+1} : A_{i+1} \to B_{i+1}^\sigma,\ R_i - A_{i+1}$-bimodule isomorphisms $\chi_i : e_iA_i(1 - e_i) \to M_i^\sigma$ and $A_{i+1} - R_i$-bimodule isomorphisms $\delta_i : (1 - e_i)A_ie_i \to N_i^\sigma$, with respect to ρ_i and φ_{i+1}, such that for $i = 1, \dots, m-1$ and $a_i = \begin{bmatrix} r_i & x_i \\ y_i & a_{i+1} \end{bmatrix} \in A_i$,*

$$\varphi_i(a_i) = \begin{bmatrix} \rho_i(r_i) & \rho_i(r_i)m_i + \chi_i(x_i) - m_i\varphi_{i+1}(a_{i+1}) \\ n_i\rho_i(r_i) + \delta_i(y_i) - n_i\varphi_{i+1}(a_{i+1}) & \varphi_{i+1}(a_{i+1}) \end{bmatrix}.$$

Moreover, all isomorphisms between isomorphic rings A and B can be described in this manner. (Keep in mind that $\varphi_1 = \varphi,\ \varphi_m = \rho_m;\ A_m = R_m$.) In particular, the auto-

morphism group of a strict n-Peirce ring can be inductively described in terms of ones of 1-Peirce subrings and of related bimodules.

Proof. Assume that A and B are isomorphic via φ. Then $\varphi(e_1)$ is a strict 1-Peirce idempotent in B. Therefore, as in the proof of Theorem 2.9, for the unique generalized matrix representation $\varphi(e_1) = \begin{bmatrix} s_1 & m \\ n & b_2 \end{bmatrix}$, with uniquely determined elements $s_1 \in S_1,\ b_2 \in B_2,\ m \in f_1B(1-f_1)$ and $n \in (1-f_1)Bf_1$, by putting $\alpha = \begin{bmatrix} s_1 & s_1m \\ ns_1 & 0 \end{bmatrix}$ and $\beta = \begin{bmatrix} 0 & mb_2 \\ b_2n & b_2 \end{bmatrix}$, one has that $\varphi(e_1) = \alpha + \beta,\ \alpha\beta = \beta\alpha = 0$, and α and β are Peirce trivial idempotents of $\varphi(e_1)B\varphi(e_1)$ as well as of S_1 and B_2, respectively. Since $\varphi(e_1)B\varphi(e_1) \in \mathbf{P}_1$, one of α and β must be 0. If $\beta = 0$, then $\varphi(e_1) = \alpha$, and hence in this case one has $s_1 = f_1$, the identity element of S_1, and one puts $\sigma(1) = 1$. If $\alpha = 0$, then $\varphi(e_1) = \beta$, and b_2 is a strict 1-Peirce idempotent of both $B_1 = S$ and B_2.

In this situation, one can repeat the process. Therefore after finitely many steps there exists a natural number $j = \sigma(1)$ such that f_j is a strict 1-Peirce idempotent of S, for each $k < j$ there are elements $x_k \in f_kSf_j$ and $v_k \in f_jSf_k$, and for each $k > j$ there are elements $u_k \in f_jSf_k$ and $y_k \in f_kSf_j$ such that

$$\varphi(e_1) = \begin{bmatrix}
0 & 0 & \cdots & \cdots & 0 & x_1 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & x_2 & \vdots & \ddots & & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & & \ddots & & \vdots \\
\vdots & & \ddots & \ddots & 0 & \vdots & \vdots & & & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 0 & x_{j-1} & 0 & \cdots & \cdots & \cdots & 0 \\
v_1 & v_2 & \cdots & \cdots & v_{j-1} & 1 & u_{j+1} & u_{j+2} & \cdots & \cdots & u_n \\
0 & \cdots & \cdots & \cdots & 0 & y_{j+1} & 0 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & & & \vdots & y_{j+2} & 0 & 0 & 0 & \cdots & 0 \\
\vdots & & \ddots & & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & & \ddots & \vdots & \vdots & \vdots & & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & \cdots & 0 & y_n & 0 & \cdots & \cdots & 0 & 0
\end{bmatrix},$$

or equivalently, for

$$m_1 = x_1 + \cdots + x_{j-1} + u_{j+1} + \cdots + u_n \in M_1^\sigma = M_{\sigma(1)} = f_jB(1-f_j)$$

and

$$n_1 = v_1 + \cdots + v_{j-1} + y_{j+1} + \cdots + y_n \in N_1^\sigma = N_{\sigma(1)} = (1-f_j)Bf_j,$$

$$\varphi(e_1) = \begin{bmatrix} 1 & m_1 \\ n_1 & 0 \end{bmatrix} \in \begin{bmatrix} f_jBf_j & f_jB(1-f_j) \\ (1-f_j)Bf_j & (1-f_j)B(1-f_j) \end{bmatrix}.$$

Therefore φ induces ring isomorphisms $\rho_1 : e_1Ae_1 = R_1 \to \varphi(e_1)B\varphi(e_1) \cong B_1^\sigma = f_jBf_j = S_j = B_1^\sigma$ and $\varphi_2 : A_2 = (1-e_1)A(1-e_1) \to (1-\varphi(e_1))B(1-\varphi(e_1)) \cong (1-f_j)B(1-f_j) = B_2^\sigma$, and the restrictions of φ to $e_1A(1-e_1)$ and $(1-e_1)Ae_1$ define the bimodule isomorphisms χ_i and δ_i to M_1^σ and N_1^σ, respectively. Consequently, if

$$a = a_1 = \begin{bmatrix} r_1 & x_1 \\ y_1 & a_2 \end{bmatrix} \in A = A_1 = \begin{bmatrix} e_1Ae_1 & e_1A(1-e_1) \\ (1-e_1)Ae_1 & (1-e_1)A(1-e_1) \end{bmatrix},$$

then

$$\varphi(a) = \varphi_1(a_1) = \varphi(r_1) + \varphi(x_1) + \varphi(y_1) + \varphi(a_2) =$$
$$= \begin{bmatrix} \rho_1(r_1) & \rho_1(r_1)m_1 + \chi_1(x_1) - m_1\varphi_2(a_2) \\ n_1\rho_1(r_1) + \delta_1(y_1) - n_1\varphi_2(a_2) & \varphi_2(a_2) \end{bmatrix}.$$

The theorem follows now easily by reduction. □

Observing that the proof of Theorem 2.27 also shows, for each index i, that

$$\varphi(e_i) = \begin{bmatrix} 1 & m_i \\ n_i & 0 \end{bmatrix} \in \begin{bmatrix} S_{\sigma(i)} & f_{\sigma(i)}B(f_{\sigma(i+1)} + \cdots + f_{\sigma(n)}) \\ (f_{\sigma(i+1)} + \cdots + f_{\sigma(n)})Bf_{\sigma(i)} & B_\sigma^{i+1} \end{bmatrix},$$

one obtains the following result in view of Lemma 2.18.

Theorem 2.28. *Let R be a strict n-Peirce ring defined by two sequences $e_1, \dots, e_n$ and $f_1, \dots, f_n$ of pairwise orthogonal 1-Peirce idempotents with sum 1 in each case. Then there is a permutation σ of $\{1, 2, \dots, n\}$ and a unit $s \in R$ such that $se_is^{-1} = f_{\sigma(i)}$ for all $i = 1, \dots, n$.*

Observe that not every permutation can occur in the description of the isomorphisms between and automorphisms of n-Peirce rings. Of course, such permutations form a subgroup of the symmetric group and this subgroup leaves invariant the class of all complete dyadic sets of partitions defining R.

We conclude this section with some remarks on the automorphism group of a ring $R \in \mathbf{P}_n$ together with a complete set $I = \{e_1, e_2, \dots, e_n\}$ of pairwise orthogonal 1-Peirce idempotents. Any automorphism ϕ of R transforms I to the complete set $I_\phi = \{\phi(e_1), \phi(e_2), \dots, \phi(e_n)\}$ of pairwise orthogonal 1-Peirce idempotents. Therefore by Theorem 2.19 ϕ determines uniquely the permutation σ_ϕ and a unit s_ϕ such that $\phi(e_i) = s_\phi^{-1}e_{\sigma_\phi(i)}s_\phi$ for every i. However, s_ϕ is not uniquely determined by ϕ. To simplify notation, we write σ and s for σ_ϕ and s_ϕ, respectively. These permutations σ form a subgroup Ω_R of the symmetric group leaving invariant the class of all complete dyadic sets of partitions. It is clear that ϕ induces the automorphisms ρ_i between subrings $R_i = e_iRe_i$ and $R_i^\sigma = e_{\sigma(i)}Re_{\sigma(i)}$ and the (ρ_i, ρ_j)-bimodules isomorphisms χ_{ij} between bimodules ${}_{R_i}e_iRe_{R_j} = R_{ij}$ and $R_{ij}^\sigma = e_{\sigma(i)}Re_{\sigma(j)}$. One can describe automorphisms of R in terms of permutations from Ω_R and isomorphisms ρ_i, χ_{ij} and units s in the way similar to that given in Theorem 2.27 by using a complete dyadic set of partitions defining R.

3. Lifting process

If $e \in \mathfrak{P}_{\mathrm{t}}(R)$, then $(ReR(1-e)R)^2 = 0 = (R(1-e)ReR)^2$, whence R is a direct product of the rings eRe and $(1-e)R(1-e)$, provided that R is semiprime. This observation implies the following result, which is basic in the lifting process concerning the structure of rings. Recall that $R/\rho(R)$ is a semiprime ring for any supernilpotent radical ρ (see [9]). The collection of supernilpotent radicals includes the prime, nil, Levitzki, Jacobson, and Brown-McCoy radicals.

Theorem 3.1. *A ρ-semisimple n-Peirce ring, for a supernilpotent radical ρ, is a direct product of n ρ-semisimple* 1*-Peirce (i.e., indecomposable) rings. In particular, a semiprime n-Peirce ring is a direct product of n semiprime* 1*-Peirce (i.e., indecomposable) rings.*

Since prime rings are clearly 1-Peirce rings, it is quite natural to ask: How large is the class of semiprime 1-Peirce rings? The following example indicates that the class of semiprime 1-Peirce rings is quite extensive.

Example 3.2. (1) Let K be a field, and let $R = K[x, y, z]/I$ be the factor ring of the commutative polynomial ring in three variables x, y, z by the ideal I generated by the monomial xyz. Then R is a semiprime 1-Peirce ring which is not prime, because R has only the two trivial idempotents 0 and 1.

(2) The ring $R = \{\frac{m}{n} : \ m, n \in \mathbb{Z}, \ (n, 2) = (n, 3) = (n, 5) = 1\}$ is a semilocal domain with nonzero Jacobson radical, namely the ideal generated by 30. Let E be the minimal injective cogenerator of R, i.e., E is a direct sum of three quasi-cyclic abelian groups $C(2^\infty), C(3^\infty)$ and $C(5^\infty)$. Then the trivial extension of R by E is also a ring having only the two trivial idempotents, and with nonzero nil radical. Consequently, this ring is a non-prime 1-Peirce ring.

(3) More generally, if R is a ring with only the two trivial idempotents, e.g., a polynomial ring over a not necessarily commutative domain with not necessarily commuting variables, and if M is any (R, R)-bimodule, then the trivial extension of R by M is also a ring with only the two trivial idempotents. This observation shows that the class $\mathbf{P}_1$ is a large and diverse class.

Definition 3.3. An idempotent e in a ring R is called ***J**-trivial* if both $eR(1-e)Re$ and $(1-e)ReR(1-e)$ are contained in the Jacobson radical of R, $\mathbf{J}(R)$. Observe that e is a

***J**-trivial* idempotent in R if and only if $e + \mathbf{J}(R)$ is central in $R/\mathbf{J}(R)$. A ring R is called a *1-**J** ring* if 0 and 1 are the only **J**-trivial idempotents of R. Inductively, for a natural number $n > 1$, a ring R is called an *n-**J** ring* if there is a **J**-trivial idempotent $e \in R$ such that eRe is an m-**J** ring for some $1 \leq m < n$ and $(1 - e)R(1 - e)$ is an $(n - m)$-**J** ring. A **J**-trivial idempotent $e \in R$ is called an n-**J** idempotent if eRe is an n-**J** ring. In particular, a ring R is called *1-primary* if $R/\mathbf{J}(R)$ is isomorphic to the endomorphism ring of (a not necessarily finite dimensional) vector space over a division ring, and R is called *n-primary* ($n > 1$) if there is a **J**-trivial idempotent $e \in R$ such that eRe is an m-primary ring for some $1 \leq m < n$ and $(1 - e)R(1 - e)$ is an $(n - m)$-primary ring.

Note that n-**J** rings are well-defined in either the class of Jacobson-semisimple rings or in the class of rings where idempotents can be lifted modulo the Jacobson radical, according to Theorem 2.9. However, it is possible that there exists a ring which is at the same time both an m- and an n-**J** ring for different natural numbers m and n. Therefore it is an interesting question to determine classes of rings where the notion of n-**J** ring is well-defined.

In general, for a ring R idempotents do not lift module $\mathbf{J}(R)$ without additional conditions on R or $\mathbf{J}(R)$. However in the investigation of n-**J** rings, we are primarily interested in lifting ***J**-trivial* idempotents. Thus we make the following definition.

Definition 3.4. A ring R is called a *weakly lifting ring* if central idempotents in $R/\mathbf{J}(R)$ can be lifted to **J**-trivial idempotents in R.

The next result is obvious.

Lemma 3.5. *If R is a weakly lifting ring and $e^2 = e \in R$ is any **J**-trivial idempotent, then eRe is also a weakly lifting ring.*

We are now in a position to generalize the classical structure theory of semi-perfect rings as follows.

Corollary 3.6.

*(1) If R is an n-**J** ring, then $R/\mathbf{J}(R)$ is a direct sum of n semisimple rings which are, in general, not in $\mathbf{P}_1$.*
(2) If R is, in addition, a weakly lifting ring, then all these direct summands are 1*-Peirce (indecomposable) rings. Furthermore, in this case of a weakly lifting n-**J** ring R, all sets $\{f_1, \ldots, f_m\}$ of pairwise orthogonal **J**-trivial idempotents with sum 1 such that all the subrings f_iRf_i are 1-**J** rings, have n elements, i.e., $m = n$. Moreover, these sets are permuted by inner automorphisms.*
*(3) A ring R is an n-primary ring if and only if there are n pairwise orthogonal **J**-trivial idempotents $e_1, \ldots, e_n$, whose sum is 1, such that all the e_iRe_i, $i = 1, \ldots, n$,*

are 1-primary rings. Any set $\{f_1, \ldots, f_m\}$ *of pairwise orthogonal* **J***-trivial idempotents, with sum 1, such that all the* $f_i R f_i$, $i = 1, \ldots, m$, *are 1-primary rings, has* n *elements, i.e.,* $m = n$. *Furthermore, if* $g^2 = g \in R$ *is any* **J***-trivial idempotent, then there is a uniquely determined natural number* $k \leq n$ *such that* g *can be written as a sum of* k *pairwise orthogonal* **J***-trivial idempotents* g_j *such that all the* $g_j R g_j$, $j = 1, \ldots, k$, *are 1-primary rings. The projective module* ${}_R Rg$ *is isomorphic to the projective module* ${}_R Re$, *where* e *is a sum of* k *appropriate idempotents* e_{i_t}, $t = 1, \ldots, k$.

(4) *An* n*-primary ring is semiperfect if idempotents can be lifted modulo the Jacobson radical and the semisimple factor is a finite direct sum of matrix rings.*

The proof of this result can be carried out in the same way as it was carried out in Section 2 for similar results on (strict) n-Peirce rings. Since a semisimple n-**J** ring is a direct sum of n semisimple 1-Peirce rings, one has the following result.

Theorem 3.7. *If* R *is a weakly lifting* n-**J** *ring, then its semisimple factor is a direct sum of* n *semisimple* 1*-Peirce rings* $\bar{R}_i$, n *is an invariant of* R *and every* **J***-trivial idempotent* e *of* R *maps to the identity of a product of some* $\bar{R}_i$ *in the semisimple factor* $\bar{R}$ *of* R. *Conversely, if the semisimple factor of a ring* R *is a direct sum of* n *semisimple 1-Peirce rings and the corresponding pairwise orthogonal idempotents can be lifted to pairwise orthogonal idempotents, then* R *is a weakly lifting* n-**J** *ring.*

Another short way to verify this result is by passing to the semisimple factor which is a direct sum of n semisimple 1-Peirce rings, then applying the corresponding results on n-Peirce rings and thereafter lifting them by using [10, Proposition III.8.1].

We now list some problems related to this classical topic of lifting idempotents.

Problems 3.8. Semiperfect rings are characterized as complemented rings. It would be nice to give a constructive proof that (pairwise orthogonal) idempotents of such rings, even of rings satisfying AB5* can be lifted to (pairwise orthogonal) idempotents. Recall that a ring *satisfies the condition* AB5* *on the right* if the lattice of right ideals is lower continuous, i.e., for any right ideal K and any set I_α of right ideals downward directed by inclusion one has $K + \cap I_\alpha = \cap(K + I_\alpha)$. An open question is whether there are n-primary rings which are not semiperfect, i.e., which are not matrix rings over local rings. It is quite an interesting enterprise to develop the theory of such n-**J** rings where idempotents can be lifted. For example, all commutative local rings are semiperfect, because they have only the trivial idempotents 0 and 1, which obviously can be lifted modulo the Jacobson radical. However, if a ring does not have an identity, then it is not known whether idempotents modulo the Jacobson radical can be lifted, even in the case of left chain rings. Posner [16] discussed an interesting relation between the question of lifting idempotents in left chain rings and the existence of left but not right primitive rings.

Since prime rings are 1-Peirce rings and semiprime n-Peirce rings are direct sums of n semiprime 1-Peirce rings, a semiprime n-Peirce ring is called a *semiprime strict n-Peirce ring* if it is a direct sum of n prime rings. Semiprime 1-Peirce rings are not necessarily prime as we have seen at the beginning of this section. The case of primitive rings are more doubtful: both left and right primitive rings are both 1-**J** rings and 1-Peirce rings, but the converse is not true in view of the ring $R = \{\frac{m}{n} : m, n \in \mathbb{Z}, (n,2) = (n,3) = (n,5) = 1\}$. It is worth noting that both primeness and (left, right) primitiveness are matrix invariants, i.e., a matrix ring over a prime ring or primitive rings is again prime or primitive, respectively. Since pairwise orthogonal idempotents can be lifted modulo the prime radical, we obviously have the following result.

Theorem 3.9. *If R is a ring such that the factor by the prime radical is a semiprime strict n-Peirce ring, then R has n pairwise orthogonal idempotents $e_1, \ldots, e_n$ whose sum is 1 such that all $e_iR(1-e_i)Re_i$ and $(1-e_i)Re_iR(1-e_i)$ are contained in the prime radical and the factor of every e_iRe_i by its prime radical is prime, $i = 1, \ldots, n$.*

Motivated by the theory of semiperfect rings one can ask for some homological characterization of the class of rings described in the above theorem. In particular, one can introduce the following notions:

Definition 3.10. An idempotent e in a ring R is called ***B**-trivial* if $eR(1-e)Re$ and $(1-e)ReR(1-e)$ are contained in the prime radical $\mathbf{B}(R)$ of R. Observe that e is a ***B**-trivial* idempotent in R if and only if $e + \mathbf{B}(R)$ is central in $R/\mathbf{B}(R)$. Note that this condition was used in [1, Lemma 3.12]. If 0 and 1 are the only **B**-trivial idempotents of R, then R is said to be a *1-**B** ring*. Inductively, for a natural number $n > 1$, a ring R is called an *n-**B** ring* if there is a **B**-trivial idempotent $e \in R$ such that eRe is an m-**B** ring for some $1 \leq m < n$ and $(1-e)R(1-e)$ is an $(n-m)$-**B** ring. A semiprime n-**B** ring is clearly a semiprime n-Peirce ring.

More generally, one can introduce the notion of trivial idempotents concerning certain radicals similar to ones defined above for the Jacobson and Baer radicals (e.g., various supernilpotent radicals, see [9]) and then develop a corresponding structure theory. Results on n-Peirce rings can be used to determine properties of rings concerning such radicals by which their factors are semiprime n-Peirce rings, whence they are direct sums of n semiprime 1-Pierce rings together with additional assumptions on lifting idempotents. For example, in the case of the Brown-McCoy radical we are interested in finite direct sums of simple rings. Therefore assuming that central idempotents modulo the Brown-McCoy radical can be lifted and the factor ring is a direct sum of finitely many simple rings of a certain kind, one can develop a structure theory based on Brown-McCoy trivial idempotents.

If $A = \mathbb{Z}_4$ and $R = \begin{bmatrix} A & A \\ 2A & A \end{bmatrix}$, then $R \in \mathbf{P}_1$, but R is not a 1-**B** ring, because $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is a **B**-trivial idempotent of R.

Note that for radicals ρ such that $\rho(R) \subseteq \mathbf{J}(R)$, a nonzero idempotent in R remains nonzero in $R/\rho(R)$. This is not so for radicals not contained in $\mathbf{J}(R)$. For example, the Brown-McCoy radical, $\mathbf{G}(R)$, may contain nontrivial idempotents. However, any nonzero inner Peirce trivial idempotent is not an element of $\mathbf{G}(R)$.

Since finitely many pairwise orthogonal idempotents can be lifted modulo the prime radical, we have the following generalization of Theorem 3.9.

Theorem 3.11. *R is an n-**B** ring if and only if its factor by the prime radical is a direct sum of n semiprime 1-Peirce rings. In particular, n is an invariant of R, i.e., there are n pairwise orthogonal **B**-trivial idempotents e_i, $i = 1, \dots, n$, in R, with sum 1, such that all e_iRe_i, $i = 1, \dots, n$, are 1-**B** rings. Moreover, every **B**-trivial idempotent $f \in R$ can be written as a sum of m pairwise orthogonal **B**-trivial idempotents f_i, $i = 1, \dots, m$, $m \leq n$, such that all the f_iRf_i are 1-**B** rings (hence semiprime indecomposable rings), and there is an idempotent $e \in R$ which is a sum of m appropriate idempotents e_i, $i = 1, \dots, n$, such that ${}_RRe$ and ${}_RRf$ are isomorphic. In particular, if $\{f_1, \dots, f_m\}$ is an arbitrary set of pairwise orthogonal **B**-trivial idempotents with sum 1 and all the f_iRf_i are 1-**B** rings, then $m = n$ and there is a permutation σ of $\{1, 2, \dots, n\}$ and an invertible element $u \in R$ such that $e_i = uf_{\sigma(i)}u^{-1}$ for all $i = 1, \dots, n$.*

By Theorem 3.11 it is an interesting problem to search for good homological characterizations of classes of rings described in Theorem 3.11, even when it is assumed that the semiprime factor ring is a direct sum of the corresponding n prime rings. It would also be interesting to compare the classes of prime rings and semiprime 1-Peirce rings.

4. Applications

In this section we apply our theory of rings in $\mathbf{P}_n$ to various important classes of rings. Observe that from our previous results each ring in $\mathbf{P}_n$ is isomorphic to a generalized matrix ring

$$R' = \begin{bmatrix} R_1 & M_{12} & \cdots & M_{1n} \\ M_{21} & R_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & M_{n-1,n} \\ M_{n1} & \cdots & M_{n,n-1} & R_n \end{bmatrix},$$

where each $R_i \in \mathbf{P}_1$, each M_{ij} is an (R_i, R_j)-bimodule and $M_{ij}M_{ji} = 0_{R_i}$ for all $i \neq j$, and $R_1, \dots, R_n$ are unique up to isomorphism and permutation. The fact that $M_{ij}M_{ji} =$

0_{R_i} simplifies the matrix multiplication of elements of R'. For example, it is relatively easy to compute idempotents. Also the calculation of various radicals which contain the prime radical (e.g., the Jacobson, nil, and Brown-McCoy radicals) is reduced to computing the radicals of the rings R_i, since $\mathcal{D}(R)^-$ is nilpotent.

Since the notion of a (quasi-)Baer ring will play a role in the main results of this section, recall: a ring R is *(quasi-)Baer* if for each nonempty $X \subseteq R$ (X an ideal of R) there is an $e = e^2 \in R$ such that $\underline{r}(X) = eR$, where $\underline{r}(X)$ denotes the right annihilator of X in R. Note that the quasi-Baer property is a Morita invariant, whereas the Baer property is not. The class of quasi-Baer rings is ubiquitous, since it contains all: Baer rings (hence endomorphism rings of vector spaces over division rings), AW*-algebras (in particular, von Neumann algebras), local multiplier C*-algebras [7, Theorem 10.3.10], regular right selfinjective rings, prime rings, and biregular right selfinjective rings. Moreover, the class of quasi-Baer rings is closed under direct products, matrix rings, triangular matrix rings, and various polynomial extensions. Furthermore, each semiprime ring has a quasi-Baer hull contained in its (Martindale) symmetric ring of quotients; for more details, see [7].

Lemma 4.1.

(1) *[1, Lemma 3.4] Let R be a ring and $e \in R$. Then e is an inner Peirce trivial idempotent if and only if $h : R \to eRe$, defined by $h(x) = exe$, is a surjective ring homomorphism.*

(2) *[1, Lemma 5.13] R is a prime ring if and only if R is a quasi-Baer 1-Peirce ring.*

Lemma 4.2. *Let R be a ring and $0 \neq c \in \mathfrak{P}_{\mathrm{t}}(R)$.*

(1) *Let $f = f^2 \in R$ be primitive and $c_1 \in \mathfrak{P}_{\mathrm{t}}(cRc)$ such that $cfc \neq 0$ and $c_1fc_1 \neq 0$. Then:*

  (a) *$fcf = f = fc_1f$; and*

  (b) *cfc is a primitive idempotent of R.*

(2) *Let $\{f_1, \dots, f_k\}$ be a complete set of primitive orthogonal idempotents of R. Then there exists $H \subseteq \{1, \dots, k\}$ such that $\{cf_hc \mid h \in H\}$ is a set of primitive orthogonal idempotents of R such that $c = \sum_{h \in H} cf_hc$.*

Proof. (1) By [1, Proposition 5.4(1)], (a) holds. Since c is inner Peirce trivial, $cfc = (cfc)^2$. To show that cfc is primitive, we prove that cfc is the only nonzero idempotent in $cfcRcfc$. Let $0 \neq cxc = (cxc)^2 \in cfcRcfc$, where $x = fcycf$ for some $y \in R$. Observe that $cxc = cxccxc = cx^2c$, since c is inner Peirce trivial. Consider

$$(fcxcf)^2 = fcxcffcxcf$$

$$\begin{aligned}&= fcx(cfc)xcf\\&= fcxcxcf\\&= fcx^2cf\\&= fcxcf \in fRf.\end{aligned}$$

Observe that $c(fcxcf)c = cfc(cxc)cfc = cxc \neq 0$. Hence $fcxcf \neq 0$. Since f is primitive, $f = fcxcf$. Then

$$\begin{aligned}cfc &= c(fcxcf)c\\&= (cfc)(cxc)(cfc)\\&= cxc.\end{aligned}$$

Therefore cfc is primitive, so (b) holds.

(2) Take $H = \{h \in \{1, \ldots, k\} \mid cf_hc \neq 0\}$. Let $h, j \in H$ such that $h \neq j$. Then $(cf_hc)(cf_jc) = cf_hf_jc = 0$, because c is inner Peirce trivial. Using (1) we now have that (2) holds. □

Theorem 4.3. *Let $R \in \mathbf{P}_n$. Then R has a complete set of orthogonal idempotents, $\{e_1, \ldots, e_n\}$, and a generalized matrix representation*

$$R \cong \begin{bmatrix} R_1 & M_{12} & \cdots & M_{1n} \\ M_{21} & R_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & M_{n-1,n} \\ M_{n1} & \cdots & M_{n,n-1} & R_n \end{bmatrix},$$

where each $R_i = e_iRe_i \in \mathbf{P}_1$, each $M_{ij} = e_iRe_j$ with $M_{ij}M_{ji} = 0_{R_i}$ for all $i \neq j$, and $R_1, \ldots, R_n$ are unique up to isomorphism and permutation. Moreover, if R satisfies any condition which transfers from R to a homomorphic image or to eRe, where $e = e^2 \in R$, then each R_i also satisfies the condition.

Proof. This result is a consequence of [1, Theorem 5.7(1)], Theorem 2.19(1) and Lemma 4.1. □

To indicate the applicability of Theorem 4.3, the following is a list of some of the classes of rings which are closed with respect to homomorphic images or contain eRe whenever $e = e^2 \in R$ and R is in the following class: right Noetherian, right (semi-)Artinian, PI (i.e., satisfies a polynomial identity), (quasi-)Baer, right (semi-)hereditary, (bi-, π-, semi-)regular, I-ring (i.e., every non-nil right ideal contains a nonzero idempotent), bounded index of nilpotency, right selfinjective, etc. Furthermore, in Theorem 4.3, if R satisfies any of the above conditions and is quasi-Baer, then each R_i is a prime ring satisfying the condition.

In the next result, see [13] for details on Krull dimension.

Corollary 4.4. *If R satisfies any of the following conditions, then $R \in \mathbf{P}_n$ with a complete set of orthogonal idempotents, $\{e_1, \ldots, e_n\}$, and a generalized matrix representation as in Theorem 4.3 and each R_i satisfies the same condition as R.*

(1) R has DCC on $\{ReR \mid e = e^2 \in R$ is Peirce trivial$\}$.
(2) R has a complete set of primitive orthogonal idempotents.
(3) R has no infinite set of orthogonal idempotents.
(4) R_R has Krull dimension.
(5) R is semilocal.
(6) R is semiperfect.
(7) R is left (or right) perfect.
(8) R is semiprimary.

Proof. From [1, Theorem 5.7(1)] and Theorem 2.19, $R \in \mathbf{P}_n$ with a complete set of orthogonal idempotents, $\{e_1, \ldots, e_n\}$, and the indicated generalized matrix representation, where each $R_i = e_iRe_i \in \mathbf{P}_1$, each $M_{ij} = e_iRe_j$ with $M_{ij}M_{ji} = 0_{R_i}$ for all $i \neq j$, and $R_1, \ldots, R_n$ are unique up to isomorphism and permutation. It only remains to show that if R satisfies any of the conditions (1) - (8), then so does each R_i.

For condition (1), the result follows from [1, Theorem 5.11]. So assume condition (2) that R has a complete set of primitive orthogonal idempotents, $\{f_1, \ldots, f_k\}$. Using Corollary 2.14 and Lemma 4.2, then $e_if_je_i = 0$ or $e_if_je_i$ is primitive for each $i = 1, \ldots, n$ and $j = 1, \ldots, k$. Then for each R_i there exists $H_i \subseteq \{1, \ldots, k\}$ such that $\{e_if_he_i \mid h \in H\}$ is a complete set of primitive orthogonal idempotents for R_i.

If R satisfies any of conditions (3) - (8), then each R_i contains no infinite set of orthogonal idempotents and satisfies the same condition as R. □

Corollary 4.5. *Assume R is left perfect in Corollary 4.4. Then each R_i is either simple Artinian or $[Soc(R_{i\,R_i})]^2 = 0$. If R is also quasi-Baer, then each R_i is simple Artinian and R is semiprimary.*

Proof. Observe that a ring in $\mathbf{P}_1$ is semicentral reduced. Now from [5, Theorem 3.13] each R_i is either simple Artinian or $[\mathrm{Soc}(R_{i\,R_i})]^2 = 0$. The remainder of the proof follows from Lemma 4.1(2). Since $\mathbf{J}(R) = \mathcal{D}(R)^-$, [1, Proposition 4.4] and Corollary 2.15 yield that R is semiprimary (also see [6, Theorem 2.3]). □

The following examples illustrate Corollary 4.5.

Examples 4.6.

(1) Let F be a field, S the ring of k-by-k upper triangular matrices over F, and R the n-by-n upper triangular matrix ring over S, where $k, n \geq 1$. Then R is a semiprimary

quasi-Baer ring in $\mathbf{P}_{kn}$ which is not a Baer ring, where each R_i is isomorphic to F (see [15]).

(2) Let A be an Artinian ring in $\mathbf{P}_1$ such that $A/\mathbf{J}(A)$ is a simple ring, $\big(\mathbf{J}(A)\big)^3 = 0$, and $\big(\mathbf{J}(A)\big)^2 \neq 0$ (e.g., $A = \mathbb{Z}/8\mathbb{Z}$). Let $R = \begin{bmatrix} A & A/\mathbf{J}(A) \\ 0 & A/\mathbf{J}(A) \end{bmatrix}$. Then $R \in \mathbf{P}_2$ with $R_1 = A$ and $[\mathrm{Soc}({R_1}_{R_1})]^2 = 0$, and R_2 is a simple Artinian ring.

(3) Let A be as in (2). Let

$$R = \begin{bmatrix} A & \big(\mathbf{J}(A)\big)^2 & \mathbf{J}(A) \\ \big(\mathbf{J}(A)\big)^2 & A & \big(\mathbf{J}(A)\big)^2 \\ \big(\mathbf{J}(A)\big)^2 & \mathbf{J}(A) & A \end{bmatrix}.$$

Then $R \in \mathbf{P}_3$, where each $R_i = A$, hence $[\mathrm{Soc}({R_i}_{R_i})]^2 = 0$ for all i.

Theorem 4.7. *If R is an n-$\boldsymbol{B}$ ring such that $R/\mathbf{B}(R)$ is quasi-Baer, then R is an $n \times n$ generalized matrix ring with rings having prime factors by the prime radical on the diagonal.*

Proof. The condition that the factor by the prime radical is quasi-Baer implies that this factor is a direct sum of prime rings. Therefore the result follows immediately from Theorem 3.9. □

We conclude the paper with the following question:

Question 4.8. Characterize the rings R in $\mathbf{P}_n$ whose maximal right ring of quotients is in $\mathbf{P}_k$ for some positive integer k?

Acknowledgments

The first author was supported by the National Research Development and Innovation Office, Hungary, NKHIH K119934, FAPESP, Brazil, Proc 2015/07195-7 during his stay at the University of São Paulo, and by the Vietnamese Institute of Mathematics for his stay in Hanoi, Vietnam.

The third author was supported by the National Research Foundation, South Africa, UID 81141. Any opinion, findings and conclusions or recommendations expressed in this material are those of the authors and therefore the National Research Foundation does not accept any liability in regard thereto.

The authors are grateful for the referee's efforts and encouraging comments.

References

[1] P.N. Ánh, G.F. Birkenmeier, L. van Wyk, Idempotents and structures of rings, Linear Multilinear Algebra 64 (10) (2016) 2002–2029.

[2] P.N. Ánh, L. van Wyk, Automorphism groups of generalized triangular matrix rings, Linear Algebra Appl. 434 (2011) 1018–1025.

[3] P.N. Ánh, L. van Wyk, Isomorphisms between strongly triangular matrix rings, Linear Algebra Appl. 438 (2013) 4374–4381.

[4] G.F. Birkenmeier, H.E. Heatherly, J.Y. Kim, J.K. Park, Triangular matrix representations, J. Algebra 230 (2000) 558–595.

[5] G.F. Birkenmeier, J.Y. Kim, J.K. Park, Semicentral reduced algebras, in: G.F. Birkenmeier, J.K. Park, J.S. Park (Eds.), International Symposium on Ring Theory, Birkhäuser, Boston, 2001, pp. 67–84.

[6] G.F. Birkenmeier, J.Y. Kim, J.K. Park, Triangular matrix representations of semiprimary rings, J. Algebra Appl. 2 (2002) 123–131.

[7] G.F. Birkenmeier, J.K. Park, S.T. Rizvi, Extensions of Rings and Modules, Birkhäuser/Springer, New York, 2013.

[8] C. Boboc, S. Dăscălescu, L. van Wyk, Isomorphisms between Morita context rings, Linear Multilinear Algebra 60 (2012) 545–563.

[9] B.J. Gardner, R. Wiegandt, Radical Theory of Rings, Marcel Dekker, New York, 2004.

[10] N. Jacobson, Structure of Rings, revised edition, American Mathematical Society Colloquium Publications, vol. 37, American Mathematical Society, Providence, R.I., 1964.

[11] P.A. Krylov, A.A. Tuganbaev, Modules over formal matrix rings, Fundam. Prikl. Mat. 15 (8) (2009) 145–211 (in Russian); translation in J. Math. Sci. (N.Y.) 171 (2) (2010) 248–295.

[12] L. Márki, J. Meyer, J. Szigeti, L. van Wyk, Matrix representations of finitely generated Grassmann algebras and some consequences, Isr. J. Math. 208 (2015) 373–384.

[13] J.C. McConnell, J.C. Robson, Noncommutative Noetherian Rings. With the Cooperation of L.W. Small, revised edition, Graduate Studies in Mathematics, vol. 30, American Mathematical Society, Providence, RI, 2001.

[14] B. Peirce, Linear associative algebra, Am. J. Math. 4 (1881) 97–229.

[15] A. Pollingher, A. Zaks, On Baer and quasi-Baer rings, Duke Math. J. 37 (1970) 127–138.

[16] E.C. Posner, Left valuation rings and simple radical rings, Trans. Am. Math. Soc. 107 (1963) 458–465.

[17] J. Szigeti, Linear algebra in lattices and nilpotent endomorphisms of semisimple modules, J. Algebra 319 (2008) 296–308.

[18] J. von Neumann, Continuous Geometry. Foreword by Israel Halperin, Princeton Mathematical Series, vol. 25, Princeton University Press, Princeton, N.J., 1960.