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Maximal commutative subalgebras of Leavitt path algebras

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Dedicated to Richard Wiegandt for his ninetieth birthday.

Let K be a field, and let E be a row-finite (directed) graph. We present a construction of a wealth of maximal commutative subalgebras of the Leavitt path algebra $L_K(E)$, which is a far-reaching generalization of the construction of the commutative core as a maximal commutative subalgebra of $L_K(E)$.

Keywords: Maximal commutative subalgebra; Leavitt path algebra.

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1. Introduction

Leavitt path algebras $L_K(E)$ of row-finite graphs E (where K is an arbitrary field) were introduced in [5, 10]. These algebras have become a subject of significant interest, both for algebraists and for analysts working in C^*-algebras. The Cuntz–Krieger

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algebras $C^*(E)$, which are the C^*-algebra counterparts of these Leavitt path algebras, where E denotes a graph, were investigated in [42].

As noted in [8], the interplay between these two classes of graph algebras has been extensive and mutually beneficial. Graph C^*-algebra results have helped to guide the development of Leavitt path algebras by suggesting the veracity of some conjectures and by hinting at directions in which studies should be focused. Similarly, Leavitt path algebras have provided a deeper understanding of graph C^*-algebras by assisting in identifying those aspects of $C^*(E)$ that are algebraic in nature.

On the algebraic side of the picture, the algebras $L_K(E)$ are natural generalizations of the algebras investigated by Leavitt in [35], and they are a specific type of path K-algebras associated with a graph E modulo certain relations. The family of algebras which can be realized as Leavitt path algebras of graphs includes full $n \times n$ matrix algebras $M_n(K)$ for $n \in \mathbb{N} \cup \{\infty\}$ (where $M_\infty(K)$ denotes matrices of countably infinite size with only a finite number of nonzero entries), the Toeplitz algebra T, the Laurent polynomial ring $K[x, x^{-1}]$, and the classical Leavitt algebras $L(1, n)$ for $n \geq 2$.

Constructions such as direct sums, direct limits, and matrices over the mentioned examples, can be also realized in this setting. Moreover, in addition to the fact that this class of algebras includes a wealth of well-known algebras, one of the main interests in their study is the pictorial representations that their corresponding graphs provide.

Recently, other algebras, which are more general than the Leavitt path algebras, have also enjoyed a lot of interest. We are thinking here of, for example, Steinberg algebras (see [18, 30]) and Kumjian–Pask algebras (see [11, 16, 17]).

One of the objects which has been studied intensively in all the mentioned settings is maximal commutative subalgebras of the considered structures. See, for example [17, 28, 30, 32, 38]. A common feature of the considerations in the aforementioned papers is the interest in the object, which in the case of Leavitt path algebras $L_K(E)$, is called the commutative core of $L_K(E)$.

Given a mathematical object, it is often rather natural to consider its maximal subobjects as a means of understanding it better. Maximal subalgebras of (not necessarily associative) algebras, and in particular maximal commutative subalgebras, have classically guided such studies. A well-known example of this principle comes to the fore, of course, in the structure theory of finite-dimensional semisimple Lie algebras, where their Cartan subalgebras feature prominently: over the complex number field, these are simply maximal commutative subalgebras, as seen in, for example, [25, 36].

Similar ideas have subsequently been applied to maximal substructures of other, possibly non-associative, algebraic structures, such as Malcev algebras, Jordan algebras, associative superalgebras, or classical groups. See, for example [26, 27, 40, 41].

On the associative side, a classical result of Schur (see [43]), which has attracted considerable historical interest, states that, for any algebraically closed field K of characteristic 0, the dimension over K of any commutative subalgebra of the full $n \times n$ matrix algebra $M_n(K)$ is at most $\lfloor \frac{n^2}{4} \rfloor + 1$, where $\lfloor \ \rfloor$ denotes the integer floor function. Jacobson showed in [31] that the mentioned upper bound holds for commutative subalgebras of $M_n(K)$ for all fields K. A concise proof of this result was presented later by Mirzakhani in [37].

Moreover, this upper bound is sharp. Indeed, following [44], let K be any field, let $n \geq 2$, and let k_1 and k_2 be positive integers such that $k_1 + k_2 = n$. Define the rectangular array B by

$$B = \{(i, j) \in \mathbb{N} \times \mathbb{N} : 1 \leq i \leq k_1 < j \leq n\},$$

and consider the subset

$$J = \left\{ \sum_{(i,j) \in B} b_{ij} E_{(i,j)} : b_{ij} \in K \ \text{ for all } (i, j) \in B \right\}$$

of $M_n(K)$, where $E_{(i,j)}$ denotes the matrix unit in $M_n(K)$ associated with position (i, j). The reader will immediately observe that J comprises the subset of $M_n(K)$ comprising the block upper triangular matrices corresponding with B.

It is very easy to see that the subalgebra

$$\mathcal{A} = KI_n + J \tag{1}$$

of $M_n(K)$, where

$$KI_n := \{aI_n : a \in K\},$$

and where I_n denotes the $n \times n$ identity matrix, is a commutative subalgebra of $M_n(K)$. Taking $k_1 = k_2 = \frac{n}{2}$ if n is even (respectively, taking $k_1 = \frac{n-1}{2}$ and $k_2 = \frac{n+1}{2}$ if n is odd), we obtain $\mathcal{A}$ with dimension equal to $\lfloor \frac{n^2}{4} \rfloor + 1$.

In the above vein, the famous result by Amitsur and Levitzky (see [9]), stating that the full matrix algebra $M_n(R)$ over any commutative ring R satisfies the standard polynomial identity

$$\sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(2n)} = 0$$

of degree $2n$ (with S_{2n} denoting the symmetric group on $2n$ symbols), and no polynomial identity of lower degree, is very significant. An immediate consequence is that every subring of $M_n(R)$ also satisfies the standard polynomial identity of degree $2n$.

In [44], special attention is paid to subalgebras of $M_n(K)$, K any field, satisfying some extra polynomial identities which are not satisfied by $M_n(K)$. Apart from the standard polynomial identity, arguably the most important polynomial identity is the so-called Lie nilpotency of index m (for some positive integer m). The m-Lie nilpotency

$$[[\cdots[[x_1,x_2],x_3],\ldots,x_m],x_{m+1}]=0$$

is not even a polynomial identity on the 2×2 matrix algebra $M_2(K)$. (Note that an algebra or a ring is commutative if and only if it is Lie nilpotent of index 1, i.e. if and only if $[x_1,x_2]=0$ for all elements x_1 and x_2 in the algebra or ring.) One of the results in [44] points out the difference, as far as the upper bounds considered therein are concerned, between the situations where $m\leq 6$ and the situations where $m>6$.

The importance of Lie nilpotency is buttressed by the fact that the (countably) infinite dimensional Grassmann algebra G has "only one identity" in the sense that the polynomial identity $[[x_1,x_2],x_3]=0$ generates the T-ideal of the polynomial identities satisfied by G. The latter is a highly nontrivial result by Krakowski and Regev (see [34]).

It is extremely important to note that the mentioned Grassmann algebra G plays a fundamental role in Kemer's monumental structure theory of T-ideals, as well as in his solution of the famous Specht problem about the finite generation of the polynomial identities of associative algebras over a field K of characteristic zero. A remarkable consequence of the mentioned structure theory is that any T-ideal contains all the polynomial identities of $M_n(G)$ for some n.

In general, the dimension of a subalgebra of $M_n(K)$ cannot be arbitrary. For instance, the dimension of any proper subalgebra of $M_n(K)$ is less than or equal to $(n-1)^2+1$. It seems to be a very challenging problem to describe the integers between 1 and n^2 which can appear as the dimension of a certain subalgebra of $M_n(K)$. Note that the mentioned results by Schur and Jacobson produce the integer $\lfloor\frac{n^2}{4}\rfloor+1$.

There is no doubt that commutativity is extremely important, and Lie nilpotency is the most natural generalization of it. In the light of the foregoing setting it is also worth drawing the reader's attention to, for example, the works [21, 29, 33, 37, 46–48].

In [28], Gil Canto and Nasr-Isfahani constructed and investigated a maximal (with respect to inclusion) commutative subalgebra of a Leavitt path algebra over a commutative ring R (see [28, Proposition 4.5 and Theorem 4.13]). For a given graph E, this commutative subalgebra, called the *commutative core*, is denoted by $\mathcal{M}_R(E)$ and is generated over R by all elements of the following form:

- $\alpha\alpha^*$, where α is a path in E;
- $\alpha c\alpha^*$, where α is a path and c is a cycle without exits; and
- $\alpha c^*\alpha^*$, where α is a path and c is a cycle without exits.

Recall that the Leavitt path algebra $L_K(\mathcal{E})$, for a field K and the graph

$$\mathcal{E}\equiv \bullet_{v_1}\xrightarrow{e_1}\bullet_{v_2}\xrightarrow{e_2}\bullet_{v_3}\cdots\cdots\bullet_{v_{n-1}}\xrightarrow{e_{n-1}}\bullet_{v_n}\ ,$$

is isomorphic[a] to the full $n \times n$ matrix algebra $M_n(K)$. Therefore, in the light of the foregoing discussion, it would be interesting to see what the algebra $\mathcal{M}_K(\mathcal{E})$ looks like (see the construction by Gil Canto and Nasr-Isfahani) if we consider $L_K(\mathcal{E})$. It occurs that we get exactly the commutative subalgebra of $L_K(\mathcal{E})$ generated by all the vertices $v_1, \ldots, v_n$, which, in matrix language, produce the commutative subalgebra of $M_n(K)$ generated by the matrix units $E_{(1,1)}, E_{(2,2)}, \ldots, E_{(n,n)}$. Thus, in the case of a full matrix algebra it can be said that $\mathcal{M}_K(E)$ is a kind of trivial example of a maximal commutative subalgebra.

Motivated by the above facts, in this paper, we will provide a construction of a class of maximal commutative subalgebras of a Leavitt path algebra $L_K(E)$ such that in the case of the matrix algebra $M_n(K)$ (seen as an isomorphic copy of $L_K(\mathcal{E})$) one of the elements of this class is the commutative subalgebra $\mathcal{A} = KI_n + J$ mentioned in (1). In [14], we achieved our goal of constructing a class of maximal commutative subalgebras of $L_K(E)$ in the case where the considered Leavitt path algebra is prime.

Maximal commutative subrings of non-commutative rings, entailing quite a number of types of constructions, have also been investigated in, for example [15, 17, 22, 39].

This paper is organized as follows:

Section 2 deals with the preliminaries, including some basic definitions and notation that will be used throughout the sequel. The main part of the paper consists of Sec. 3 (see Theorem 3.8) and Sec. 4 (see Theorem 4.4). It includes constructions of maximal commutative subalgebras of $L_K(E)$, which begin by considering a pair (A, B) of subsets of the set E^0 of vertices of the graph E, such that (A, B) constitutes a partition $\{A, B\}$ of the set E^0.

In Sec. 3, we assume that the partition has properties that seem very strong at first glance. Namely, we assume that for every vertex u in A we can find a vertex v in B such that the trees of these two vertices intersect nontrivially, and we assume that for every vertex v' in B we can find a vertex u' in A such that the trees of these two vertices intersect nontrivially. In fact, in Sec. 5, we will show that in a certain sense every partition of E^0 comprising two subsets of E^0 can be improved to get a partition satisfying the mentioned conditions.

In Sec. 4, invoking the main result of Sec. 3, we will construct maximal commutative subalgebras of $L_K(E)$ using any (i.e. without assuming the mentioned seemingly strong conditions) given partition $\{A, B\}$ of E^0.

2. Preliminaries

A (directed) graph $E = (E^0, E^1, s, r)$ consists of two sets, namely, E^0 and E^1, and functions $s, r : E^1 \to E^0$. The elements of E^0 are called *vertices*, and the elements

[a]The isomorphism we are thinking about here maps each vertex v_i to the matrix unit $E_{(i,i)}$, each path $e_i \ldots e_{j-1}$ to $E_{(i,j)}$, and each ghost path $e^*_{j-1} \ldots e^*_i$ to $E_{(j,i)}$, $1 \le i < j \le n$.

of E^1 are called *edges*. For each edge e, $s(e)$ is the *source* of e and $r(e)$ is the *range* of e. A vertex which emits no edges is called a *sink*, and $\mathbb{S}$ denotes the set of all sinks in E. A vertex v is called an *infinite emitter* if $s^{-1}(v)$ is an infinite set, and a regular vertex otherwise. If every vertex $v \in E^0$ is regular, then the graph E is called *row-finite*.

A *path* η in a graph E is a sequence of edges $\eta = e_1e_2 \dots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. In this case, n is called the *length* of η, and it is denoted by $|\eta|$. By η^0 we denote the set of vertices $\{s(e_1), s(e_2), \dots, s(e_n), r(e_n)\}$.

We define $s(\eta) = s(e_1)$ and $r(\eta) = r(e_n)$. The set of all paths of E is denoted by $\mathrm{Path}(E)$.

A path $\eta = e_1 \dots e_n$ is *closed* if $s(e_1) = r(e_n)$. A closed path $\eta = e_1 \dots e_n$ is *simple* in case $s(e_i) \neq s(e_1)$ for all $2 \leq i \leq n$. Such a simple closed path η is said to be based at $v = s(e_1)$. A simple closed path $\eta = e_1 \dots e_n$ is a *cycle* in case there are no repeats in the list of vertices $s(e_1), s(e_2), \dots, s(e_n)$. An *exit* of a cycle $\eta = e_1 \dots e_n$ is an edge f such that $s(f) = s(e_i)$ for some i and $f \neq e_i$. If no such f exists for a cycle η, then we say that η is a cycle *without exits*.

For a cycle $c = e_1e_2 \dots e_n$ in E and $u_i = s(e_i)$ we will write c_{u_i} to denote the cycle $e_ie_{i+1} \dots e_{i-1}$. By $\mathcal{C}$ we denote the set of all those cycles c without exits such that there is a finite number of paths ending at c and not containing c. Note that if c is a cycle in $\mathcal{C}$ and $u, v \in c^0$ with $u \neq v$, then c_u and c_v will be distinct elements in $\mathcal{C}$.

Recall (see [19, Definition 2.1]) that a cycle c in E is an *extreme cycle* if c has exits and for every path λ starting at a vertex in c^0 there is a path μ such that $\lambda\mu \neq 0$ and $r(\mu) \in c^0$.

If there is a path from a vertex u to a vertex v, we write $u \geq v$. Let P and Q be subsets of E^0. Then $P \geq Q$ means that there are vertices $u \in P$ and $v \in Q$ such that $u \geq v$. In this context, for a vertex u, the notation $u \geq P$ and $P \geq u$ should be clear. Finally, $T(P)$ denotes the set of all vertices $v \in E^0$ such that $P \geq v$.

A subset H of E^0 is called *hereditary* if, whenever $v \geq w$ and $v \in H$, then $w \in H$. A hereditary set is *saturated* if every regular vertex which feeds into H and only into H is again in H, that is, if, whenever $s^{-1}(v) \neq \emptyset$ is finite and $r(s^{-1}(v)) \subseteq H$, then $v \in H$.

If H is a hereditary subset of E^0, then the *saturated closure* of H (see [10]), denoted by $\overline{H}$, is defined as $\bigcup_{i \in \mathbb{N} \cup \{0\}} \Lambda_i(H)$, where $\Lambda_0(H) = H$ and recursively,

$$\Lambda_i(H) = \Lambda_{i-1}(H) \cup \{v \in E^0 : v \text{ is a regular vertex, and } r(s^{-1}(v)) \subseteq \Lambda_{i-1}(H)\}.$$

Following [20, Definition 1.3], for a given non-empty hereditary subset H of E^0, let $F_E(H)$ denote the set

$$\{e_1 \dots e_n : e_i \in E^1, s(e_1) \in E^0 \backslash H, r(e_i) \in E^0 \backslash H \text{ for } i < n, \text{ and } r(e_n) \in H\}.$$

Following [2], we will say that a directed graph E is connected if, for all vertices $v, w \in E^0$, there is a sequence $\mu = \mu_1\mu_2 \dots \mu_n$, with $\mu_i \in E^1 \cup (E^1)^*$ for all i, such that $s(\mu) = v$ and $r(\mu) = w$. Intuitively, E is connected if E cannot be written as the union of two disjoint subgraphs, or equivalently, E is connected in case the corresponding undirected graph of E is so in the usual sense. It is not difficult to show that if E is the disjoint union of subgraphs F_i, then $L_K(E) \cong \bigoplus L_K(F_i)$.

Now, we recall the following definition (see [4]):

Definition 2.1. Let E be an arbitrary graph and K be a field. The Leavitt path algebra associated with E, denoted by $L_K(E)$, is the K-algebra generated by the set E^0, together with $\{e, e^* : e \in E^1\}$, satisfying the following relations:

(V) $vw = \delta_{v,w} v$ for all $v, w \in E^0$,
(E1) $s(e)e = er(e) = e$ for all $e \in E^1$,
(E2) $e^* s(e) = r(e)e^* = e^*$ for all $e \in E^1$,
(CK1) $e^* f = \delta_{e,f} r(f)$ for all $e, f \in E^1$,
(CK2) $v = \sum_{\{e \in E^1 :\, s(e)=v\}} ee^*$ for every vertex v which is not a sink and emits a finite number of edges.

For an edge $e \in E^1$, the element e^* is called a *ghost edge*, and for a path $\eta = e_1 e_2 \dots e_n$, by σ^* we denote the so-called *ghost path* $e_n^* e_{n-1}^* \dots e_1^*$. If c is a cycle and s is a negative integer, then $(c^*)^{-s}$ is denoted by c^s.

It is well known that the algebra $L_K(E)$ is $\mathbb{Z}$-graded. We have $L_K(E) = \bigoplus_{n \in \mathbb{Z}} L_K(E)_n$, where

$$L_K(E)_n = \mathrm{span}_K\{\alpha\beta^* : \alpha, \beta \in \mathrm{Path}(E) \text{ and } |\alpha| - |\beta| = n\}.$$

Moreover, $L_K(E) = \mathrm{span}_K\{\alpha\beta^* : \alpha, \beta \in \mathrm{Path}(E)\}$.

An element x in $L_K(E)_n$ is said to be of degree n, and we use the notation $\deg(x) = n$.

Note that if α and β are paths such that $\alpha^*\beta \neq 0$, then either $\alpha = \beta\alpha'$ for some path α' (in the case $|\alpha| \geq |\beta|$) or $\beta = \alpha\beta'$ for some path β' (in the case $|\alpha| < |\beta|$). In the first case we have $\alpha^*\beta = (\alpha')^*$, and in the second $\alpha^*\beta = \beta'$.

For general notation, terminology and results in Leavitt path algebras we refer the reader to, for example [1, 4, 5, 13].

3. Fully Downward Directed Partitions

In this section, we will present a first construction which leads us to a class of maximal commutative subalgebras of a Leavitt path algebra $L_K(E)$ associated with an arbitrary graph E (where K is a field). If E is the disjoint union of subgraphs F_i, and M_i is a maximal commutative subalgebra of $L_K(F_i)$ for every i, then $\bigoplus M_i$ is a maximal commutative subalgebra of $\bigoplus L_K(F_i)$ which is isomorphic to a maximal commutative subalgebra of $L_K(E)$. Therefore, it is fully justified that henceforth we consider only connected graphs. To be precise, throughout this section, we assume that the (arbitrary) considered graph E is connected.

Definition 3.1. Consider a pair (A, B) of two disjoint (not necessarily non-empty) subsets of E^0 which constitutes a partition $E^0 = A \cup B$ of E^0. For short we say that (A, B) is a partition of E^0. Let

$$\mathcal{D}(A) = \{u \in A : T(u) \cap T(v) \neq \emptyset \text{ for some } v \in B\}$$

and

$$\mathcal{D}(B) = \{v \in B : T(u) \cap T(v) \neq \emptyset \text{ for some } u \in A\}.$$

(We note that if $A = \emptyset$ or $B = \emptyset$, then both $\mathcal{D}(A) = \emptyset$ and $\mathcal{D}(B) = \emptyset$.) We call the pair $(\mathcal{D}(A), \mathcal{D}(B))$ the downward directed part of the pair (A, B). If $(\mathcal{D}(A), \mathcal{D}(B)) = (A, B)$ (which, of course, implies that $A \neq \emptyset$ and $B \neq \emptyset$), then we say that the pair (A, B) (or the partition of E^0 constituted by (A, B)) is fully downward directed.

Since E is connected (by assumption), it is obvious that if $A \neq \emptyset$ and $B \neq \emptyset$ (for a partition $A \cup B = E^0$ of E^0), then also $\mathcal{D}(A) \neq \emptyset$ and $\mathcal{D}(B) \neq \emptyset$.

Example 3.2. Let E be the graph

$$E \equiv \bullet_u \longleftarrow \bullet_v \longrightarrow \bullet_w \ ,$$

and consider the partition (A, B) of E^0, with $A = \{u, v\}$ and $B = \{w\}$. Then $\mathcal{D}(A) = \{v\}$ and $\mathcal{D}(B) = \{w\}$, and so the partition (A, B) is not fully downward directed. However, it is easy to "improve" it in order to get one which is so. Indeed, it suffices to move the vertex v to the set B and the vertex w to A.

In the last section, we will show that an "improvement", as described above, is always possible.

Definition 3.3. Let (A, B) be a partition of the set of vertices E^0. We call the subalgebra of $L_K(E)$ generated by all monomials $\alpha\beta^*$, such that $s(\alpha) \in A$ and $r(\beta^*) \in B$, the *diagonal subalgebra* of $L_K(E)$ related to (A, B), and we denote it by $L_K\big(\mathcal{D}(A), \mathcal{D}(B)\big)$. If (A, B) is fully downward directed, then $L_K\big(\mathcal{D}(A), \mathcal{D}(B)\big)$ $(= L_K(A, B))$ is called the *full diagonal subalgebra* of $L_K(E)$ related to (A, B).

For the graph considered in Example 3.2 with $A = \{u, v\}$, $B = \{w\}$ we get $L_K(\mathcal{D}(A), \mathcal{D}(B)) = \langle f \rangle$, where f is the edge from v to w. If we take $A = \{u, w\}$ and $B = \{v\}$, then $L_K(\mathcal{D}(A), \mathcal{D}(B)) = \langle f^*, g^* \rangle$, where g is the edge from v to u.

Remark 3.4. Henceforth, if we consider an element $\mathfrak{b} \in L_K(E)$, then we simultaneously fix a presentation $\mathfrak{b} = \sum_{j \in J} k_j \alpha_j \beta_j^*$, where J is a finite set of indices, and for every $j \in J$, $\alpha_j, \beta_j \in \mathrm{Path}(E)$, $k_j \in K$. We also assume that the presentation we work with is chosen so that the cardinality of J is as small as possible. Moreover, we assume that for every monomial $\alpha_j \beta_j^* = e_1 e_2 \ldots e_n f_1^* f_2^* \ldots f_m^*$ appearing in the considered presentation of $\mathfrak{b}$, where $e_1, \ldots, e_n, f_1, \ldots, f_m$ are edges and $n, m \geq 1$, we have $e_n f_1^* \neq s(e_n)$. In such a case, we will say that $\mathfrak{b}$ is in a reduced form (or, $\mathfrak{b}$ is reduced).

Definition 3.5. For a given graph E and a field K, consider a homogeneous element $\mathfrak{b} \in L_K(E)$ which is in a fixed reduced form $\mathfrak{b} = \sum_{j \in J} k_j \alpha_j \beta_j^*$ with non-zero $k_j \in K$ for every $j \in J$. Then for each $n \geq 0$ define,

$$S_n(\mathfrak{b}) := \{\alpha_j \beta_j^* \in supp(\mathfrak{b}) : |\alpha_j| = n\}.$$

By [14, Proposition 4.5] we have the following proposition:

Proposition 3.6. *Let K be a field and E be a graph, and let $\mathfrak{b}$ be a non-zero homogeneous element of $L_K(E)$ with a reduced presentation $\mathfrak{b} = \sum_{j \in J} k_j \alpha_j \beta_j^*$, where $0 \neq k_j \in K$ for every $j \in J$. If n_0 is the smallest integer such that $S_{n_0}(\mathfrak{b})$ is a non-empty subset of $supp(\mathfrak{b})$, then for every $\alpha_i \beta_i^* \in S_{n_0}(\mathfrak{b})$ there are paths γ, $\bar{\gamma}$ and a non-zero $k \in K$ such that*

$$\gamma^* \mathfrak{b} \bar{\gamma} = k \cdot \gamma^* \alpha_i \beta_i^* \bar{\gamma} = k \cdot s(\gamma^*) = k \cdot r(\bar{\gamma}).$$

Moreover, $\gamma = \alpha_i \sigma$ and $\bar{\gamma} = \beta_i \sigma$ for some $\sigma \in Path(E)$.

With the notation as in the above proposition we have the following straightforward, but very useful, fact, which should be viewed in the light of [14, Corollary 4.8] and which will be used freely in the sequel.

Corollary 3.7. *Let a pair (A, B) be a fully downward directed partition of E^0, and let $v \in E^0$ be a vertex such that $v\mathfrak{b}v \neq 0$. If $v \in A$, then there is a monomial $\alpha\beta^*$ such that $\bar{\gamma}\alpha\beta^* \in L_K(A, B)$ and*

$$\mathfrak{b}\bar{\gamma}\alpha\beta^* \neq 0.$$

On the other hand, if $v \in B$, then there is a monomial $\delta\sigma^*$ such that $\delta\sigma^*\gamma^* \in L_K(A, B)$ and

$$\delta\sigma^*\gamma^*\mathfrak{b} \neq 0.$$

Let $Z(L_K(E))$ denote the center of $L_K(E)$. As the main result of this section we will prove the following theorem:

Theorem 3.8. *Let E be a row-finite graph and let K be a field. If a pair (A, B) is a fully downward directed partition of E^0, then for the full diagonal subalgebra $L_K(A, B)$ of $L_K(E)$, the vector space*

$$\mathcal{A}(A, B) = L_K(A, B) + Z(L_K(E))$$

is a maximal commutative subalgebra of $L_K(E)$.

However, we first have to develop the required machinery in order to obtain a proof of this result.

With reference to [19], we consider the following three sets of vertices, which according to the mentioned paper, "... are the three primary colors of the center of a Leavitt path algebra":

- the set $P_l(E)$, which consists of all vertices whose trees contain neither bifurcations nor cycles;

- the set $P_c(E)$, comprising all vertices in cycles without exits;
- the set $P_{ec}(E)$ of all vertices which are vertices of extreme cycles.

Next, let

$$P = P_l(E) \cup P_c(E) \cup P_{ec}(E).$$

Now, we recall some definitions and results from [19].

Definition 3.9. We define the relation $\sim^1$ on E^0 as follows: $u \sim^1 v$ if and only if $u = v$ or the following two conditions hold:

(i) $u \geq v$ or $v \geq u$, and there are no bifurcations at any vertex in $T(u) \cup T(v)$;
(ii) there exists a cycle c such that $c^0 \geq u$ and $c^0 \geq v$.

The relation $\sim^1$ is reflexive and symmetric, but not necessarily transitive. Therefore, we consider the transitive closure of $\sim^1$ and denote it by $\sim$. The notation $[v]$ will be used for the (equivalence) class of a vertex v with respect to the equivalence relation $\sim$. By [19, Lemma 3.12], $[v]$ is a hereditary set for every $v \in P$.

Next, set $X = P/\sim$, and consider the sets

$$P_f = \{v \in P : |\overline{[v]}| < \infty \text{ and } |F_E(\overline{[v]})| < \infty\}$$

and

$$X_f = \{[u] \in X : v \in P_f \text{ for all } v \in [u]\}.$$

Taking into account the main results of [19] we can formulate the following result:

Theorem 3.10. *If E is a row-finite graph, then the center $Z(L_K(E))$ is graded and*

$$\mathcal{B}_0 = \{r_{[v]} : [v] \in X_f\} \tag{2}$$

is a basis for $(Z(L_K(E)))_0$, where $r_{[v]} = \sum_{u\in\overline{[v]}} u + \sum_{\alpha\in F_E(\overline{[v]})} \alpha\alpha^*$. Furthermore, for every non-zero integer n,

$$\mathcal{B}_n = \left\{ \sum_{\substack{m,\ \alpha,\ u \\ m\cdot|c|=n \\ \alpha\in F_E(c^0)\cup c^0 \\ u\in c^0}} \alpha c_u^m \alpha^* : c \in \mathcal{C} \right\}$$

is a basis for $\big(Z\big(L_K(E)\big)\big)_n$.

Remark 3.11. Using Theorem 3.10, it is not hard to see that $x = \sum_{v\in E^0} vxv$ for every element x of the center $Z(L_K(E))$ of $L_K(E)$, which together with the definition of $L_K(A, B)$ simply give that $\mathcal{A}(A, B)$ is an algebra. It should be also clear to the reader that the algebra $\mathcal{A}(A, B)$ is commutative. Thus, what remains to be done is to show maximality of $\mathcal{A}(A, B)$ as a commutative subalgebra of $L_K(E)$.

In order to construct a proof of this part we may assume, for a contradiction, that there is a homogeneous element $\mathfrak{a} \in L_K(E) \setminus \mathcal{A}(A, B)$ which commutes with all the elements of $\mathcal{A} = \mathcal{A}(A, B)$. We may also assume that $\mathfrak{a}$ is an element with the above properties such that the cardinality of the set $\{(u, v) \in E^0 \times E^0) : u\mathfrak{a}v \neq 0\}$ is as small as possible.

Throughout this section, we assume that E is a row-finite graph, (A, B) is a fully downward directed pair, and $\mathfrak{a}$ is an element with the properties listed in Remark 3.11.

Lemma 3.12. *For the element* $\mathfrak{a}$, *we have* $\mathfrak{a} = \sum_{v\in E^0} v\mathfrak{a}v$.

Proof. It is obvious that $\mathfrak{a} = \sum_{(u,v)\in E^0\times E^0} u\mathfrak{a}v$. By Remark 3.11, we may assume that $u\mathfrak{a}v = 0$ for every pair $(u, v) \in A \times B$.

Now, suppose for a contradiction, that $x\mathfrak{a}y \neq 0$ for some $(x, y) \in (E^0 \times E^0) \setminus (A \times B)$ such that $x \neq y$. Consider the case where $x \in A$. Then also $y \in A$. By Proposition 3.6, there are paths γ and $\overline{\gamma}$ such that

$$\gamma^* x\mathfrak{a}y\overline{\gamma} = kw$$

for some non-zero $k \in K$ and a vertex w. If $w \in B$, then $\overline{\gamma} \in \mathcal{A}$ and we get $kw = \gamma^* x\mathfrak{a}y\overline{\gamma} = \gamma^* xy\overline{\gamma}\mathfrak{a} = 0$; a contradiction. Hence, $w \in A$. Then there is a monomial $\alpha\beta^* \in \mathcal{A}$ such that $s(\alpha) = w$. Moreover, $\overline{\gamma}\alpha\beta^*$ is a non-zero element of $\mathcal{A}$, and we get

$$k\alpha\beta^* = \gamma^* x\mathfrak{a}y\overline{\gamma}\alpha\beta^* = \gamma^* xy\overline{\gamma}\alpha\beta^*\mathfrak{a} = 0;$$

a contradiction. Similarly, we can show that if $x \in B$, then we also arrive at a contradiction. □

Lemma 3.13. *Let* α, β *be paths such that* $\mathfrak{a}\alpha\beta^* \neq 0$, *with* $\alpha\beta^* \in L_K(A, B)$. *Then*

$$\mathfrak{a}\alpha\beta^* = \alpha'\mathfrak{a}\alpha''\beta^* = \alpha(\beta')^*\mathfrak{a}(\beta'')^*$$

for all paths $\alpha', \alpha'', \beta', \beta''$ *such that* $\alpha = \alpha'\alpha''$ *and* $\beta = \beta''\beta'$.

Proof. By the assumption we made, $\alpha\beta^*\mathfrak{a} = \mathfrak{a}\alpha\beta^* \neq 0$.

Let $\alpha = \alpha'\alpha''$ for some paths α' and α''. If $r(\alpha') \in B$, then $\alpha' \in L_K(A, B)$ and we get $\mathfrak{a}\alpha\beta^* = \alpha'\mathfrak{a}\alpha''\beta^*$. On the other hand, if $r(\alpha') \in A$, then $\alpha''\beta^* \in L_K(A, B)$ and we have $\mathfrak{a}\alpha\beta^* = \alpha\beta^*\mathfrak{a} = \alpha'\mathfrak{a}\alpha''\beta^*$. It should be clear that the remaining equality can be established similarly. □

Lemma 3.14. *If* c *is a cycle in* E *which has exits, then* c *is an extreme cycle if and only if every element of* $T(c^0)$ *is in a cycle which has exits and* $v \geq w$ *for all* $v, w \in T(c^0)$.

Proof. Note that the "if" part is straightforward. Now, suppose that c is an extreme cycle. If $v \in T(c^0)$, then by assumption, $v \geq c^0$, and it follows that there is a closed path τ such that $s(\tau) = v = r(\tau)$. It is not hard to see that in this case there is a cycle c' such that $v \in (c')^0$. Obviously, c' has exits.

Consider any two vertices $v, w \in T(c^0)$. We have $c^0 \geq v$ and $c^0 \geq w$. By assumption we also have $v \geq c^0$, and as $\geq$ is transitive relation, we get finally $v \geq w$. $\square$

Next, we consider the set

$$\mathcal{T} = \{(u, v) \in E^0 \times E^0 : v \in T(u)\},$$

and we fix a function $\pi : \mathcal{T} \to \mathrm{Path}(E)$ such that $s(\pi(u, v)) = u$ and $r(\pi(u, v)) = v$ for all $(u, v) \in \mathcal{T}$. Then $\pi_{u,v}$ stands for $\pi(u, v)$.

Lemma 3.15. *If a vertex $u \in E^0$ is such that $u\mathfrak{a}u \neq 0$, then there is $v \in T(u)$ such that $T(v) \subseteq P_l(E) \cup P_c(E) \cup P_{ec}(E)$ and for every $w \in T(v)$, $w\mathfrak{a}w \neq 0$. Moreover, if* $\deg(\mathfrak{a}) = 0$, *then there is $\bar{v} \in P_l(E) \cup P_c(E) \cup P_{ec}(E)$ such that $\bar{v}\mathfrak{a}\bar{v} = k\bar{v}$ for some non-zero $k \in K$.*

Proof. Let $u\mathfrak{a}u \neq 0$ for a vertex $u \in E^0$. By Proposition 3.6 there are paths $\gamma, \bar{\gamma}$ such that $\gamma^* u\mathfrak{a}u\bar{\gamma} = ku'$ for some non-zero $k \in K$ and a vertex u'.

Case 1. Assume that $u \in A$. Consider the set $T(u')$ and an element $w \in T(u')$. For the path $\pi_{u',w}$ we have $\gamma^*\mathfrak{a}\bar{\gamma}\pi_{u',w} = k\pi_{u',w} \neq 0$.

Suppose that $w \in B$. Then $\bar{\gamma}\pi_{u',w} \in \mathcal{A}$ and $\gamma^*\bar{\gamma}\pi_{u',w}\mathfrak{a} = \gamma^*\mathfrak{a}\bar{\gamma}\pi_{u',w} \neq 0$ which yields $w\mathfrak{a}w \neq 0$. If $w \in A$ then there is $\alpha\beta^* \in \mathcal{A}$ such that $s(\alpha) = w$. Then $\bar{\gamma}\pi_{u',w}\alpha\beta^* \in \mathcal{A}\backslash\{0\}$ and using Lemma 3.13 we get $\gamma^*\bar{\gamma}\pi_{u',w}\mathfrak{a}\alpha\beta^* = \gamma^*\mathfrak{a}\bar{\gamma}\pi_{u',w}\alpha\beta^* \neq 0$ and the fact that $w\mathfrak{a}w \neq 0$ follows. So if $u \in A$, then $w\mathfrak{a}w \neq 0$ for any $w \in T(u')$.

Case 2. Suppose that $u \in B$. Consider any $w \in T(u')$ and $\pi_{u',w}$. Suppose also that $w \in B$. As (A, B) is a fully downward directed partition of E^0, then there is $\alpha\beta^* \in \mathcal{A}$ such that $r(\beta^*) = w$. Then $\alpha\beta^*\pi^*_{u',w}\gamma^*$ is a non-zero element of $\mathcal{A}$, and we have

$$\alpha\beta^*\mathfrak{a}\pi^*_{u',w}\gamma^*\bar{\gamma} = \alpha\beta^*\pi^*_{u',w}\gamma^*\mathfrak{a}\bar{\gamma} = k\alpha\beta^*\pi^*_{u',w} \neq 0,$$

which implies that $w\mathfrak{a}w \neq 0$.

If $w \in A$, then $\pi^*_{u',w}\gamma^* \in \mathcal{A}$ and we get

$$\mathfrak{a}\pi^*_{u',w}\gamma^*\bar{\gamma} = \pi^*_{u',w}\gamma^*\mathfrak{a}\bar{\gamma} = k\pi^*_{u',w} \neq 0.$$

So also in this case we have $w\mathfrak{a}w \neq 0$.

Summarizing the above considered cases, for any $w \in T(u')$ we have $w\mathfrak{a}w \neq 0$.

It should be clear from the foregoing arguments that the set $T(u')$ is finite. Then using Lemma 3.14 we can conclude that there is $v \in T(u')$ such that $v \in \mathbb{S} \cup P_c(E) \cup$

$P_{ec}(E) \subseteq P_l(E) \cup P_c(E) \cup P_{ec}(E)$. Since $T(v) \subseteq T(u')$, it is clear that the first part of the result has been proved.

Now, assume that $\deg(\mathfrak{a}) = 0$, and let $v \in T(u)$ be such that $T(v) \subseteq P_l(E) \cup P_c(E) \cup P_{ec}(E)$ and for every $w \in T(v)$, $w\mathfrak{a}w \neq 0$.

Obviously, if $v \in P_l(E)$, then there is $\bar{v} \in T(v) \cap \mathbb{S}$ such that $\bar{v}\mathfrak{a}\bar{v} = k\bar{v}$ for some non-zero $k \in K$.

Suppose that $v \in c^0$, where c is either a cycle without exits or an extreme cycle. By the above we have $w\mathfrak{a}w \neq 0$ for every $w \in T(v) \subseteq T(u')$.

Suppose now that $v \in A$. Then there is $x \in B$ such that for a non-zero monomial $\alpha\beta^* \in \mathcal{A}$ we have $s(\alpha) = v$ and $r(\beta^*) = x$. Consider any positive integer m and the element $c_v^m\alpha\beta^*$, which is a non-zero element in $\mathcal{A}$. Then

$$(c_v^m)^*\mathfrak{a}c_v^m\alpha\beta^* = (c_v^m)^*c_v^m\alpha\beta^*\mathfrak{a} = \alpha\mathfrak{a}\beta^*.$$

Since $\alpha^*\alpha\mathfrak{a}\beta^*\beta = \bar{v}\mathfrak{a}\bar{v}$ for $\bar{v} = r(\alpha)$, and $\bar{v} \in T(v)$, we have $(c_v^m)^*\mathfrak{a}c_v^m \neq 0$. Thus, by [14, Lemma 3.3(c)], for a big enough positive integer n we get $(c_v^n)^*\mathfrak{a}c_v^n = kv$ for some non-zero $k \in K$. Then

$$\bar{v}\mathfrak{a}\bar{v} = \alpha^*\alpha\mathfrak{a}\beta^*\beta = \alpha^*(c_v^n)^*c_v^n\alpha\mathfrak{a}\beta^*\beta = \alpha^*(c_v^n)^*\mathfrak{a}c_v^n\alpha\beta^*\beta = k\alpha^*\alpha\beta^*\beta = k\bar{v}.$$

Note that by Lemma 3.14 we can see that, $\bar{v} \in P_c(E) \cup P_{ec}(E)$, as required.

If $v \in B$, then similar considerations enable us to find $\bar{v} \in P_c(E) \cup P_{ec}(E)$ such that $\bar{v}\mathfrak{a}\bar{v} = k\bar{v}$ for some non-zero $k \in K$. $\square$

Any element $v \in P_l(E) \cup P_c(E) \cup P_{ec}(E)$ such that $w\mathfrak{a}w \neq 0$ for every $w \in T(v)$ will be called a *leading vertex of* $\mathfrak{a}$. Notice that if v is a leading vertex of an element $\mathfrak{a}$ then every element of $T(v)$ is also like that.

Now, we would like to show (in Lemma 3.18) that in the case where $\deg(\mathfrak{a}) \neq 0$, there is a leading vertex of $\mathfrak{a}$ with some special properties, which are useful in our further considerations. In order to achieve our goal, we need two preliminary results.

Lemma 3.16. *Let K be a field, E be a graph and λ be a closed path which is neither a cycle without exits nor a positive power of a cycle without exits. If*

$$\lambda\sigma = \sigma\sigma' \tag{3}$$

for some paths σ, σ', then there is a path δ such that $\sigma\delta \neq 0$ and

$$(\sigma\delta)^*\lambda(\sigma\delta) = 0 = (\sigma\delta)^*\lambda^*(\sigma\delta).$$

Proof. Let $e_1, e_2, \ldots, e_n$ be edges such that $\lambda = e_1e_2\ldots e_n$. By the assumption on λ there is an i, $0 \leq i \leq n-1$, such that, for some edge f, $s(f) = s(e_i)$ and $f \neq e_i$.

It is not hard to check that there is a non-negative integer k such that $\sigma = \lambda^k\tau$ for a path τ such that $|\tau| < |\lambda|$. Moreover, $\lambda = \tau\bar{\lambda}$ for some path $\bar{\lambda}$.

Let $\delta = \bar{\lambda}e_1e_2\ldots e_{i-1}f$. Straightforward calculations show that $\sigma\delta \neq 0$ and $(\sigma\delta)^*\lambda(\sigma\delta) = 0 = (\sigma\delta)^*\lambda^*(\sigma\delta)$. $\square$

Lemma 3.17. *Let K be a field, E be a graph, σ be a path, and λ be a nontrivial closed path in E. If either $\sigma^*\lambda\sigma \neq 0$ or $\sigma^*\lambda^*\sigma \neq 0$, then $\lambda\sigma = \sigma\sigma'$ for some $\sigma' \in \mathrm{Path}(E)$.*

Proof. First notice that $\lambda\sigma \neq 0$.

Assume now that $|\lambda| < |\sigma|$. Then, regardless of whether $\sigma^*\lambda\sigma \neq 0$ or $\sigma^*\lambda^*\sigma \neq 0$, we get $\sigma = \lambda\sigma''$ for some path σ''. Thus $\sigma^*\lambda\sigma = \sigma''^*\lambda^*\lambda\sigma = \sigma''^*\sigma$ and $\sigma^*\lambda^*\sigma = \sigma^*\lambda^*\lambda\sigma'' = \sigma^*\sigma''$ and it follows that $\sigma = \sigma''\sigma'$ for some $\sigma' \in \mathrm{Path}(E)$. Hence $\lambda\sigma = \lambda\sigma''\sigma' = \sigma\sigma'$.

If $|\lambda| \geq |\sigma|$, then using the assumption that either $\sigma^*\lambda\sigma \neq 0$ or $\sigma^*\lambda^*\sigma \neq 0$, we conclude that $\lambda = \sigma\sigma''$ for some $\sigma'' \in \mathrm{Path}(E)$. In this case $\lambda\sigma = \sigma\sigma'$ for $\sigma' = \sigma''\sigma$. $\square$

Lemma 3.18. *If $\deg(\mathfrak{a}) \neq 0$, then there is a leading vertex v of $\mathfrak{a}$ such that $v\mathfrak{a}v = kc_v^s$ for a cycle c_v without exits, non-zero $k \in K$ and a non-zero integer s.*

Proof. The existence of a leading vertex v of $\mathfrak{a}$ follows from Lemma 3.15. Obviously, if $\deg(\mathfrak{a}) \neq 0$, then v cannot be an element of $P_l(E)$ by the same lemma.

Suppose, for a contradiction, that v is a vertex of an extreme cycle c. Consider the case $\deg(\mathfrak{a}) > 0$. Using Proposition 3.6, there are $\alpha\beta^* \in supp(\mathfrak{a})$ and paths $\gamma = \alpha\sigma, \bar{\gamma} = \beta\sigma$ such that

$$\gamma^*(v\mathfrak{a}v)\bar{\gamma} = \gamma^*\alpha\beta^*\bar{\gamma} = ku \tag{4}$$

for some non-zero $k \in K$ and a vertex $u \in E^0$. As $v = s(\alpha)$, $v \in c^0$ and $v \geq u$, by Lemma 3.14 we have $u \in (c')^0$ for some extreme cycle c'.

Suppose that $v \in A$. Then there is a monomial $\tau\eta^*$ such that $\bar{\gamma}\tau\eta^* \in \mathcal{A}\backslash\{0\}$. Therefore we have the non-zero element $\gamma^*\mathfrak{a}\bar{\gamma}\tau\eta^* = \gamma^*\bar{\gamma}\mathfrak{a}\tau\eta^*$, which implies that $\sigma^*\alpha^*\beta\sigma \neq 0$, and so $\alpha = \beta\lambda$ for some non-trivial path λ. As $\alpha\beta^* = \beta\lambda\beta^* \neq 0$, it follows that λ is a closed path with all vertices in $T(v)$. This implies that λ can neither be a cycle without exits nor a positive power of a cycle without exits.

Since $\gamma^*\bar{\gamma} \neq 0$, we have $\sigma^*\lambda^*\sigma = \sigma^*\lambda^*\beta^*\beta\sigma \neq 0$. By Lemmas 3.16 and 3.17 there is a path δ such that $\sigma\delta \neq 0$ and

$$\delta^*\sigma^*\lambda^*\sigma\delta = 0. \tag{5}$$

Note that $s(\delta) = u$. Then

$$ku = \delta^*\gamma^*(v\mathfrak{a}v)\bar{\gamma}\delta.$$

As $v \in A$ and $s(\bar{\gamma}) = v$, we can find a monomial $\bar{\eta}\theta^*$ such that $\bar{\gamma}\delta\bar{\eta}\theta^*$ is a non-zero element of $\mathcal{A}$. Then using Lemma 3.13, we get

$$\begin{aligned} k\bar{\eta}\theta^* &= \underbrace{\delta^*\gamma^*(v\mathfrak{a}v)\bar{\gamma}\delta}_{ku}\bar{\eta}\theta^* = \delta^*\gamma^*\bar{\gamma}\delta\mathfrak{a}\bar{\eta}\theta^* = \delta^*\underbrace{\sigma^*\lambda^*\beta^*}_{\gamma^*}\underbrace{\beta\sigma}_{\bar{\gamma}}\delta\mathfrak{a}\bar{\eta}\theta^* \\ &= \delta^*\sigma^*\lambda^*\sigma\delta\mathfrak{a}\bar{\eta}\theta^* \overset{\text{by (5)}}{=} 0; \end{aligned} \tag{6}$$

a contradiction.

Similarly, we get a contradiction if we assume that $v \in B$. Thus $v \in c^0$ for some cycle c without exits.

As c has no exits, looking at (4) we get $u \in c^0$ and $\gamma\gamma^* = v = \bar{\gamma}\bar{\gamma}^*$. So we have

$$v\mathfrak{a}v = \gamma\gamma^*\mathfrak{a}\bar{\gamma}\bar{\gamma}^* = k\gamma\bar{\gamma}^*.$$

Since $\deg(\mathfrak{a}) > 0$ and $v \in c^0$, we deduce that $\gamma = \bar{\gamma}\lambda$ for some path λ. As $s(\lambda) = r(\lambda)$ we get $\lambda = c^s_{r(\bar{\gamma})}$ for some positive integer s. So $v\mathfrak{a}v = k\bar{\gamma}c^s_{r(\bar{\gamma})}\bar{\gamma}^* = kc^s_v \neq 0$. Thus the proof is complete in the case $v \in A$ and $\deg(\mathfrak{a}) > 0$. It is no hard to check that the other cases can be treated similarly, and so we have finished the proof. □

Lemma 3.19. *If* $\deg(\mathfrak{a}) = 0$ *and* $u\mathfrak{a}u \neq 0$ *for a vertex* $u \in E^0$, *then* $w\mathfrak{a}w \neq 0$ *for every vertex* w *such that* $u \in T(w)$. *Moreover, if* $w \in c^0$ *for some cycle* c, *then* $(c^m_w)^*\mathfrak{a}c^m_w \neq 0$ *for every positive integer* m.

Proof. By Lemma 3.15 there is a $u' \in T(u)$ such that

$$u'\mathfrak{a}u' = ku' \tag{7}$$

for some non-zero $k \in K$. Obviously, if $u \in T(w)$ for some $w \in E^0$, then also $u' \in T(w)$. In fact, we will only consider the case where $w \in c^0$ for some cycle c, which we assume from now on. It should be clear from the proof that the rest is also true.

Consider the case $w \in A$, and take the path $\pi_{w,u'}$.

If $u' \in B$, then $\pi_{w,u'} \in \mathcal{A}$, and if $u' \in A$, then there is a monomial $\alpha\beta^* \in \mathcal{A}$ such that $s(\alpha) = u'$. In the first case, $c^m_w\pi_{w,u'} \in \mathcal{A}$ for any positive integer m, and using (7), we have $(c^m_w)^*\mathfrak{a}c^m_w\pi_{w,u'} = (c^m_w)^*c^m_w\pi_{w,u'}\mathfrak{a} = \pi_{w,u'}\mathfrak{a} = k\pi_{w,u'} \neq 0$. Otherwise, we consider $0 \neq c^m_w\pi_{w,u'}\alpha\beta^* \in \mathcal{A}$, and we get

$$(c^m_w)^*\mathfrak{a}c^m_w\pi_{w,u'}\alpha\beta^* = \pi_{w,u'}\mathfrak{a}\alpha\beta^* = k\pi_{w,u'}\alpha\beta^* \neq 0.$$

If $w \in B$, then similar considerations lead us to the required facts. □

Lemma 3.20. *If* $\deg(\mathfrak{a}) \neq 0$, $w\mathfrak{a}w \neq 0$ *and* $w \in c^0$ *for some cycle* c, *then* c *has no exits.*

Proof. Suppose, for a contradiction, that $\deg(\mathfrak{a}) > 0$ and for some cycle c with exits and for some $w \in c^0$, $w\mathfrak{a}w \neq 0$ (the other cases can be treated similarly). By Lemma 3.18 there is a $u \in T(w)$ such that

$$u\mathfrak{a}u = k\bar{c}^s_u \tag{8}$$

for some non-zero $k \in K$ and a non-zero integer s, where $\bar{c}$ is a cycle without exits.

Consider the case $w \in A$, and take the path $\pi_{w,u}$.

Let m be a positive integer. We consider two cases: either $u \in B$ or $u \in A$. In the first situation, using (8), we have

$$(c^m_w)^*\mathfrak{a}c^m_w\pi_{w,u} = \pi_{w,u}\mathfrak{a} = k\pi_{w,u}\bar{c}^s_u \neq 0. \tag{9}$$

Otherwise, as $u \in \bar{c}^0$ and $\bar{c}$ is a cycle without exits, it is not difficult to see that there is a $\beta^* \in \mathcal{A}$ such that $s(\beta^*) = u$. Then also $c_w^m \pi_{w,u}\beta^* \in \mathcal{A}$ and

$$(c_w^m)^* \mathfrak{a} c_w^m \pi_{w,u}\beta^* = \pi_{w,u}\mathfrak{a}\beta^* = k\pi_{w,u}\bar{c}_u^s \beta^* \neq 0. \tag{10}$$

In both (9) and (10) we have, for every positive integer m,

$$(c_w^m)^* \mathfrak{a} c_w^m \pi_{w,u} = k\pi_{w,u}\bar{c}_u^s \neq 0. \tag{11}$$

On the other hand, by [14, Lemma 3.3], for a fixed, but large enough n,

$$(c_w^n)^* \mathfrak{a} c_w^n = k' c_w^t \tag{12}$$

for some positive integer t and some non-zero $k' \in K$ (in fact, in this case one can see that $k' = k$). Note that $t \cdot |c_w| = s \cdot |\bar{c}_u|$. It follows from (11) and (12) that

$$c_w^t \pi_{w,u} = \pi_{w,u}\bar{c}_u^s. \tag{13}$$

Now, by Lemma 3.16, there is a path δ such that $\pi_{w,u}\delta \neq 0$ (so $s(\delta) = u$), and $(\pi_{w,u}\delta)^* c_w^t \pi_{w,u}\delta = 0$. But, by (13),

$$(\pi_{w,u}\delta)^* c_w^t \pi_{w,u}\delta = (\pi_{w,u}\delta)^* \pi_{w,u}\bar{c}_u^s \delta = \delta^* \bar{c}_u^s \delta \neq 0,$$

since $\bar{c}_u$ does not have exits and $s(\delta) = u$; a contradiction.

If $w \in B$, then similar arguments get us home. □

Lemma 3.21. *If* $\deg(\mathfrak{a}) \neq 0$ *and* $w\mathfrak{a}w \neq 0$ *for some* $w \in c^0$, *with* c *a cycle without exits, then* $u\mathfrak{a}u \neq 0$ *for every vertex* u *such that* $w \in T(u)$.

Proof. Since for the considered $w \in c^0$, $w\mathfrak{a}w \neq 0$ and c is a cycle without exits, we may assume that $w\mathfrak{a}w = kc_w^s$ for some non-zero $k \in K$ and a non-zero integer s. Suppose that $w \in T(u)$ for a vertex u, and suppose that $u \in A$. Consider $\pi_{u,w}$. It follows readily that there is a path δ^* such that $s(\delta^*) = w$ and $r(\delta) \in B$. Then $\pi_{u,w}\delta^* \in \mathcal{A}$. Hence, we have $\mathfrak{a}\pi_{u,w}\delta^* = \pi_{u,w}\mathfrak{a}\delta^* = k\pi_{u,w}c_w^s\delta^* \neq 0$. Since the other cases can be treated similarly, the proof is complete. □

Lemma 3.22. *Let* $u, v \in E^0$. *If* $u \sim^1 v$ *and* $u\mathfrak{a}u \neq 0$, *then* $v\mathfrak{a}v \neq 0$.

Proof. Consider firstly the case $\deg(\mathfrak{a}) = 0$. It should be clear, using Lemma 3.15, that $v\mathfrak{a}v \neq 0$ if $u \geq v$ or $v \geq u$, and there are no bifurcations at any vertex in $T(u) \cup T(v)$.

Suppose now that there is a cycle $\bar{c}$ such that $\bar{c}^0 \geq u$ and $\bar{c}^0 \geq v$. By Lemma 3.19 for every $w \in \bar{c}^0$, $(\bar{c}_w^m)^* \mathfrak{a}\bar{c}_w^m \neq 0$ for every positive integer m. So it follows from [14, Lemma 3.3] that for a big enough positive integer n

$$(\bar{c}_w^n)^* \mathfrak{a}\bar{c}_w^n = k'w \tag{14}$$

for some non-zero $k' \in K$.

Take the path $\pi_{w,v}$ and consider the following cases:

Case 1. $w, v \in A$: In this case there is a monomial $\alpha\beta^* \in \mathcal{A}$ such that $s(\alpha) = v$. Then $\bar{c}_w^n \pi_{w,v}\alpha\beta^* \in \mathcal{A}$, and using Lemma 3.13 we get

$$(\bar{c}_w^n)^* \bar{c}_w^n \pi_{w,v}\mathfrak{a}\alpha\beta^* = (\bar{c}_w^n)^* \mathfrak{a}\bar{c}_w^n \pi_{w,v}\alpha\beta^* = k'\pi_{w,v}\alpha\beta^* \neq 0,$$

which implies that $v\mathfrak{a}v \neq 0$.

Case 2. $w \in A,\ v \in B$: Then $\bar{c}_w^n \pi_{w,v} \in \mathcal{A}$. and similar arguments as in Case 1 suffice.

Case 3. $w, v \in B$: Then there is a monomial $\alpha\beta^* \in \mathcal{A}$ such that $r(\beta^*) = v$. So also $\alpha\beta^*\pi_{w,v}^*(\bar{c}_w^n)^* \in \mathcal{A}$, and we get

$$\alpha\beta^*\mathfrak{a}\pi_{w,v}^*(\bar{c}_w^n)^*\bar{c}_w^n = \alpha\beta^*\pi_{w,v}^*(\bar{c}_w^n)^*\mathfrak{a}\bar{c}_w^n = k'\alpha\beta^*\pi_{w,v}^* \neq 0,$$

which gives $v\mathfrak{a}v \neq 0$.

Case 4. $w \in B,\ v \in A$. Now $\pi_{w,v}^*(\bar{c}_w^n)^* \in \mathcal{A}$, and similar arguments as in the Case 3 can be used to show what we need.

If $\deg(\mathfrak{a}) \neq 0$ then by Lemmas 3.18, 3.20 and 3.21 there is only one possibility, namely there is a cycle c without exits such that $u, v \in c^0$. But then the result follows from Lemma 3.15. The proof is complete. □

By the above considerations we have the following proposition:

Proposition 3.23. *If* $\deg(\mathfrak{a}) = 0$, *then there is a vertex* $v \in P$ *such that the sets* $\overline{[v]}$ *and* $F_E(\overline{[v]})$ *are both finite,* $u\mathfrak{a}u \neq 0$ *for every* $u \in \overline{[v]}$, *and* $v\mathfrak{a}v = kv$ *for some non-zero* $k \in K$.

Proof. By Lemma 3.15 there is a vertex $v \in P$ such that $v\mathfrak{a}v = kv$ for some non-zero $k \in K$. If $u \in [v]$, then there is a sequence of vertices $v = v_0 \sim^1 v_1 \sim^1 \cdots \sim^1 v_q = u$, but then Lemma 3.22 gives $u\mathfrak{a}u \neq 0$. If $u \in \overline{[v]}$, then there is a path η such that $s(\eta) = u$ and $r(\eta) \in [v]$. This fact, in conjunction with Lemma 3.19, shows that $u\mathfrak{a}u \neq 0$. Obviously, $\overline{[v]}$ must be a finite set. Also by Lemma 3.19 there are finitely many vertices $w \in E^0 \setminus \overline{[v]}$ such that there is a path μ with the property $s(\mu) = w$ and $r(\mu) \in \overline{[v]}$. Note that no such w can be in a cycle, because then $w \in [v]$; a contradiction. Finally, we deduce that the set $F_E(\overline{[v]})$ is finite. □

Proposition 3.24. *If* $\deg(\mathfrak{a}) \neq 0$, *then there is a cycle* $c \in \mathcal{C}$ *such that*

(i) *for every* $u \in c^0$, $u\mathfrak{a}u \neq 0$,
(ii) *for some* $v \in c^0$, $v\mathfrak{a}v = kc_v^s$ *for some non-zero* $k \in K$ *and some non-zero integer* s,
(iii) *for every* $\alpha \in F_E(c^0)$, $w\mathfrak{a}w \neq 0$ *where* $w = s(\alpha)$.

Proof. By Lemma 3.18 there is a cycle c without exits such that conditions (i) and (ii) hold. Now, Lemmas 3.20 and 3.21 give $c \in \mathcal{C}$ and (iii). □

We are now in a position to prove Theorem 3.8.

Proof. Lemma 3.12 is needed. Let $\mathfrak{a}$ be an element such that the set $\{u \in E^0 : u\mathfrak{a}u \neq 0\}$ has the smallest cardinality among all the elements of $L_K(E)$ which commute with $\mathcal{A}$ and do not belong to $\mathcal{A}$.

Case 1. Suppose that $\deg(\mathfrak{a}) = 0$. By Proposition 3.23 there is a leading vertex v such that $v\mathfrak{a}v = kv$ for some non-zero $k \in K$. Moreover, $|\overline{[v]}| < \infty$ and $|F_E(\overline{[v]})| < \infty$. Consider the element $r_{[v]}$ belonging to $\mathcal{B}_0$ defined in (2). Note that $r_{[v]} \in \sum_{w \in E^0} wL_K(E)w$, and if $w \in E^0$ is such that $wr_{[v]}w \neq 0$, then $w\mathfrak{a}w \neq 0$. Obviously, $\mathfrak{a} - kr_{[v]}$ commutes with all the elements of $\mathcal{A}$, and $\mathfrak{a} - kr_{[v]} \notin \mathcal{A}$. By the assumption we made at the beginning of this proof and the fact that

$$v(\mathfrak{a} - kr_{[v]})v = v\mathfrak{a}v - kvr_{[v]}v = kv - kv = 0,$$

we deduce that $\mathfrak{a} = kr_{[v]}$; a contradiction.

Case 2. Suppose that $\deg(\mathfrak{a}) = n$, where n is a non-zero integer. Then there is a cycle c without exits and a leading vertex v of $\mathfrak{a}$ which belongs to c^0 such that $v\mathfrak{a}v = kc_v^s$ for some non-zero $k \in K$ and some non-zero integer s such that $s \cdot |c| = \deg(\mathfrak{a})$. By Proposition 3.24, $c \in \mathcal{C}$, and one of the non-zero elements of $\mathcal{B}_n$ is

$$r = \sum_{\substack{\alpha \in F_E(c^0) \cup c^0 \\ u \in c^0}} \alpha c_u^s \alpha^*.$$

As in the previous case, $\mathfrak{a} - kr$ commutes with all the elements of $\mathcal{A}$, and $\mathfrak{a} - kr \notin \mathcal{A}$. Since $v(\mathfrak{a} - kr)v = 0$, we get $\mathfrak{a} = kr$; a contradiction.

We conclude that $\mathcal{A}$ is indeed a maximal commutative subalgebra of $L_K(E)$. $\square$

In the following example, we want to present for a concrete graph what the algebra $\mathcal{A}$ looks like.

Example 3.25. Consider the following graph:

$$E \equiv \bullet^{A}_{v_1} \xleftarrow{\;e\;} \bullet^{B}_{v_2} \xrightarrow{\;f\;} \bullet^{A}_{v_3} \circlearrowright c\,.$$

Then the set

$$\mathcal{B} = \{v_1 + v_2 + v_3\} \cup \{fc^n f^* + c^n \,|\, n \in \mathbb{Z}\backslash\{0\}\},$$

is a basis for $Z(L_K(E))$ and the set

$$\mathcal{C} = \{f^*, e^*\} \cup \{c^n f^* \,|\, n \in \mathbb{Z}\backslash\{0\}\},$$

is a basis for $L_K(A, B)$. And we have $\mathcal{A} = L_K(A, B) + Z(L_K(E))$.

4. Downward Directed Pairs

In this section, we will consider the case where a pair (A, B) of subsets of E^0 which constitutes a partition of E^0 is not a fully downward directed pair. In such a case we to take into account the following two subsets of E^0:

$$\mathcal{U}(A) = \{u \in A\colon \text{ if } v \in E^0 \text{ and } T(u) \cap T(v) \neq \emptyset, \text{ then } v \in A\},$$

$$\mathcal{U}(B) = \{v \in B\colon \text{ if } u \in E^0 \text{ and } T(u) \cap T(v) \neq \emptyset, \text{ then } u \in B\}.$$

It is obvious that $\mathcal{U}(A) = A \backslash \mathcal{D}(A)$ (and $\mathcal{U}(B) = B \backslash \mathcal{D}(B)$), but we "think" of $\mathcal{U}(A)$ and $\mathcal{U}(B)$ in the above form.

Elements of the set $\mathcal{U} = \mathcal{U}(A) \cup \mathcal{U}(B)$ will be called *uniform elements* of the pair (A, B).

Now, we would like to present a useful lemma whose proof we leave to the reader.

Lemma 4.1. *With the above notation, we have the following properties of the sets $\mathcal{U}(A)$ and $\mathcal{U}(B)$:*

(i) *The collection $\{\mathcal{D}(A), \mathcal{D}(B), \mathcal{U}(A), \mathcal{U}(B)\}$ of subsets of E^0 is a partition of E^0.*
(ii) *$\mathcal{U}$ is a hereditary and saturated subset of E^0.*
(iii) *If for a vertex $v \in E^0$, $v \geq \mathcal{U}(A)$, then $v \ngeq \mathcal{U}(B)$ and $v \in A$.*
(iv) *If for a vertex $v \in E^0$, $v \geq \mathcal{U}(B)$, then $v \ngeq \mathcal{U}(A)$ and $v \in B$.*
(v) *If $v \geq \mathcal{U}(A)$, then $u \geq \mathcal{U}(A)$ for every $u \in E^0$ such that $v \in T(u)$.*
(vi) *If $v \geq \mathcal{U}(B)$, then $u \geq \mathcal{U}(B)$ for every $u \in E^0$ such that $v \in T(u)$.*

Let $I(\mathcal{U})$ be the ideal of $L_K(E)$ generated by all the elements (vertices) of $\mathcal{U}$, and let $E/\mathcal{U}$ denote the quotient graph

$$(E^0 \backslash \mathcal{U}, \{e \in E^1 : r(e) \notin \mathcal{U}\}, r|_{(E/\mathcal{U})^1}, s|_{(E/\mathcal{U})^1}).$$

Sticking to the above notation, by [12, Lemma 2.3] we have the following lemma:

Lemma 4.2. *Define $\Psi : L_K(E) \to L_K(E/\mathcal{U})$ by setting*

$$\Psi(v) = \chi_{(E/\mathcal{U})^0}(v)v, \quad \Psi(e) = \chi_{(E/\mathcal{U})^1}(e)e \quad \textit{and} \quad \Psi(e^*) = \chi_{((E/\mathcal{U})^1)^*}(e^*)e^*$$

for every vertex $v \in E^0$ and every edge $e \in E^1$, where $\chi_{(E/\mathcal{U})^0} : E^0 \to K$ and $\chi_{(E/\mathcal{U})^1} : E^1 \to K$ are the characteristic functions. Then the map Ψ extends to a K-algebra epimorphism of $\mathbb{Z}$-graded algebras, with $\mathrm{Ker}(\Psi) = I(\mathcal{U})$, which gives $L_K(E \backslash \mathcal{U}) \cong L_K(E)/I(\mathcal{U})$.

Lemma 4.3. *If c is a cycle in $E/\mathcal{U}$ which does not have exits in this graph, then neither does c have exits in E.*

Proof. Suppose that c is a cycle in $E/\mathcal{U}$ which, in this graph, does not have exits, and suppose that c has exits in E. Let e be an edge which is an exit for c in E. Then $s(e) \in c^0 \subseteq E^0 \backslash \mathcal{U}$ and $r(e) \in \mathcal{U} = \mathcal{U}(A) \cup \mathcal{U}(B)$. Without loss of generality, we may assume that $r(e) \in \mathcal{U}(A)$. Then Lemma 4.1 implies that for every edge f which is an exit for c, $r(f) \in \mathcal{U}(A)$ and $c^0 \subseteq \mathcal{D}(A)$. Also by Lemma 4.1, for every vertex u such that $u \geq c^0$, we have $u \in \mathcal{U}(A)$. However, then we deduce that $s(e) \in \mathcal{U}(A)$; a contradiction. $\square$

Although $L_K(E\backslash\mathcal{U})$ need not be a subalgebra of $L_K(E)$, as is easily seen, we want to consider the elements of the center $Z(L_K(E\backslash\mathcal{U}))$ of $L_K(E\backslash\mathcal{U})$ as elements of $L_K(E)$. We denote the vector space of $L_K(E)$ spanned by all such elements simply by $\Psi^{-1}(Z(L_K(E\backslash\mathcal{U})))$, which will hopefully not cause any confusion.

Recall (see [28]) that the *commutative core* of a Leavitt path algebra $L_K(E)$, denoted by $\mathcal{M}_K(E)$, is a subalgebra $L_K(E)$, and it is generated by all elements of the form

$$\alpha\alpha^*, \quad \alpha c\alpha^* \quad \text{and} \quad \alpha c^*\alpha^*,$$

where α is a path and c is a cycle without exits. We consider the subset $G(\mathcal{U})$ of the above set of generators of $\mathcal{M}_K(E)$ comprising all the elements of the above form with $r(\alpha) \in \mathcal{U}$. The subalgebra of $L_K(E)$ generated by $G(\mathcal{U})$ is denoted by $\mathcal{M}_K(\mathcal{U})$.

As the main theorem of this section we prove the following theorem:

Theorem 4.4. *Let E be a row-finite graph and let K be a field. Consider a pair (A, B) of sets which constitutes a partition of E^0. Then the vector space*

$$\mathcal{A}(A, B) = L_K(\mathcal{D}(A), \mathcal{D}(B)) + \Psi^{-1}(Z(L_K(E \setminus \mathcal{U}))) + \mathcal{M}_K(\mathcal{U})$$

is a maximal commutative subalgebra of $L_K(E)$.

Proof. If $\mathcal{U} = \emptyset$, then the result follows from Theorem 3.8. Thus, we may assume that $\mathcal{U} \neq \emptyset$.

Firstly, we will show that any two elements of $\mathcal{A}(A, B)$ commute with each other. Note that

$$\Psi(\mathcal{A}(A, B)) = \Psi(L_K(\mathcal{D}(A), \mathcal{D}(B)) + \Psi^{-1}(Z(L_K(E\backslash\mathcal{U})))),$$

and that it follows from Theorem 3.8 that $\Psi(\mathcal{A}(A, B))$ is a maximal commutative subalgebra of $L_K(E\backslash\mathcal{U})$. Therefore, for every monomial

$$\alpha\beta^* \in L_K((\mathcal{D}(A), \mathcal{D}(B))$$

and every

$$x \in \Psi^{-1}(Z(L_K(E\backslash\mathcal{U}))) + \mathcal{M}_K(\mathcal{U}),$$

$\alpha\beta^* \cdot x - x \cdot \alpha\beta^* \in I(\mathcal{U})$. It is evident from Theorem 3.10 and the construction of $\mathcal{M}_K(\mathcal{U})$ that

$$x = \sum_{v \in E^0} vxv. \tag{15}$$

Suppose, for a contradiction, that $\alpha\beta^* \cdot x - x \cdot \alpha\beta^* \neq 0$. Then for some paths γ and $\overline{\gamma}$, some non-zero $k \in K$ and a some vertex $u \in \mathcal{U}$,

$$\gamma^*(\alpha\beta^* \cdot x - x \cdot \alpha\beta^*)\overline{\gamma} = ku.$$

By (15), with $s(\alpha) \in A$ and $r(\beta^*) \in B$, we have $s(\alpha) \geq u$ and $r(\beta^*) \geq u$; a contradiction (see Lemma 4.1). Since the product of any two elements of $L_K((\mathcal{D}(A), \mathcal{D}(B))$ is zero, the foregoing arguments imply that all the elements of $L_K((\mathcal{D}(A), \mathcal{D}(B))$ commute with all the elements of $\mathcal{A}(A, B)$.

By the construction of the commutative core $\mathcal{M}_K(E)$ (see [28]), Theorem 3.10, Lemma 4.3 and the construction of $\mathcal{M}_K(\mathcal{U})$, it is not difficult to see that all the elements of $\Psi^{-1}(Z(L_K(E\backslash\mathcal{U}))) + \mathcal{M}_K(\mathcal{U})$ belong to the commutative core $\mathcal{M}_K(E)$ of $L_K(E)$, and so they commute with one another. We conclude that indeed all the elements of $\mathcal{A}(A, B)$ commute with one another.

Next, suppose, for a contradiction, that there is an element $\mathfrak{a}$ of $L_K(E)$ which commutes with all elements of $\mathcal{A}(A, B)$, but which does not belong to $\mathcal{A}(A, B)$. By Theorem 3.8, the algebra $\Psi(\mathcal{A}(A, B))$ is a maximal commutative subalgebra of $L_K(E\backslash\mathcal{U})$, and so $\Psi(\mathfrak{a}) \in \Psi(\mathcal{A}(A, B))$. Thus there is an $\mathfrak{a}_I \in I(\mathcal{U})$ such that $\mathfrak{a} = x + \mathfrak{a}_I$, where

$$x \in L_K((\mathcal{D}(A), \mathcal{D}(B)) + \Psi^{-1}(Z(L_K(E\backslash\mathcal{U}))).$$

In particular, $\mathfrak{a}_I$ commutes with all the elements of $\mathcal{M}_K(\mathcal{U})$. It is easy to check that $\mathfrak{a}_I = \sum_{j\in J} \sigma_j \delta_j^*$ for some finite set J of indices and paths σ_j and δ_j with the property that

$$r(\sigma_j) = s(\delta_j^*) \in \mathcal{U} \tag{16}$$

for every $j \in J$. For a contradiction we may assume that none of the monomials $\sigma_j \delta_j^*$ belong to $\mathcal{M}_K(\mathcal{U})$.

Let $\gamma, \overline{\gamma}^* \in \text{Path}(E)$ be such that $\gamma^* \mathfrak{a}_I \overline{\gamma} = ku$ for some non-zero $k \in K$ and some $u \in E^0$. By Proposition 3.6, there is an $i \in J$ and a path τ such that

$$\gamma = \sigma_i \tau \quad \text{and} \quad \overline{\gamma} = \delta_i \tau. \tag{17}$$

As $\mathcal{U}$ is a hereditary set, it follows from (16) that

$$\gamma\gamma^*, \quad \overline{\gamma}\overline{\gamma}^* \in \mathcal{M}_K(\mathcal{U}), \tag{18}$$

which in conjunction with the fact that $\mathfrak{a}_I$ commutes with all the elements of $\mathcal{M}_K(\mathcal{U})$, implies that

$$\gamma^* \overline{\gamma} \neq 0. \tag{19}$$

Hence, by (17), $\tau^* \sigma_i^* \delta_i \tau \neq 0$, and finally, $\sigma_i^* \delta_i \neq 0$.

If $|\sigma_i| = |\delta_i|$, then by the above, $\sigma_i \delta_i^* = \sigma_i \sigma_i^* \in \mathcal{M}_K(\mathcal{U})$; a contradiction. Thus either $|\sigma_i| > |\delta_i|$ or $|\delta_i| < |\sigma_i|$.

Firstly, we will consider the case $|\sigma_i| > |\delta_i|$. In this situation $\sigma_i = \delta_i \lambda$ for some path λ, and $\sigma_i \delta_i^* = \delta_i \lambda \delta_i^*$. As $\sigma_i \delta_i^* \notin \mathcal{M}_K(\mathcal{U})$ and $\delta_i \lambda \delta_i^* \neq 0$, it follows that λ is a closed path which is neither of the form c^ℓ nor $(c^*)^\ell$, where c is a cycle without exits and ℓ is a positive integer. Thus, assuming that

$$\lambda = e_1 e_2 \dots e_m$$

for some edges $e_1, e_2, \dots, e_m$ with $s(e_1) = r(e_m)$, there is a j, $1 \leq j \leq m$, and an edge f such that $s(e_j) = s(f)$ and $e_j \neq f$. Note that $\lambda^0 \subseteq \mathcal{U}$.

From (19) we get, for $z = \delta_i\lambda\tau\tau^*\delta_i^*$, the following:

$$z = \delta_i\lambda\tau\tau^*\delta_i^* = \gamma\overline{\gamma}^* = \gamma\gamma^*\mathfrak{a}_I\overline{\gamma\gamma}^* \in \mathcal{M}_K(\mathcal{U}) \setminus \{0\}. \tag{20}$$

Thus, for $\overline{\gamma\gamma}^* = \delta_i\tau\tau^*\delta_i^*$, by (18) we have $\delta_i\tau\tau^*\delta_i^* z = z\delta_i\tau\tau^*\delta_i^*$, which implies that

$$\delta_i\tau\tau^*\lambda\tau\tau^*\delta_i^* = \delta_i\lambda\tau\tau^*\delta_i^* = z \neq 0.$$

By the above, $\tau^*\lambda\tau \neq 0$, and it is not hard to check that

$$\tau^*\lambda\tau = e_ie_{i+1}\dots e_me_1e_2\dots e_{i-1}$$

for some $i \in \{1, 2, \dots, m\}$.

Consider $\eta = e_ie_{i+1}\dots e_{j-1}f$, which is of length less than or equal to m, and

$$y = \delta_i\tau\eta\eta^*\tau^*\delta_i^* = \delta_i\tau\eta(\delta_i\tau\eta)^*,$$

which by the construction is a non-zero element of $\mathcal{M}_K(\mathcal{U})$ (by (16), $r(\delta_i) \in \mathcal{U}$). Then we deduce from (20) that $zy = yz$. Notice that $zy = \delta_i\lambda\tau\eta\eta^*\tau^*\delta_i^* \neq 0$. On the other hand, in the product yz we have $\eta^*\tau^*\lambda\tau$ as a factor, which equals

$$(f^*e_{j-1}^*\dots e_{i+1}^*e_i^*)(e_ie_{i+1}\dots e_me_1e_2\dots e_{i-1}) = 0,$$

since $f \neq e_j$. Hence $yz = 0$; a contradiction.

In a similar way we get a contradiction if we consider the case $|\sigma_i| < |\delta_i|$. Finally, we deduce that indeed $\mathcal{A}(A, B)$ is a maximal commutative subalgebra of $L_K(E)$. □

Remark 4.5. Note that if a pair (A, B) gives a fully downward directed partition of E^0, then $\mathcal{U} = \emptyset$ and $\mathcal{A}(A, B)$ in Theorem 4.4 is equal to $L_K(A, B) + Z(L_K(E))$. On the other hand, if $\mathcal{U} = E^0$ (it could happen, for example, if we take $A = E^0$ and $B = \emptyset$), then $\mathcal{A}(A, B) = \mathcal{M}_K(E)$ is the commutative core of $L_K(E)$ considered in [28].

5. Partitions

In this section, we want to show that, considering a graph E, and starting with a pair (A, B) which constitutes a partition of E^0, we can "improve" it to get a fully downward directed pair (A', B') of E^0 with the property that $\mathcal{D}(A) \subseteq \mathcal{D}(A')$ and $\mathcal{D}(B) \subseteq \mathcal{D}(B')$. In such a case we will say that (A', B') is a *downward directed correction of* the pair (A, B).

Let E be a graph. For a pair (A, B) which constitutes a partition of E^0 we will consider, as in the previous section, the sets $\mathcal{U}(A)$ and $\mathcal{U}(B)$.

Definition 5.1. We define the relation $\approx'$ on E^0 as follows: for $u, v \in E^0$, $u \approx' v$ if $T(u) \cap T(v) \neq \emptyset$.

The relation $\approx'$ is reflexive and symmetric, but not necessary transitive. Therefore, by $\approx$ we denote the transitive closure of $\approx'$. The (equivalence) class of a vertex v with respect to $\approx$ is denoted by $[v]$.

Definition 5.2. For vertices u and v such that $u \approx v$, by the distance between u and v we mean the smallest k such that there are $u_0, \dots, u_k \in E^0$ with

$$u = u_0 \approx' u_1 \approx' \cdots \approx' u_k = v.$$

For a class $[v]$, let

$$[v]^A = [v] \cap \mathcal{U}(A) \quad \text{and} \quad [v]^B = [v] \cap \mathcal{U}(B),$$

and let

$$[E^A] = \{[v] : [v]^A \neq \emptyset\} \quad \text{and} \quad [E^B] = \{[v] : [v]^B \neq \emptyset\}.$$

Consider any functions $f_A : [E^A] \to \mathcal{U}(A)$ and $f_B : [E^B] \to \mathcal{U}(B)$ such that

$$f_A([v]) \in [v]^A \quad \text{and} \quad f_B([v]) \in [v]^B.$$

It should be clear that

$$\mathcal{U}(A) = \bigcup_{[v]\in[E^A]} [v]^A \quad \text{and} \quad \mathcal{U}(B) = \bigcup_{[v]\in[E^B]} [v]^B. \tag{21}$$

For any $[v] \in [E^A]$ and $i \geq 0$ we set

$$[v]^i_A = \{u \in [v] : \text{ distance between } u \text{ and } f_A([v]) \text{ is equal to } i\}$$

and, similarly, for any $[v] \in [E^B]$ and $i \geq 0$, we set

$$[v]^i_B = \{u \in [v] : \text{ distance between } u \text{ and } f_B([v]) \text{ is equal to } i\}.$$

Next, we consider the following sets:

$$[v]^{i,A}_A = [v]^i_A \cap \mathcal{U}(A) \quad \text{and} \quad [v]^{i,B}_B = [v]^i_B \cap \mathcal{U}(B).$$

Note that, using this notation, for a class $[v] \in [E^A]$,

$$[v]^0_A = [v]^{0,A}_A = \{f_A([v])\},$$

and for a class $[v] \in [E^B]$,

$$[v]^0_B = [v]^{0,B}_B = \{f_B([v])\}.$$

We want to stress that it is not hard to see that for some positive integers i, $[v]^{i,A}_A$ or $[v]^{i,B}_B$ can possibly be the empty set.

By (21) and the above definitions we have

$$\mathcal{U}(A) = \bigcup_{[v]\in[E^A]} [v]^A = \bigcup_{[v]\in[E^A]} \bigcup_{i\geq 0} [v]^{i,A}_A$$

and

$$\mathcal{U}(B) = \bigcup_{[v]\in[E^B]} [v]^B = \bigcup_{[v]\in[E^B]} \bigcup_{i\geq 0} [v]^{i,B}_B.$$

Now, fix a class $[v] \in [E^A]$, and consider the set

$$I(A,[v]) = \{i \geq 0 : [v]_A^{i,A} \neq \emptyset\}.$$

For $x \in [v]_A^i$ define the set

$$NA(x) = \{w \in [v]_A^{i-1} : w \approx' x\}.$$

Let a and b be distinct symbols, and define the function

$$F_{A,[v]} : [v] \to \{a, b\}$$

in the following way: for every $x \in [v] \backslash [v]^A$,

$$F_{A,[v]}(x) = \begin{cases} a, & \text{if } x \in A, \\ b, & \text{if } x \in B, \end{cases}$$

and

$$F_{A,[v]}(f_A([v])) = b.$$

Then consider $n > 0$ and suppose that the set $[v]_A^{n,A}$ is not empty. Suppose also that for every element $x \in \bigcup_{i=0}^{n-1} [v]_A^i$ the image $F_{A,[v]}(x)$ of x is known. Now, consider the following sets:

$$[v]_A^{n,A,a} = \{x \in [v]_A^{n,A} : b \notin F_{A,[v]}(NA(x))\},$$
$$[v]_A^{n,A,b} = \{x \in [v]_A^{n,A} : b \in F_{A,[v]}(NA(x))\}.$$

Obviously, $[v]_A^{n,A} = [v]_A^{n,A,a} \cup [v]_A^{n,A,b}$ and $[v]_A^{n,A,a} \cap [v]_A^{n,A,b} = \emptyset$. Then considering the set $[v]_A^{n,A}$ we set

$$F_{A,[v]}(x) = \begin{cases} b, & \text{if } x \in [v]_A^{n,A,a}, \\ a, & \text{if } x \in [v]_A^{n,A,b}. \end{cases}$$

For every class $[v] \in [E^B]$, the set $I(B,[v])$ and the function $F_{B,[v]}$ are defined analogously.

Finally, we consider the following two sets:

$$\mathcal{X}(A) = \bigcup_{[v] \in [E^A]} \bigcup_{i \geq 0} [v]_A^{i,A,a},$$

and

$$\mathcal{X}(B) = \bigcup_{[v] \in [E^B]} \bigcup_{i \geq 0} [v]_B^{i,B,b}.$$

With the notation as above we have the following theorem:

Theorem 5.3. *Let E be a graph, and let (A, B) be a pair constituting a partition of E^0. Then there is a pair (A', B') which is a downward directed correction of (A, B).*

Proof. One can readily verify that the pair (A', B') constitutes a required partition of E^0, where

$$A' = \mathcal{D}(A) \cup \mathcal{X}(B) \cup (\mathcal{U}(A)\backslash\mathcal{X}(A)) \quad \text{and} \quad B' = \mathcal{D}(B) \cup \mathcal{X}(A) \cup (\mathcal{U}(B)\backslash\mathcal{X}(B)).$$

□

Example 5.4. In the graph in Fig. 1, we label vertices and also we indicate sets (according to the partition) where these vertices belong to.

All notations for the example we work on now, are explained in the above consideration. So we want to ask the reader to use them freely in the entire construction.

For the above graph $\mathcal{D}(A) = \{x_0, x_1\}$, $\mathcal{D}(B) = \{x_2\}$, $\mathcal{U}(A) = \{v_0, v_1, \ldots\}$ $\mathcal{U}(B) = \{u_0, u_1, u_2, u_3, u_4\}$.

Note that for all vertices $w \in E^0$ there is $[w] = E^0$, so $[w]^A = \mathcal{U}(A)$ and $[w]^B = \mathcal{U}(B)$. Let for every $w \in E^0$, $f_A([w]) = v_0$ and $f_B([w]) = u_4$. Then

$[w]^0_A = [w]^{0,A}_A = \{v_0\}$ and by the definition of $F_{A,[w]}$ on $f_A([w])$ we got $F_{A,[w]}(v_0) = b$. The next layer set is $[w]^1_A = \{x_0, v_1\}$, especially $[w]^{1,A}_A = \{v_1\}$. Notice that $NA(v_1) = \{v_0\}$, so $F_{A,[w]}(v_1) = a$. Next, $[w]^2_A = \{x_1, x_2, v_2, v_3, v_4\}$, so $[w]^{2,A}_A = \{v_2, v_3, v_4\}$ and the neighborhood is $NA(v_2) = NA(v_3) = NA(v_4) = \{v_1\}$, so $F_{A,[w]}(v_2) = F_{A,[w]}(v_3) = F_{A,[w]}(v_4) = b$. Similarly, $[w]^3_A = \{u_0, u_1, u_2, u_3, u_4, v_5, v_6\}$ and inside the set A we have $[w]^{3,A}_A = \{v_5, v_6\}$ and then $NA(v_5) = NA(v_6) = \{v_2, v_3, v_4\}$, so we got $F_{A,[w]}(v_5) = F_{A,[w]}(v_6) = a$. We get the sets $[w]^i_A$ and $[w]^{i,A}_A$ analogously for any $i > 3$.

In the next step we have $[w]^0_B = [w]^{0,B}_B = \{u_4\}$, so by the definition of $F_{B,[w]}(f_A([w]))$ there is $F_{B,[w]}(u_4) = a$. Furthermore, $[w]^1_B = \{u_0, u_1, u_2, u_3, x_2\}$ and inside the set B we have $[w]^{1,B}_B = \{u_0, u_1, u_2, u_3\}$ and then $NB(u_0) = NB(u_1) = NB(u_2) = NB(u_3) = \{u_4\}$, so $F_{B,[w]}(u_0) = F_{B,[w]}(u_1) = F_{B,[w]}(u_2) = F_{B,[w]}(u_3) = b$. Then we have $[w]^2_B = \{x_0, x_1\}$ but then $[w]^{2,B}_B = \emptyset$. It is not hard to see that for $i \geq 2$ we get $[w]^{i,B}_B = \emptyset$ (what is the set $[w]^i_B$ can be omitted here).

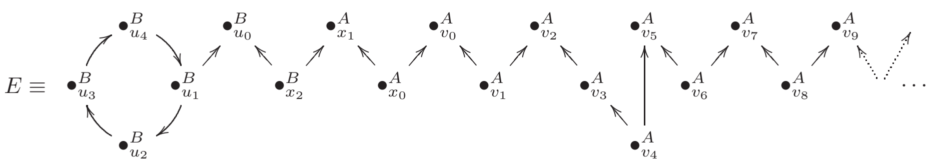

Fig. 1. Example of graph with partition before correction.

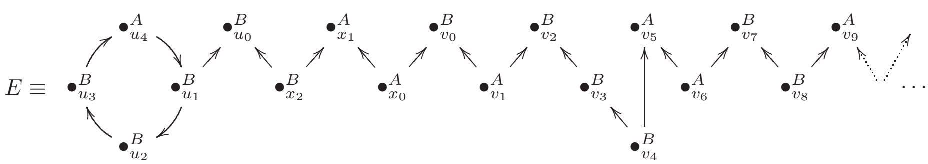

Fig. 2. Example of graph with partition after correction.

After the described algorithm we get the graph in Fig. 2 which in our terminology is after the correction.

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