A power Cayley-Hamilton identity for $n \times n$ matrices over a Lie nilpotent ring of index $k$

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**Abstract**

For an $n \times n$ matrix $A$ over a Lie nilpotent ring $R$ of index $k$, with $k \geq 2$, we prove that an invariant “power” Cayley-Hamilton identity

$$\left( I_n \lambda_0^{(2)} + A \lambda_1^{(2)} + \cdots + A^{n^2-1} \lambda_{n^2-1}^{(2)} + A^{n^2} \lambda_{n^2}^{(2)} \right)^{2^{k-2}} = 0$$

of degree $n^{2^{k-2}}$ holds. The right coefficients $\lambda_i^{(2)} \in R$, $0 \leq i \leq n^2$ are not uniquely determined by $A$, and the cosets of $D$, with $D$ the double commutator ideal $[R[[R, R]], R]$ of $R$, appear in the so-called second right characteristic polynomial $p_{A,2}(x)$ of the natural image $\overline{A}$ of $A$ in the $n \times n$ matrix ring $M_n(R/D)$ over the factor ring $R/D$:

$$p_{A,2}(x) = (\lambda_0^{(2)} + D) + (\lambda_1^{(2)} + D)x + \cdots + (\lambda_{n^2-1}^{(2)} + D)x^{n^2-1} + (\lambda_{n^2}^{(2)} + D)x^{n^2}.$$
1. Introduction

The Cayley-Hamilton theorem and the corresponding trace identity play a fundamental role in proving classical results about the polynomial and trace identities of the $n \times n$ matrix algebra $M_n(K)$ over a field $K$ (see, for example, [2], [3] and [13]).

In case of $\text{char}(K) = 0$, Kemer’s pioneering work (see [5]) on the T-ideals of associative algebras revealed the importance of the identities satisfied by the $n \times n$ matrices over the Grassmann (exterior) algebra

$$E = K \langle v_1, v_2, \ldots, v_i, \ldots \mid v_i v_j + v_j v_i = 0 \text{ for all } 1 \leq i \leq j \rangle$$

generated by the infinite sequence of anticommutative indeterminates $(v_i)_{i \geq 1}$.

Accordingly, the importance of matrices over non-commutative rings features prominently in the theory of PI-rings; indeed, this fact has been obvious for a long time in other branches of algebra, for example, in the structure theory of semisimple rings. Thus any Cayley-Hamilton type identity for such matrices seems to be of general interest.

In the general case (when $R$ is an arbitrary non-commutative ring with 1) Paré and Schelter proved (see [9]) that a matrix $A \in M_n(R)$ satisfies a monic identity in which the leading term is $A^m$ for some large integer $m$, i.e., $m \geq 2^{n-1}$. The other summands in the identity are of the form $r_0 A r_1 A r_2 \cdots r_{l-1} A r_l$, with left scalar coefficient $r_0 \in R$, right scalar coefficient $r_1 \in R$ and “sandwich” scalar coefficients $r_2, \ldots, r_{l-1} \in R$. An explicit monic identity for $2 \times 2$ matrices arising from the argument of [9] was given by Robson in [12]. Further results in this direction can be found in [10] and [11].

Obviously, by imposing extra algebraic conditions on the base ring $R$, we can expect “stronger” identities in $M_n(R)$. A number of examples show that certain polynomial identities satisfied by $R$ can lead to “canonical” constructions providing invariant Cayley-Hamilton identities for $A$ of degree much lower than $2^{n-1}$.

If $R$ satisfies the polynomial identity

$$[[[\cdots[[x_1, x_2], x_3], \ldots], x_k], x_{k+1}] = 0$$

of Lie nilpotency of index $k$ (with $[x, y] = xy - yx$), then for a matrix $A \in M_n(R)$, a left (and right) Cayley-Hamilton identity of degree $n^k$ was constructed in [14] (see also [7]). Since $E$ is Lie nilpotent of index $k = 2$, this identity for a matrix $A \in M_n(E)$ is of degree $n^2$.

In [1], Domokos considered a slightly modified version of the mentioned identity, in which the left (as well as the right) coefficients are invariant under the conjugate action of $\text{GL}_n(K)$ on $M_n(E)$. For a $2 \times 2$ matrix $A \in M_2(E)$, the left scalar coefficients of this Cayley-Hamilton identity are expressed as polynomials (over $K$) of the traces $\text{tr}(A)$, $\text{tr}(A^2)$ and $\text{tr}(A^3)$.

If $\frac{1}{2} \in R$ and $R$ satisfies the so-called weak Lie solvability identity

$$[[x, y], [x, z]] = 0,$$
then for a $2 \times 2$ matrix $A \in M_2(R)$, a Cayley-Hamilton trace identity (of degree 4 in $A$) with sandwich coefficients was exhibited in [8]. If $R$ satisfies the identity

$$[x_1, x_2, \ldots, x_{2^s}]_{\text{solv}} = 0$$

of general Lie solvability, then a recursive construction (also in [8]) gives a similar Cayley-Hamilton trace identity (the degree of which depends on $s$) for a matrix $A \in M_2(R)$.

In the present paper we consider an $n \times n$ matrix $A \in M_n(R)$ over a ring $R$ (with 1) satisfying the identity

$$[[x_1, y_1], z_1][[x_2, y_2], z_2] \cdots [[x_t, y_t], z_t] = 0,$$

and we prove that an invariant “power” Cayley-Hamilton identity of the form

$$(I_n \lambda_0^{(2)} + A \lambda_1^{(2)} + \cdots + A^{n^2-1} \lambda_{n^2-1}^{(2)} + A^n \lambda_n^{(2)})^t = 0$$

holds, with certain right coefficients

$$\lambda_i^{(2)} \in R, \ 0 \leq i \leq n^2 - 1, \ \text{and} \ \lambda_n^{(2)} = n\{(n-1)\}^{1+n},$$

which are only partially determined by $A$. The cosets $\lambda_i^{(2)} + D$, with $D$ the double commutator ideal $R[[R, R], R]R$ of $R$, appear in the second right characteristic polynomial $p_{\lambda,2}(x)$ of the natural image $\lambda \in M_n(R/D)$ of $A$ over the factor ring $R/D$:

$$p_{\lambda,2}(x) = (\lambda_0^{(2)} + D) + (\lambda_1^{(2)} + D)x + \cdots + (\lambda_{n^2-1}^{(2)} + D)x^{n^2-1} + (\lambda_n^{(2)} + D)x^{n^2}.$$

We note that $[[x_1, y_1], z_1][[x_2, y_2], z_2] \cdots [[x_t, y_t], z_t] = 0$ is a typical identity of the ring $U_t(R)$ of $t \times t$ upper triangular matrices over a ring $R$ satisfying the identity $[[x, y], z] = 0$ (i.e., Lie nilpotency of index 2).

Finally, using a theorem of Jennings (see [4]), we prove that if $R$ is Lie nilpotent of index $k$, then an identity of the form

$$(I_n \lambda_0^{(2)} + A \lambda_1^{(2)} + \cdots + A^{n^2-1} \lambda_{n^2-1}^{(2)} + A^n \lambda_n^{(2)})^{2k-2} = 0 \quad (*)$$

holds for $A \in M_n(R)$. The total degree of this identity (in $A$) is $n^22^{k-2}$, a much smaller integer than the degree $n^k$ of $A$ in the right Cayley-Hamilton identity

$$I_n \lambda_0^{(k)} + A \lambda_1^{(k)} + \cdots + A^{n^k-1} \lambda_{n^k-1}^{(k)} + A^n \lambda_n^{(k)} = 0 \quad (**).$$

arising from the $k$-th right characteristic polynomial

$$p_{A,k}(x) = \lambda_0^{(k)} + \lambda_1^{(k)}x + \cdots + \lambda_{n^k-1}^{(k)}x^{n^k-1} + \lambda_n^{(k)}x^{n^k} \in R[x]$$
of \( A \) (see [14] and [16]). The advantage of (**) is that all of the coefficients are on the right side of the powers of \( A \), while the expansion of the power in (*) yields a sum of products of the form \( A^{i_1}\lambda_{i_1}A^{i_2}\lambda_{i_2}\cdots A^{i_s}\lambda_{i_s} \), with \( s = 2^{k-2} \).

In order to provide a self-contained treatment, we present the necessary prerequisites in sections 2 and 3.

2. Some results on Lie nilpotent rings

Let \( R \) be a ring, and let \([x, y] = xy - yx\) denote the additive commutator of the elements \( x, y \in R \). It is well known that \((R, +, [ , ])\) is a Lie ring, \([y, x] = -[x, y]\) and \([[x, y], z] + [[y, z], x] + [[z, x], y] = 0\) (the Jacobian identity).

For a sequence \( x_1, x_2, \ldots, x_k \) of elements in \( R \) we use the notation \([x_1, x_2, \ldots, x_k]_k\) for the left normed commutator (Lie-)product:

\[
[x_1]_1 = x_1 \quad \text{and} \quad [x_1, x_2, \ldots, x_k]_k = [\ldots[[x_1, x_2], x_3], \ldots, x_k].
\]

Clearly, we have

\[
[x_1, x_2, \ldots, x_k, x_{k+1}]_{k+1} = [[x_1, x_2, \ldots, x_k]_k, x_{k+1}] = [[x_1, x_2], x_3, \ldots, x_k, x_{k+1}]_k.
\]

A ring \( R \) is called Lie nilpotent of index \( k \) (or having property \( L_k \)) if

\[
[x_1, x_2, \ldots, x_k, x_{k+1}]_{k+1} = 0
\]

is a polynomial identity on \( R \). If \( R \) has property \( L_k \), then \([x_1, x_2, \ldots, x_k]_k\) is central for all \( x_1, x_2, \ldots, x_k \in R \).

A concise proof of Theorem 2.1 due to Jennings can be found in [17].

**Theorem 2.1 ([4]).** Let \( k \geq 3 \) be an integer and \( R \) be a ring with \( L_k \). Then

\[
[x_1, x_2, \ldots, x_k]_k \cdot [y_1, y_2, \ldots, y_k]_k = 0
\]

for all \( x_i, y_i \in R, 1 \leq i \leq k \). Thus the two-sided ideal

\[
N = R\{[x_1, x_2, \ldots, x_k]_k \mid x_i \in R, 1 \leq i \leq k\} = \{[x_1, x_2, \ldots, x_k]_k \mid x_i \in R, 1 \leq i \leq k\} R
\]

generated by the (central) elements \([x_1, x_2, \ldots, x_k]_k\) is nilpotent, with \( N^2 = \{0\} \).

**Corollary 2.2 ([4]).** If \( R \) is a ring with \( L_k \) \( (k \geq 2) \), then the double commutator ideal

\[
D = R[[R, R], R]R = \{\sum_{1 \leq i \leq m} r_i[[a_i, b_i], c_i]s_i \mid r_i, a_i, b_i, c_i, s_i \in R, 1 \leq i \leq m\} \vartriangleleft R
\]

is nilpotent, with \( D^{2k-2} = \{0\} \).

**Proof.** This follows from Theorem 2.1 by an easy induction on \( k \).
3. The Lie nilpotent Cayley-Hamilton theorem

A Lie nilpotent analogue of classical determinant theory was developed in [14]; further details can be found in [1], [15] and [16]. Here we present the basic definitions and results about the sequences of right determinants and right characteristic polynomials, including the so-called Lie nilpotent right Cayley-Hamilton identities.

Let \( R \) be an arbitrary (possibly non-commutative) ring or algebra with 1, and let

\[
S_n = \text{Sym}(\{1, \ldots, n\})
\]
denote the symmetric group of all permutations of the set \( \{1, 2, \ldots, n\} \). If \( A = [a_{i,j}] \) is an \( n \times n \) matrix over \( R \), then the element

\[
\text{sdet}(A) = \sum_{\tau, \rho \in S_n} \text{sgn}(\rho) a_{\tau(1), \rho(\tau(1))} \cdots a_{\tau(t), \rho(\tau(t))} \cdots a_{\tau(n), \rho(\tau(n))} = \sum_{\alpha, \beta \in S_n} \text{sgn}(\alpha)\text{sgn}(\beta) a_{\alpha(1), \beta(1)} \cdots a_{\alpha(t), \beta(t)} \cdots a_{\alpha(n), \beta(n)}
\]
of \( R \) is called the symmetric determinant of \( A \).

The \((r, s)\)-entry of the symmetric adjoint matrix \( A^* = [a^*_{r,s}] \) of \( A \) is defined as follows:

\[
a^*_{r,s} = \sum_{\tau, \rho} \text{sgn}(\rho) a_{\tau(1), \rho(\tau(1))} \cdots a_{\tau(s-1), \rho(\tau(s-1))} a_{\tau(s), \rho(\tau(s))} \cdots a_{\tau(n), \rho(\tau(n))} = \sum_{\alpha, \beta} \text{sgn}(\alpha)\text{sgn}(\beta) a_{\alpha(1), \beta(1)} \cdots a_{\alpha(s-1), \beta(s-1)} a_{\alpha(s), \beta(s)} \cdots a_{\alpha(n), \beta(n)}
\]
where the first sum is taken over all \( \tau, \rho \in S_n \) with \( \tau(s) = s \) and \( \rho(s) = r \), while the second sum is taken over all \( \alpha, \beta \in S_n \) with \( \alpha(s) = s \) and \( \beta(s) = r \). We note that the \((r, s)\) entry of \( A^* \) is exactly the signed symmetric determinant \((-1)^{r+s}\text{sdet}(A_{s,r})\) of the \((n-1) \times (n-1)\) minor \( A_{s,r} \) of \( A \) arising from the deletion of the \( s \)-th row and the \( r \)-th column of \( A \).

The trace \( \text{tr}(M) \) of a matrix \( M \in M_n(R) \) is the sum of the diagonal entries of \( M \). In spite of the fact that the well known identity \( \text{tr}(AB) = \text{tr}(BA) \) is no longer valid for matrices \( A, B \in M_n(R) \) over a non-commutative \( R \), we still have (see [16])

\[
\text{sdet}(A) = \text{tr}(AA^*) = \text{tr}(A^*A).
\]

If \( R \) is commutative, then \( \text{sdet}(A) = n!\text{det}(A) \) and \( A^* = (n-1)!\text{adj}(A) \), where \( \text{det}(A) \) and \( \text{adj}(A) \) denote the ordinary determinant and adjoint, respectively, of \( A \).

The right adjoint sequence \((P_k)_{k \geq 1}\) of a matrix \( A \in M_n(R) \) is defined by the following recursion:

\[
P_1 = A^* \quad \text{and} \quad P_{k+1} = (AP_k \cdots P_1)^*
\]
for \( k \geq 1 \). The \( k \)-th right adjoint of \( A \) is defined as

\[
\mathrm{radj}_{(k)}(A) = nP_1 \cdots P_k.
\]

The \( k \)-th right determinant of \( A \) is the trace of \( AP_1 \cdots P_k \):

\[
\mathrm{rdet}_{(k)}(A) = \mathrm{tr}(AP_1 \cdots P_k).
\]

The following theorem shows that \( \mathrm{radj}_{(k)}(A) \) and \( \mathrm{rdet}_{(k)}(A) \) can play a role similar to that played by the ordinary adjoint and determinant, respectively, in the commutative case.

**Theorem 3.1** ([14], [16]). If \( \frac{1}{n} \in R \) and the ring \( R \) is Lie nilpotent of index \( k \), then for a matrix \( A \in M_n(R) \) we have

\[
A \mathrm{adj}_{(k)}(A) = nAP_1 \cdots P_k = \mathrm{rdet}_{(k)}(A)I_n.
\]

The above Theorem 3.1 is not used explicitly in the sequel, however it helps our understanding and serves as a starting point in the proof of Theorem 3.3.

Let \( R[x] \) denote the ring of polynomials in the single commuting indeterminate \( x \), with coefficients in \( R \). The \( k \)-th right characteristic polynomial of \( A \) is the \( k \)-th right determinant of the \( n \times n \) matrix \( xI_n - A \) in \( M_n(R[x]) \):

\[
p_{A,k}(x) = \mathrm{rdet}_{(k)}(xI_n - A).
\]

**Proposition 3.2** ([14], [16]). The \( k \)-th right characteristic polynomial \( p_{A,k}(x) \in R[x] \) of \( A \in M_n(R) \) is of the form

\[
p_{A,k}(x) = \lambda_0^{(k)} + \lambda_1^{(k)} x + \cdots + \lambda_{n_k-1}^{(k)} x^{n_k-1} + \lambda_{n_k}^{(k)} x^{n_k},
\]

where \( \lambda_0^{(k)}, \lambda_1^{(k)}, \ldots, \lambda_{n_k-1}^{(k)}, \lambda_{n_k}^{(k)} \in R \) and \( \lambda_{n_k}^{(k)} = n\{(n-1)!\}^{1+n+n^2+\cdots+n^{k-1}} \).

**Theorem 3.3** ([14], [16]). If \( \frac{1}{n} \in R \) and the ring \( R \) is Lie nilpotent of index \( k \), then a right Cayley-Hamilton identity

\[
(A)p_{A,k} = I_n\lambda_0^{(k)} + A\lambda_1^{(k)} + \cdots + A^{n_k-1}\lambda_{n_k-1}^{(k)} + A^{n_k}\lambda_{n_k}^{(k)} = 0
\]

with right scalar coefficients holds for \( A \in M_n(R) \). We also have \( (A)u = 0 \), where \( u(x) = p_{A,k}(x)h(x) \) and \( h(x) \in R[x] \) is arbitrary.

**Theorem 3.4** ([1]). If \( \frac{1}{2} \in R \) and the ring \( R \) is Lie nilpotent of index 2, then for a \( 2 \times 2 \) matrix \( A \in M_2(R) \) the right Cayley-Hamilton identity in the above 3.3 can be written in the following trace form:
\[(A)p_{A,2} = I_2 \left( \frac{1}{2}tr^4(A) + \frac{1}{2}tr^2(A^2) + \frac{1}{4}tr^2(A)tr(A^2) - \frac{5}{4}tr(A^2)tr^2(A) + [tr(A^3), tr(A)] \right)
+ A\left( tr(A)tr(A^2) + tr(A^2)tr(A) - 2tr^3(A) \right)
+ A^2\left( 4tr^2(A) - 2tr(A^2) \right)
- A^3\left( 4tr(A) \right) + 2A^4 = 0. \]

**Corollary 3.5** ([1]). If \( \frac{1}{2} \in R \) and the ring \( R \) is Lie nilpotent of index 2, then, for every \( A \in M_2(R) \),

\[ tr(A) = tr(A^2) = 0 \quad \text{imply that} \quad A^4 = 0. \]

4. Matrices over a ring with \([ [x_1, y_1], z_1] [[x_2, y_2], z_2] \cdots [[x_t, y_t], z_t] = 0 \]

We shall make use of the following well known fact.

**Proposition 4.1.** If \([ [x_1, y_1], z_1] [[x_2, y_2], z_2] \cdots [[x_t, y_t], z_t] = 0 \) is a polynomial identity on a ring \( R \), then \( D^t = \{0\} \), with \( D \) the ideal \( R[[R, R], R]R \) of \( R \).

**Theorem 4.2.** If \( \frac{1}{2} \in R \) and \( A \in M_n(R) \) is a matrix over a ring \( R \) satisfying the polynomial identity \([ [x_1, y_1], z_1] [[x_2, y_2], z_2] \cdots [[x_t, y_t], z_t] = 0 \), then an invariant “power” Cayley-Hamilton identity of the form

\[ \left( I_n \lambda_0^{(2)} + A \lambda_1^{(2)} + \cdots + A^{n^2-1} \lambda_{n^2-1}^{(2)} + A^{n^2} \lambda_n^{(2)} \right)^t = 0 \]

holds, with certain right coefficients

\[ \lambda_i^{(2)} \in R, \quad 0 \leq i \leq n^2 - 1, \quad \text{and} \quad \lambda_n^{(2)} = n\{(n-1)!\}^{1+n} \]

(only partially determined by \( A \)). The cosets \( \lambda_i^{(2)} + D \) with \( D = R[[R, R], R]R \triangleleft R \) appear in the second right characteristic polynomial \( p_{A,2}(x) \) of the natural image \( \overline{A} \in M_n(R/D) \) of \( A \) over the factor ring \( R/D \):

\[ p_{A,2}(x) = (\lambda_0^{(2)} + D) + (\lambda_1^{(2)} + D)x + \cdots + (\lambda_{n^2-1}^{(2)} + D)x^{n^2-1} + (\lambda_n^{(2)} + D)x^{n^2} \in (R/D)[x]. \]

**Proof.** Consider the factor ring \( R/D \), where \( D = R[[R, R], R]R \triangleleft R \) is the double commutator ideal. If \( A = [a_{i,j}] \in M_n(R) \), then we use the notation \( \overline{A} = [a_{i,j} + D] \) for the image of \( A \) in \( M_n(R/D) \). Since \( R/D \) is Lie nilpotent of index 2, Theorem 3.3 implies that, in \( M_n(R/D) \),

\[ (\overline{A})p_{\overline{A},2} = \overline{I}_2(\lambda_0^{(2)} + D) + \overline{A}(\lambda_1^{(2)} + D) + \cdots + (\overline{A})^{n^2-1}(\lambda_{n^2-1}^{(2)} + D) + (\overline{A})^{n^2}(\lambda_n^{(2)} + D) = \overline{0}, \]

where
\[ p_{\mathcal{A},2}(x) = r\text{det}(\nu)(xT - A) = (\lambda_0^{(2)} + D) + (\lambda_1^{(2)} + D)x + \cdots + (\lambda_{n^2-1}^{(2)} + D)x^{n^2-1} + (\lambda_n^{(2)} + D)x^{n^2} \]

is the second right characteristic polynomial of \( A \) in \((R/D)[x]\). Clearly,

\[
I_n\lambda_0^{(2)} + A\lambda_1^{(2)} + \cdots + A^{n^2-1}\lambda_{n^2-1}^{(2)} + A^n\lambda_n^{(2)}
\]

\[
= T_n(\lambda_0^{(2)} + D) + \bar{A}(\lambda_1^{(2)} + D) + \cdots + (\bar{A})^{n^2-1}(\lambda_{n^2-1}^{(2)} + D) + (\bar{A})^n(\lambda_n^{(2)} + D) = 0
\]

implies that

\[
I_n\lambda_0^{(2)} + A\lambda_1^{(2)} + \cdots + A^{n^2-1}\lambda_{n^2-1}^{(2)} + A^n\lambda_n^{(2)} \in M_n(D).
\]

Now \( D^t = \{0\} \) is a consequence of Proposition 4.1, whence \((M_n(D))^t = \{0\}\) and

\[
\left( I_n\lambda_0^{(2)} + A\lambda_1^{(2)} + \cdots + A^{n^2-1}\lambda_{n^2-1}^{(2)} + A^n\lambda_n^{(2)} \right)^t = 0
\]

follows. \(\Box\)

**Remark 4.3.** If \([x_1, y_1][x_2, y_2] \cdots [x_t, y_t] = 0\) is a polynomial identity on \(R\) and \(A \in M_n(R)\), then using the commutator ideal \(T = R[R, R][R]\) and the natural image \(\bar{A} \in M_n(R/T)\) of \(A\) over the commutative ring \(R/T\), a similar argument as in the proof of Theorem 4.2 gives that

\[
\left( I_n\lambda_0^{(1)} + A\lambda_1^{(1)} + \cdots + A^{n-1}\lambda_{n-1}^{(1)} + A^n\lambda_n^{(1)} \right)^t = 0
\]

holds, where \(p_{\mathcal{A},1}(x) = (\lambda_0^{(1)} + T) + (\lambda_1^{(1)} + T)x + \cdots + (\lambda_{n-1}^{(1)} + T)x^{n-1} + (\lambda_n^{(1)} + T)x^n\) is the \(n!\) times scalar multiple of the classical characteristic polynomial of \(\bar{A}\) in \((R/T)[x]\) with \(\lambda_n^{(1)} = n!\).

**Theorem 4.4.** If \(\frac{1}{2} \in R\) and \(A \in M_n(R)\) is a matrix over a Lie nilpotent ring \(R\) of index \(k\), then an invariant “power” Cayley-Hamilton identity of the form

\[
\left( I_n\lambda_0^{(2)} + A\lambda_1^{(2)} + \cdots + A^{n^2-1}\lambda_{n^2-1}^{(2)} + A^n\lambda_n^{(2)} \right)^{2k-2} = 0
\]

holds, with certain right coefficients

\[
\lambda_i^{(2)} \in R, \; 0 \leq i \leq n^2 - 1, \quad \text{and} \quad \lambda_{n^2}^{(2)} = n\{n - 1\}^{1+n}
\]

(only partially determined by \(A\)). The cosets \(\lambda_i^{(2)} + D\) with \(D = R[[R, R], R]R \triangleleft R\) appear in the second right characteristic polynomial \(p_{\mathcal{A},2}(x)\) of the natural image \(\bar{A} \in M_n(R/D)\) of \(A\) over the factor ring \(R/D\):
\( p_{\mathcal{A},2}(x) = (\lambda_0^{(2)} + D) + (\lambda_1^{(2)} + D)x + \cdots + (\lambda_n^{(2)} + D)x^{n^2-1} + (\lambda_n^{(2)} + D)x^{n^2} \in (R/D)[x] \).

**Proof.** According to Jennings’s result (Corollary 2.2), the double commutator ideal

\[
D = R[[R, R], R]R = \{ \sum_{1 \leq i \leq m} r_i [a_i, b_i], c_i s_i \mid r_i, a_i, b_i, c_i, s_i \in R, 1 \leq i \leq m \} < R
\]

is nilpotent, with \( D^{2^k-2} = \{0\} \). Thus the application of Theorem 4.2 gives our identity. \( \square \)

**Remark 4.5.** If \( k = 2 \), then \( R[[R, R], R]R = \{0\} \), and the identity in Theorem 4.4 remains the same as the Lie nilpotent right Cayley-Hamilton identity in Theorem 3.3.

**Remark 4.6.** The Grassmann algebra

\[
E = K \langle v_1, v_2, \ldots, v_i, \ldots \mid v_i v_j + v_j v_i = 0 \text{ for all } 1 \leq i \leq j \rangle
\]

over a field \( K \) (with \( 2 \neq 0 \)) has property \( L_2 \), and

\[
[v_1, v_2] \cdot [v_3, v_4] \cdot \cdots \cdot [v_{2t-1}, v_{2t}] = 2^t v_1 v_2 \cdots v_{2t} \neq 0
\]

shows that \( L_2 \) does not imply the identity \([x_1, y_1][x_2, y_2] \cdots [x_t, y_t] = 0\) for any \( t \). Thus the identity mentioned in Remark 4.3 cannot be used directly to derive new identities for matrices over a Lie nilpotent ring of index \( k \geq 2 \). However, as the referee pointed out, the following (weak) version of Latyshev’s theorem provides a possibility to use Remark 4.3 in order to obtain another remarkable “power” Cayley-Hamilton identity.

**Theorem** ([6]). If \( S \) is a Lie nilpotent algebra (over an infinite field) of index \( k \), generated by \( m \) elements, then there exists an integer \( d = d(k, m) \) such that \( S \) satisfies the polynomial identity \([x_1, y_1][x_2, y_2] \cdots [x_d, y_d] = 0\). (In the original version \( S \) satisfies a so-called nonmatrix polynomial identity.)

If \( A \in M_n(R) \) is a matrix over a Lie nilpotent algebra (over an infinite field) \( R \) of index \( k \), then \( A \in M_n(S) \), where \( S \) is the (unitary) subalgebra generated by the \( n^2 \) entries of \( A \). Thus \([x_1, y_1][x_2, y_2] \cdots [x_d, y_d] = 0\) is a polynomial identity on \( S \) with \( d = d(k, n^2) \) and Remark 4.3 gives that an identity

\[
\left( I_n \lambda_0^{(1)} + A \lambda_1^{(1)} + \cdots + A^{n-1} \lambda_{n-1}^{(1)} + A^n \lambda_n^{(1)} \right)^{d(k, n^2)} = 0
\]

of degree \( nd(k, n^2) \) holds. Unfortunately, our knowledge about \( d(k, n^2) \) is very limited, the fact that \( d(2, 4) = 3 \) was mentioned by the referee.

**Remark 4.7.** If \( R \) is an algebra over a field \( K \) of characteristic zero, then the invariance of the identities in 4.2 and 4.4 means that \( p_{T^{-1}AT,2}(x) = p_{\mathcal{A},2}(x) \) holds for any \( T \in \text{GL}_n(K) \) (see [1]).
Corollary 4.8. If $\frac{1}{2} \in R$ and the ring $R$ is Lie nilpotent of index $k$, then, for every $A \in M_2(R)$,

$$\text{tr}(A) = \text{tr}(A^2) = 0$$

imply that $A^{2^k} = 0$.

Proof. Using $D = R[[R, R], R] \triangleleft R$, $\overline{A} \in M_2(R/D)$ and

$$\text{tr}(\overline{A}) = \text{tr}(A) + D = 0, \quad \text{tr}(\overline{A}^2) = \text{tr}(A^2) + D = 0,$$

the application of Corollary 3.5 ensures that $\overline{A}^4 = (\overline{A})^4 = 0$. Thus the nilpotency of $D$ ($D^{2^k-2} = \{0\}$) gives that $A^{2^k} = (A^4)^{2^{k-2}} = 0$. \Box

Remark 4.9. According to the following important observation of the referee, the use of Latyshev’s theorem gives an $n \times n$ variant of Corollary 4.8. If $A \in M_n(R)$ is a matrix over a Lie nilpotent algebra (over a field $K$ of characteristic zero) $R$ of index $k$, then we prove that

$$\text{tr}(A) = \text{tr}(A^2) = \cdots = \text{tr}(A^n) = 0$$

implies that $A^{nd(k, n^2)} = 0$. Indeed, $A \in M_n(S)$, where $S \subseteq R$ is the (unitary) subalgebra of $R$ generated by the entries of $A$. Now consider the natural image $\tilde{A} \in M_n(S/S[S, S]S)$ of $A$. The application of the well known fact that

$$\text{tr}(\tilde{A}) = \text{tr}(\tilde{A}^2) = \cdots = \text{tr}(\tilde{A}^n) = \tilde{0}$$

implies that $\tilde{A}^n = (\tilde{A})^n = \tilde{0}$ (it is a consequence of the Newton trace formulae for the coefficients of the characteristic polynomial $p_{\tilde{A}, 1}(x) \in (S/S[S, S]S)[x]$, where the factor $S/S[S, S]S$ is a commutative algebra over $K$). Since $(S/S[S, S]S)^{d(k, n^2)} = \{0\}$ by Latyshev’s theorem and $A^n \in M_n(S[S, S]S)$, we obtain the desired equality.

Declaration of competing interest

There is no competing interest.

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