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A power Cayley-Hamilton identity for $n \times n$ matrices over a Lie nilpotent ring of index k

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For an $n \times n$ matrix A over a Lie nilpotent ring R of index k, with $k \geq 2$, we prove that an invariant "power" Cayley-Hamilton identity

$$\left(I_n\lambda_0^{(2)} + A\lambda_1^{(2)} + \cdots + A^{n^2-1}\lambda_{n^2-1}^{(2)} + A^{n^2}\lambda_{n^2}^{(2)}\right)^{2^{k-2}} = 0$$

of degree $n^2 2^{k-2}$ holds. The right coefficients $\lambda_i^{(2)} \in R$, $0 \leq i \leq n^2$ are not uniquely determined by A, and the cosets $\lambda_i^{(2)} + D$, with D the double commutator ideal $R[[R,R],R]R$ of R, appear in the so-called second right characteristic polynomial $p_{\overline{A},2}(x)$ of the natural image $\overline{A}$ of A in the $n \times n$ matrix ring $\mathrm{M}_n(R/D)$ over the factor ring R/D:

$$p_{\overline{A},2}(x) = (\lambda_0^{(2)} + D) + (\lambda_1^{(2)} + D)x + \cdots + (\lambda_{n^2-1}^{(2)} + D)x^{n^2-1} + (\lambda_{n^2}^{(2)} + D)x^{n^2}.$$

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1. Introduction

The Cayley-Hamilton theorem and the corresponding trace identity play a fundamental role in proving classical results about the polynomial and trace identities of the $n \times n$ matrix algebra $\mathrm{M}_n(K)$ over a field K (see, for example, [2], [3] and [13]).

In case of $\mathrm{char}(K) = 0$, Kemer's pioneering work (see [5]) on the T-ideals of associative algebras revealed the importance of the identities satisfied by the $n \times n$ matrices over the Grassmann (exterior) algebra

$$E = K \langle v_1, v_2, ..., v_i, ... \mid v_i v_j + v_j v_i = 0 \text{ for all } 1 \leq i \leq j \rangle$$

generated by the infinite sequence of anticommutative indeterminates $(v_i)_{i \geq 1}$.

Accordingly, the importance of matrices over non-commutative rings features prominently in the theory of PI-rings; indeed, this fact has been obvious for a long time in other branches of algebra, for example, in the structure theory of semisimple rings. Thus any Cayley-Hamilton type identity for such matrices seems to be of general interest.

In the general case (when R is an arbitrary non-commutative ring with 1) Paré and Schelter proved (see [9]) that a matrix $A \in \mathrm{M}_n(R)$ satisfies a monic identity in which the leading term is A^m for some large integer m, i.e., $m \geq 2^{2^{n-1}}$. The other summands in the identity are of the form $r_0 A r_1 A r_2 \cdots r_{l-1} A r_l$, with left scalar coefficient $r_0 \in R$, right scalar coefficient $r_l \in R$ and "sandwich" scalar coefficients $r_2, \ldots, r_{l-1} \in R$. An explicit monic identity for 2×2 matrices arising from the argument of [9] was given by Robson in [12]. Further results in this direction can be found in [10] and [11].

Obviously, by imposing extra algebraic conditions on the base ring R, we can expect "stronger" identities in $\mathrm{M}_n(R)$. A number of examples show that certain polynomial identities satisfied by R can lead to "canonical" constructions providing invariant Cayley-Hamilton identities for A of degree much lower than $2^{2^{n-1}}$.

If R satisfies the polynomial identity

$$[[[\ldots[[x_1, x_2], x_3], \ldots], x_k], x_{k+1}] = 0$$

of Lie nilpotency of index k (with $[x, y] = xy - yx$), then for a matrix $A \in \mathrm{M}_n(R)$, a left (and right) Cayley-Hamilton identity of degree n^k was constructed in [14] (see also [7]). Since E is Lie nilpotent of index $k = 2$, this identity for a matrix $A \in \mathrm{M}_n(E)$ is of degree n^2.

In [1], Domokos considered a slightly modified version of the mentioned identity, in which the left (as well as the right) coefficients are invariant under the conjugate action of $\mathrm{GL}_n(K)$ on $\mathrm{M}_n(E)$. For a 2×2 matrix $A \in \mathrm{M}_2(E)$, the left scalar coefficients of this Cayley-Hamilton identity are expressed as polynomials (over K) of the traces $\mathrm{tr}(A)$, $\mathrm{tr}(A^2)$ and $\mathrm{tr}(A^3)$.

If $\frac{1}{2} \in R$ and R satisfies the so-called weak Lie solvability identity

$$[[x, y], [x, z]] = 0,$$

then for a 2×2 matrix $A \in \mathrm{M}_2(R)$, a Cayley-Hamilton trace identity (of degree 4 in A) with sandwich coefficients was exhibited in [8]. If R satisfies the identity

$$[x_1, x_2, ..., x_{2^s}]_{\mathrm{solv}} = 0$$

of general Lie solvability, then a recursive construction (also in [8]) gives a similar Cayley-Hamilton trace identity (the degree of which depends on s) for a matrix $A \in \mathrm{M}_2(R)$.

In the present paper we consider an $n \times n$ matrix $A \in \mathrm{M}_n(R)$ over a ring R (with 1) satisfying the identity

$$[[x_1, y_1], z_1][[x_2, y_2], z_2] \cdots [[x_t, y_t], z_t] = 0,$$

and we prove that an invariant "power" Cayley-Hamilton identity of the form

$$\left(I_n \lambda_0^{(2)} + A\lambda_1^{(2)} + \cdots + A^{n^2-1}\lambda_{n^2-1}^{(2)} + A^{n^2}\lambda_{n^2}^{(2)}\right)^t = 0$$

holds, with certain right coefficients

$$\lambda_i^{(2)} \in R,\ 0 \le i \le n^2 - 1, \quad \text{and} \quad \lambda_{n^2}^{(2)} = n\{(n-1)!\}^{1+n},$$

which are only partially determined by A. The cosets $\lambda_i^{(2)} + D$, with D the double commutator ideal $R[[R, R], R]R$ of R, appear in the second right characteristic polynomial $p_{\overline{A},2}(x)$ of the natural image $\overline{A} \in \mathrm{M}_n(R/D)$ of A over the factor ring R/D:

$$p_{\overline{A},2}(x) = (\lambda_0^{(2)} + D) + (\lambda_1^{(2)} + D)x + \cdots + (\lambda_{n^2-1}^{(2)} + D)x^{n^2-1} + (\lambda_{n^2}^{(2)} + D)x^{n^2}.$$

We note that $[[x_1, y_1], z_1][[x_2, y_2], z_2] \cdots [[x_t, y_t], z_t] = 0$ is a typical identity of the ring $\mathrm{U}_t(R)$ of $t \times t$ upper triangular matrices over a ring R satisfying the identity $[[x, y], z] = 0$ (i.e., Lie nilpotency of index 2).

Finally, using a theorem of Jennings (see [4]), we prove that if R is Lie nilpotent of index k, then an identity of the form

$$\left(I_n \lambda_0^{(2)} + A\lambda_1^{(2)} + \cdots + A^{n^2-1}\lambda_{n^2-1}^{(2)} + A^{n^2}\lambda_{n^2}^{(2)}\right)^{2^{k-2}} = 0 \qquad (*)$$

holds for $A \in \mathrm{M}_n(R)$. The total degree of this identity (in A) is $n^2 2^{k-2}$, a much smaller integer than the degree n^k of A in the right Cayley-Hamilton identity

$$I_n \lambda_0^{(k)} + A\lambda_1^{(k)} + \cdots + A^{n^k-1}\lambda_{n^k-1}^{(k)} + A^{n^k}\lambda_{n^k}^{(k)} = 0 \qquad (**)$$

arising from the k-th right characteristic polynomial

$$p_{A,k}(x) = \lambda_0^{(k)} + \lambda_1^{(k)}x + \cdots + \lambda_{n^k-1}^{(k)}x^{n^k-1} + \lambda_{n^k}^{(k)}x^{n^k} \in R[x]$$

of A (see [14] and [16]). The advantage of $(**)$ is that all of the coefficients are on the right side of the powers of A, while the expansion of the power in $(*)$ yields a sum of products of the form $A^{i_1}\lambda_{i_1}A^{i_2}\lambda_{i_2}\cdots A^{i_s}\lambda_{i_s}$, with $s=2^{k-2}$.

In order to provide a self-contained treatment, we present the necessary prerequisites in sections 2 and 3.

2. Some results on Lie nilpotent rings

Let R be a ring, and let $[x,y]=xy-yx$ denote the additive commutator of the elements $x,y\in R$. It is well known that $(R,+,[\ ,\ ])$ is a Lie ring, $[y,x]=-[x,y]$ and $[[x,y],z]+[[y,z],x]+[[z,x],y]=0$ (the Jacobian identity).

For a sequence $x_1,x_2,\ldots,x_k$ of elements in R we use the notation $[x_1,x_2,\ldots,x_k]_k$ for the left normed commutator (Lie-)product:

$$[x_1]_1=x_1 \qquad \text{and} \qquad [x_1,x_2,\ldots,x_k]_k=[\ldots[[x_1,x_2],x_3],\ldots,x_k].$$

Clearly, we have

$$[x_1,x_2,\ldots,x_k,x_{k+1}]_{k+1}=[[x_1,x_2,\ldots,x_k]_k,x_{k+1}]=[[x_1,x_2],x_3,\ldots,x_k,x_{k+1}]_k.$$

A ring R is called Lie nilpotent of index k (or having property L_k) if

$$[x_1,x_2,\ldots,x_k,x_{k+1}]_{k+1}=0$$

is a polynomial identity on R. If R has property L_k, then $[x_1,x_2,\ldots,x_k]_k$ is central for all $x_1,x_2,\ldots,x_k\in R$.

A concise proof of Theorem 2.1 due to Jennings can be found in [17].

Theorem 2.1 *([4]). Let $k\geq 3$ be an integer and R be a ring with L_k. Then*

$$[x_1,x_2,\ldots,x_k]_k\cdot[y_1,y_2,\ldots,y_k]_k=0$$

for all $x_i,y_i\in R$, $1\leq i\leq k$. Thus the two-sided ideal

$$N=R\big\{[x_1,x_2,\ldots,x_k]_k\mid x_i\in R,1\leq i\leq k\big\}=\big\{[x_1,x_2,\ldots,x_k]_k\mid x_i\in R,1\leq i\leq k\big\}R$$

generated by the (central) elements $[x_1,x_2,\ldots,x_k]_k$ is nilpotent, with $N^2=\{0\}$.

Corollary 2.2 *([4]). If R is a ring with L_k $(k\geq 2)$, then the double commutator ideal*

$$D=R[[R,R],R]R=\{\textstyle\sum_{1\leq i\leq m}r_i[[a_i,b_i],c_i]s_i\mid r_i,a_i,b_i,c_i,s_i\in R,1\leq i\leq m\}\lhd R$$

is nilpotent, with $D^{2^{k-2}}=\{0\}$.

Proof. This follows from Theorem 2.1 by an easy induction on k. $\square$

3. The Lie nilpotent Cayley-Hamilton theorem

A Lie nilpotent analogue of classical determinant theory was developed in [14]; further details can be found in [1], [15] and [16]. Here we present the basic definitions and results about the sequences of right determinants and right characteristic polynomials, including the so-called Lie nilpotent right Cayley-Hamilton identities.

Let R be an arbitrary (possibly non-commutative) ring or algebra with 1, and let

$$\mathrm{S}_n = \mathrm{Sym}(\{1, \ldots, n\})$$

denote the symmetric group of all permutations of the set $\{1, 2, \ldots, n\}$. If $A = [a_{i,j}]$ is an $n \times n$ matrix over R, then the element

$$\begin{aligned}\mathrm{sdet}(A) &= \sum_{\tau,\rho \in \mathrm{S}_n} \mathrm{sgn}(\rho) a_{\tau(1),\rho(\tau(1))} \cdots a_{\tau(t),\rho(\tau(t))} \cdots a_{\tau(n),\rho(\tau(n))} \\ &= \sum_{\alpha,\beta \in \mathrm{S}_n} \mathrm{sgn}(\alpha)\mathrm{sgn}(\beta) a_{\alpha(1),\beta(1)} \cdots a_{\alpha(t),\beta(t)} \cdots a_{\alpha(n),\beta(n)}\end{aligned}$$

of R is called the symmetric determinant of A.

The (r, s)-entry of the symmetric adjoint matrix $A^* = [a^*_{r,s}]$ of A is defined as follows:

$$\begin{aligned}a^*_{r,s} &= \sum_{\tau,\rho} \mathrm{sgn}(\rho) a_{\tau(1),\rho(\tau(1))} \cdots a_{\tau(s-1),\rho(\tau(s-1))} a_{\tau(s+1),\rho(\tau(s+1))} \cdots a_{\tau(n),\rho(\tau(n))} \\ &= \sum_{\alpha,\beta} \mathrm{sgn}(\alpha)\mathrm{sgn}(\beta) a_{\alpha(1),\beta(1)} \cdots a_{\alpha(s-1),\beta(s-1)} a_{\alpha(s+1),\beta(s+1)} \cdots a_{\alpha(n),\beta(n)} \ ,\end{aligned}$$

where the first sum is taken over all $\tau, \rho \in \mathrm{S}_n$ with $\tau(s) = s$ and $\rho(s) = r$, while the second sum is taken over all $\alpha, \beta \in \mathrm{S}_n$ with $\alpha(s) = s$ and $\beta(s) = r$. We note that the (r, s) entry of A^* is exactly the signed symmetric determinant $(-1)^{r+s}\mathrm{sdet}(A_{s,r})$ of the $(n-1) \times (n-1)$ minor $A_{s,r}$ of A arising from the deletion of the s-th row and the r-th column of A.

The trace $\mathrm{tr}(M)$ of a matrix $M \in \mathrm{M}_n(R)$ is the sum of the diagonal entries of M. In spite of the fact that the well known identity $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ is no longer valid for matrices $A, B \in \mathrm{M}_n(R)$ over a non-commutative R, we still have (see [16])

$$\mathrm{sdet}(A) = \mathrm{tr}(AA^*) = \mathrm{tr}(A^*A).$$

If R is commutative, then $\mathrm{sdet}(A) = n!\mathrm{det}(A)$ and $A^* = (n-1)!\mathrm{adj}(A)$, where $\det(A)$ and $\mathrm{adj}(A)$ denote the ordinary determinant and adjoint, respectively, of A.

The right adjoint sequence $(P_k)_{k\geq 1}$ of a matrix $A \in \mathrm{M}_n(R)$ is defined by the following recursion:

$$P_1 = A^* \qquad \text{and} \qquad P_{k+1} = (AP_1 \cdots P_k)^*$$

for $k \geq 1$. The k-th right adjoint of A is defined as

$$\mathrm{radj}_{(k)}(A) = nP_1 \cdots P_k.$$

The k-th right determinant of A is the trace of $AP_1 \cdots P_k$:

$$\mathrm{rdet}_{(k)}(A) = \mathrm{tr}(AP_1 \cdots P_k).$$

The following theorem shows that $\mathrm{radj}_{(k)}(A)$ and $\mathrm{rdet}_{(k)}(A)$ can play a role similar to that played by the ordinary adjoint and determinant, respectively, in the commutative case.

Theorem 3.1 *([14], [16]). If $\frac{1}{n} \in R$ and the ring R is Lie nilpotent of index k, then for a matrix $A \in \mathrm{M}_n(R)$ we have*

$$A\mathrm{radj}_{(k)}(A) = nAP_1 \cdots P_k = \mathrm{rdet}_{(k)}(A)I_n.$$

The above Theorem 3.1 is not used explicitly in the sequel, however it helps our understanding and serves as a starting point in the proof of Theorem 3.3.

Let $R[x]$ denote the ring of polynomials in the single commuting indeterminate x, with coefficients in R. The k-th right characteristic polynomial of A is the k-th right determinant of the $n \times n$ matrix $xI_n - A$ in $\mathrm{M}_n(R[x])$:

$$p_{A,k}(x) = \mathrm{rdet}_{(k)}(xI_n - A).$$

Proposition 3.2 *([14], [16]). The k-th right characteristic polynomial $p_{A,k}(x) \in R[x]$ of $A \in \mathrm{M}_n(R)$ is of the form*

$$p_{A,k}(x) = \lambda_0^{(k)} + \lambda_1^{(k)}x + \cdots + \lambda_{n^k-1}^{(k)}x^{n^k-1} + \lambda_{n^k}^{(k)}x^{n^k},$$

where $\lambda_0^{(k)}, \lambda_1^{(k)}, \ldots, \lambda_{n^k-1}^{(k)}, \lambda_{n^k}^{(k)} \in R$ and $\lambda_{n^k}^{(k)} = n\{(n-1)!\}^{1+n+n^2+\cdots+n^{k-1}}$.

Theorem 3.3 *([14], [16]). If $\frac{1}{n} \in R$ and the ring R is Lie nilpotent of index k, then a right Cayley-Hamilton identity*

$$(A)p_{A,k} = I_n\lambda_0^{(k)} + A\lambda_1^{(k)} + \cdots + A^{n^k-1}\lambda_{n^k-1}^{(k)} + A^{n^k}\lambda_{n^k}^{(k)} = 0$$

with right scalar coefficients holds for $A \in \mathrm{M}_n(R)$. We also have $(A)u = 0$, where $u(x) = p_{A,k}(x)h(x)$ and $h(x) \in R[x]$ is arbitrary.

Theorem 3.4 *([1]). If $\frac{1}{2} \in R$ and the ring R is Lie nilpotent of index 2, then for a 2×2 matrix $A \in \mathrm{M}_2(R)$ the right Cayley-Hamilton identity in the above 3.3 can be written in the following trace form:*

$$
\begin{aligned}
(A)p_{A,2} = I_2 & \left(\frac{1}{2}\mathrm{tr}^4(A) + \frac{1}{2}\mathrm{tr}^2(A^2) + \frac{1}{4}\mathrm{tr}^2(A)\mathrm{tr}(A^2) - \frac{5}{4}\mathrm{tr}(A^2)\mathrm{tr}^2(A) + \left[\mathrm{tr}(A^3), \mathrm{tr}(A)\right]\right) \\
& + A\Big(\mathrm{tr}(A)\mathrm{tr}(A^2) + \mathrm{tr}(A^2)\mathrm{tr}(A) - 2\mathrm{tr}^3(A)\Big) + A^2\Big(4\mathrm{tr}^2(A) - 2\mathrm{tr}(A^2)\Big) \\
& - A^3\Big(4\mathrm{tr}(A)\Big) + 2A^4 = 0.
\end{aligned}
$$

Corollary 3.5 *([1]). If $\frac{1}{2} \in R$ and the ring R is Lie nilpotent of index 2, then, for every $A \in \mathrm{M}_2(R)$,*

$$\mathrm{tr}(A) = \mathrm{tr}(A^2) = 0 \quad \textit{imply that} \quad A^4 = 0.$$

4. Matrices over a ring with $[[x_1, y_1], z_1][[x_2, y_2], z_2] \cdots [[x_t, y_t], z_t] = 0$

We shall make use of the following well known fact.

Proposition 4.1. *If $[[x_1, y_1], z_1][[x_2, y_2], z_2] \cdots [[x_t, y_t], z_t] = 0$ is a polynomial identity on a ring R, then $D^t = \{0\}$, with D the ideal $R[[R, R], R]R$ of R.*

Theorem 4.2. *If $\frac{1}{2} \in R$ and $A \in \mathrm{M}_n(R)$ is a matrix over a ring R satisfying the polynomial identity $[[x_1, y_1], z_1][[x_2, y_2], z_2] \cdots [[x_t, y_t], z_t] = 0$, then an invariant "power" Cayley-Hamilton identity of the form*

$$\left(I_n\lambda_0^{(2)} + A\lambda_1^{(2)} + \cdots + A^{n^2-1}\lambda_{n^2-1}^{(2)} + A^{n^2}\lambda_{n^2}^{(2)}\right)^t = 0$$

holds, with certain right coefficients

$$\lambda_i^{(2)} \in R,\ 0 \le i \le n^2 - 1, \quad \textit{and} \quad \lambda_{n^2}^{(2)} = n\big\{(n-1)!\big\}^{1+n}$$

(only partially determined by A). The cosets $\lambda_i^{(2)} + D$ with $D = R[[R, R], R]R \lhd R$ appear in the second right characteristic polynomial $p_{\overline{A},2}(x)$ of the natural image $\overline{A} \in M_n(R/D)$ of A over the factor ring R/D:

$$p_{\overline{A},2}(x) = (\lambda_0^{(2)} + D) + (\lambda_1^{(2)} + D)x + \cdots + (\lambda_{n^2-1}^{(2)} + D)x^{n^2-1} + (\lambda_{n^2}^{(2)} + D)x^{n^2} \in (R/D)[x].$$

Proof. Consider the factor ring R/D, where $D = R[[R, R], R]R \lhd R$ is the double commutator ideal. If $A = [a_{i,j}] \in \mathrm{M}_n(R)$, then we use the notation $\overline{A} = [a_{i,j} + D]$ for the image of A in $\mathrm{M}_n(R/D)$. Since R/D is Lie nilpotent of index 2, Theorem 3.3 implies that, in $\mathrm{M}_n(R/D)$,

$$(\overline{A})p_{\overline{A},2} = \overline{I_n}(\lambda_0^{(2)} + D) + \overline{A}(\lambda_1^{(2)} + D) + \cdots + (\overline{A})^{n^2-1}(\lambda_{n^2-1}^{(2)} + D) + (\overline{A})^{n^2}(\lambda_{n^2}^{(2)} + D) = \overline{0},$$

where

$$\begin{aligned} p_{\overline{A},2}(x) &= \mathrm{rdet}_{(k)}(x\overline{I_n} - \overline{A}) \\ &= (\lambda_0^{(2)} + D) + (\lambda_1^{(2)} + D)x + \cdots + (\lambda_{n^2-1}^{(2)} + D)x^{n^2-1} + (\lambda_{n^2}^{(2)} + D)x^{n^2} \end{aligned}$$

is the second right characteristic polynomial of $\overline{A}$ in $(R/D)[x]$. Clearly,

$$\begin{aligned} &\overline{I_n\lambda_0^{(2)} + A\lambda_1^{(2)} + \cdots + A^{n^2-1}\lambda_{n^2-1}^{(2)} + A^{n^2}\lambda_{n^2}^{(2)}} \\ &\quad = \overline{I_n}(\lambda_0^{(2)} + D) + \overline{A}(\lambda_1^{(2)} + D) + \cdots + (\overline{A})^{n^2-1}(\lambda_{n^2-1}^{(2)} + D) + (\overline{A})^{n^2}(\lambda_{n^2}^{(2)} + D) = \overline{0} \end{aligned}$$

implies that

$$I_n\lambda_0^{(2)} + A\lambda_1^{(2)} + \cdots + A^{n^2-1}\lambda_{n^2-1}^{(2)} + A^{n^2}\lambda_{n^2}^{(2)} \in \mathrm{M}_n(D).$$

Now $D^t = \{0\}$ is a consequence of Proposition 4.1, whence $(\mathrm{M}_n(D))^t = \{0\}$ and

$$\left(I_n\lambda_0^{(2)} + A\lambda_1^{(2)} + \cdots + A^{n^2-1}\lambda_{n^2-1}^{(2)} + A^{n^2}\lambda_{n^2}^{(2)}\right)^t = 0$$

follows. □

Remark 4.3. If $[x_1, y_1][x_2, y_2]\cdots[x_t, y_t] = 0$ is a polynomial identity on R and $A \in \mathrm{M}_n(R)$, then using the commutator ideal $T = R[R, R]R$ and the natural image $\widetilde{A} \in \mathrm{M}_n(R/T)$ of A over the commutative ring R/T, a similar argument as in the proof of Theorem 4.2 gives that

$$\left(I_n\lambda_0^{(1)} + A\lambda_1^{(1)} + \cdots + A^{n-1}\lambda_{n-1}^{(1)} + A^n\lambda_n^{(1)}\right)^t = 0$$

holds, where $p_{\widetilde{A},1}(x) = (\lambda_0^{(1)} + T) + (\lambda_1^{(1)} + T)x + \cdots + (\lambda_{n-1}^{(1)} + T)x^{n-1} + (\lambda_n^{(1)} + T)x^n$ is the $n!$ times scalar multiple of the classical characteristic polynomial of $\widetilde{A}$ in $(R/T)[x]$ with $\lambda_n^{(1)} = n!$.

Theorem 4.4. *If $\frac{1}{2} \in R$ and $A \in \mathrm{M}_n(R)$ is a matrix over a Lie nilpotent ring R of index k, then an invariant "power" Cayley-Hamilton identity of the form*

$$\left(I_n\lambda_0^{(2)} + A\lambda_1^{(2)} + \cdots + A^{n^2-1}\lambda_{n^2-1}^{(2)} + A^{n^2}\lambda_{n^2}^{(2)}\right)^{2^{k-2}} = 0$$

holds, with certain right coefficients

$$\lambda_i^{(2)} \in R,\ 0 \le i \le n^2 - 1, \quad \text{and} \quad \lambda_{n^2}^{(2)} = n\{(n-1)!\}^{1+n}$$

(only partially determined by A). The cosets $\lambda_i^{(2)} + D$ with $D = R[[R, R], R]R \lhd R$ appear in the second right characteristic polynomial $p_{\overline{A},2}(x)$ of the natural image $\overline{A} \in M_n(R/D)$ of A over the factor ring R/D:

$$p_{\overline{A},2}(x) = (\lambda_0^{(2)} + D) + (\lambda_1^{(2)} + D)x + \cdots + (\lambda_{n^2-1}^{(2)} + D)x^{n^2-1} + (\lambda_{n^2}^{(2)} + D)x^{n^2} \in (R/D)[x].$$

Proof. According to Jennings's result (Corollary 2.2), the double commutator ideal

$$D = R[[R,R],R]R = \left\{ \sum_{1 \le i \le m} r_i[[a_i,b_i],c_i]s_i \mid r_i,a_i,b_i,c_i,s_i \in R, 1 \le i \le m \right\} \lhd R$$

is nilpotent, with $D^{2^{k-2}} = \{0\}$. Thus the application of Theorem 4.2 gives our identity. □

Remark 4.5. If $k = 2$, then $R[[R,R],R]R = \{0\}$, and the identity in Theorem 4.4 remains the same as the Lie nilpotent right Cayley-Hamilton identity in Theorem 3.3.

Remark 4.6. The Grassmann algebra

$$E = K\left\langle v_1, v_2, ..., v_i, ... \mid v_i v_j + v_j v_i = 0 \text{ for all } 1 \le i \le j \right\rangle$$

over a field K (with $2 \neq 0$) has property L_2, and

$$[v_1, v_2] \cdot [v_3, v_4] \cdot \ \cdots \ \cdot [v_{2t-1}, v_{2t}] = 2^t v_1 v_2 \cdots v_{2t} \neq 0$$

shows that L_2 does not imply the identity $[x_1, y_1][x_2, y_2] \cdots [x_t, y_t] = 0$ for any t. Thus the identity mentioned in Remark 4.3 cannot be used directly to derive new identities for matrices over a Lie nilpotent ring of index $k \ge 2$. However, as the referee pointed out, the following (weak) version of Latyshev's theorem provides a possibility to use Remark 4.3 in order to obtain another remarkable "power" Cayley-Hamilton identity.

Theorem *([6]). If S is a Lie nilpotent algebra (over an infinite field) of index k, generated by m elements, then there exists an integer $d = d(k,m)$ such that S satisfies the polynomial identity $[x_1, y_1][x_2, y_2] \cdots [x_d, y_d] = 0$. (In the original version S satisfies a so-called nonmatrix polynomial identity.)*

If $A \in \mathrm{M}_n(R)$ is a matrix over a Lie nilpotent algebra (over an infinite field) R of index k, then $A \in \mathrm{M}_n(S)$, where S is the (unitary) subalgebra generated by the n^2 entries of A. Thus $[x_1, y_1][x_2, y_2] \cdots [x_d, y_d] = 0$ is a polynomial identity on S with $d = d(k, n^2)$ and Remark 4.3 gives that an identity

$$\left(I_n \lambda_0^{(1)} + A\lambda_1^{(1)} + \cdots + A^{n-1}\lambda_{n-1}^{(1)} + A^n \lambda_n^{(1)} \right)^{d(k,n^2)} = 0$$

of degree $nd(k, n^2)$ holds. Unfortunately, our knowledge about $d(k, n^2)$ is very limited, the fact that $d(2, 4) = 3$ was mentioned by the referee.

Remark 4.7. If R is an algebra over a field K of characteristic zero, then the invariance of the identities in 4.2 and 4.4 means that $p_{\overline{T^{-1}AT},2}(x) = p_{\overline{A},2}(x)$ holds for any $T \in \mathrm{GL}_n(K)$ (see [1]).

Corollary 4.8. *If $\frac{1}{2} \in R$ and the ring R is Lie nilpotent of index k, then, for every $A \in \mathrm{M}_2(R)$,*

$$\mathrm{tr}(A) = \mathrm{tr}(A^2) = 0 \quad \textit{imply that} \quad A^{2^k} = 0.$$

Proof. Using $D = R[[R,R],R]R \lhd R$, $\overline{A} \in \mathrm{M}_2(R/D)$ and

$$\mathrm{tr}(\overline{A}) = \mathrm{tr}(A) + D = 0,\ \mathrm{tr}((\overline{A})^2) = \mathrm{tr}(\overline{A^2}) = \mathrm{tr}(A^2) + D = 0,$$

the application of Corollary 3.5 ensures that $\overline{A^4} = \left(\overline{A}\right)^4 = \overline{0}$. Thus the nilpotency of D ($D^{2^{k-2}} = \{0\}$) gives that $A^{2^k} = \left(A^4\right)^{2^{k-2}} = 0$. □

Remark 4.9. According to the following important observation of the referee, the use of Latyshev's theorem gives an $n \times n$ variant of Corollary 4.8. If $A \in \mathrm{M}_n(R)$ is a matrix over a Lie nilpotent algebra (over a field K of characteristic zero) R of index k, then we prove that

$$\mathrm{tr}(A) = \mathrm{tr}(A^2) = \cdots = \mathrm{tr}(A^n) = 0$$

implies that $A^{nd(k,n^2)} = 0$. Indeed, $A \in \mathrm{M}_n(S)$, where $S \subseteq R$ is the (unitary) subalgebra of R generated by the entries of A. Now consider the natural image $\widetilde{A} \in \mathrm{M}_n(S/S[S,S]S)$ of A. The application of the well known fact that

$$\mathrm{tr}(\widetilde{A}) = \mathrm{tr}((\widetilde{A})^2) = \cdots = \mathrm{tr}((\widetilde{A})^n) = \widetilde{0}$$

implies that $\widetilde{A^n} = (\widetilde{A})^n = \widetilde{0}$ (it is a consequence of the Newton trace formulae for the coefficients of the characteristic polynomial $p_{\widetilde{A},1}(x) \in (S/S[S,S]S)[x]$, where the factor $S/S[S,S]S$ is a commutative algebra over K). Since $(S[S,S]S)^{d(k,n^2)} = \{0\}$ by Latyshev's theorem and $A^n \in \mathrm{M}_n(S[S,S]S)$, we obtain the desired equality.

Declaration of competing interest

There is no competing interest.

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