Research Article

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Construction of a class of maximal commutative subalgebras of prime Leavitt path algebras

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Abstract: Considering prime Leavitt path algebras $L_K(E)$, with E being an arbitrary graph with at least two vertices, and K being any field, we construct a class of maximal commutative subalgebras of $L_K(E)$ such that, for every algebra A from this class, A has zero intersection with the commutative core $\mathcal{M}_K(E)$ of $L_K(E)$ defined and studied in [C. Gil Canto and A. Nasr-Isfahani, The commutative core of a Leavitt path algebra, *J. Algebra* **511** (2018), 227–248]. We also give a new proof of the maximality, as a commutative subalgebra, of the commutative core $\mathcal{M}_R(E)$ of an arbitrary Leavitt path algebra $L_R(E)$, where E is an arbitrary graph and R is a commutative unital ring.

Keywords: Leavitt path algebra, maximal commutative subalgebra

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1 Introduction and motivation

The notion of a Leavitt path algebra $L_K(E)$, with K and E denoting an arbitrary field and a row-finite graph, respectively, which was introduced in [3, 9], has subsequently received significant interest from algebraists, as well as from analysts specializing in C^*-algebras.

For example, the Cuntz–Krieger algebras $C^*(E)$, which are the C^*-algebra counterparts of these Leavitt path algebras, where E denotes a graph, were investigated in [33]. The interplay between these two classes of graph algebras, as mentioned in [6], has been both extensive and mutually beneficial.

Graph C^*-algebra results have aided in, not only steering the development of Leavitt path algebras by suggesting the correctness of some conjectures, but also by hinting at the direction of future investigation. Similarly, Leavitt path algebras have furnished a more thorough comprehension of graph C^*-algebras by abetting in pinpointing the algebraic aspects of $C^*(E)$.

The algebras $L_K(E)$ are natural generalizations of the algebras investigated by Leavitt in [27], and they are a specific type of path K-algebras associated with a graph E modulo certain relations.

A wealth of well-known algebras can be realized as Leavitt path algebras of graphs, for example, full $n \times n$ matrix algebras $\mathbb{M}_n(K)$ for $n \in \mathbb{N} \cup \{\infty\}$ (where $\mathbb{M}_\infty(K)$ denotes matrices of countably infinite size with

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only a finite number of nonzero entries), the Toeplitz algebra T, the Laurent polynomial ring $K[x, x^{-1}]$ and the classical Leavitt algebras $L(1, n)$ for $n \geq 2$. Another important interest in the study of Leavitt path algebras is the pictorial representations that their corresponding graphs provide.

Algebras which are more general than Leavitt path algebras, such as Kumjian–Pask algebras (see, for example, [11, 15]) and Steinberg algebras (see, for example, [16]), have also recently enjoyed significant interest.

In order to understand a mathematical object better, it is often rather natural to consider its maximal subobjects. Maximal subalgebras of (not necessarily associative) algebras, and in particular maximal commutative subalgebras (see, for example, [15, 22, 25, 30]), have classically guided such studies. A well-known example of this principle comes to the fore in the structure theory of finite-dimensional semisimple Lie algebras, where their Cartan subalgebras feature prominently: over the complex number field, these are simply maximal commutative subalgebras, as seen in, for example, [19, 28]. Similar ideas have subsequently been applied to maximal substructures of other, possibly non-associative, algebraic structures, such as Malcev algebras, Jordan algebras, associative superalgebras, or classical groups; see, for example, [20, 21, 31, 32].

On the associative side, a common feature of the considerations in some of the foregoing papers is the interest in the objects, which, in the case of Leavitt path algebras $L_K(E)$, are called the commutative core of $L_K(E)$.

A classical result of Schur (see [34]), which has attracted considerable historical interest, states that, for any algebraically closed field K of characteristic 0, the dimension over K of any commutative subalgebra of the full $n \times n$ matrix algebra $\mathbb{M}_n(K)$ is at most $\lfloor \frac{n^2}{4} \rfloor + 1$, where $\lfloor \cdot \rfloor$ denotes the integer floor function. Jacobson showed in [24] that the mentioned upper bound holds for commutative subalgebras of $\mathbb{M}_n(K)$ for any field K. A concise proof of this result was presented later by Mirzakhani in [29]; see, for example, also [17, 18, 23].

Moreover, the upper bound $\lfloor \frac{n^2}{4} \rfloor + 1$ is sharp. Indeed, following [35], let K be any field, let $n \geq 2$, and let k_1 and k_2 be positive integers such that $k_1 + k_2 = n$. Define the rectangular array B by

$$B = \{(i, j) \in \mathbb{N} \times \mathbb{N} : 1 \leq i \leq k_1 < j \leq n\},$$

and consider the subset

$$\mathbb{J} = \Big\{ \sum_{(i,j) \in B} b_{ij} E_{(i,j)} : b_{ij} \in K \text{ for all } (i, j) \in B \Big\} \tag{1.1}$$

of $\mathbb{M}_n(K)$, where $E_{(i,j)}$ denotes the matrix unit in $\mathbb{M}_n(K)$ associated with position (i, j). The reader will immediately observe that $\mathbb{J}$ comprises the subset of $\mathbb{M}_n(K)$ consisting of the block upper triangular matrices corresponding with B; see Figure 1.

Taking block multiplication into account, it is very easy to see that the product of any two elements in $\mathbb{J}$ is 0, and so the subalgebra

$$\mathcal{A} = KI_n + \mathbb{J} \tag{1.2}$$

of $\mathbb{M}_n(K)$, where $KI_n := \{aI_n : a \in K\}$ (with I_n denoting the $n \times n$ identity matrix), is a commutative subalgebra of $\mathbb{M}_n(K)$. Taking $k_1 = k_2 = \frac{n}{2}$ if n is even (respectively taking $k_1 = \frac{n-1}{2}$ and $k_2 = \frac{n+1}{2}$ if n is odd), we obtain $\mathcal{A}$ with dimension equal to $\lfloor \frac{n^2}{4} \rfloor + 1$.

In the above context, the famous result by Amitsur and Levitzky (see [8]), stating that $\mathbb{M}_n(R)$ (with R being any commutative ring) satisfies the standard polynomial identity (PI)

$$\sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(2n)} = 0$$

of degree $2n$ (with S_{2n} denoting the symmetric group on $2n$ symbols), and no PI of lower degree, is particularly relevant. Of course, an immediate consequence is that every subring of $\mathbb{M}_n(R)$ also satisfies the standard PI of degree $2n$.

Certain subalgebras of $\mathbb{M}_n(K)$, with K being any field, satisfying some extra PI's which are not satisfied by $\mathbb{M}_n(K)$, are studied in [35]. Apart from the standard PI, the most important PI is beyond any doubt the so-called Lie nilpotency (of index m, for some positive integer m). The m-Lie nilpotency, namely $[[\cdots[[x_1, x_2], x_3], \ldots, x_m], x_{m+1}] = 0$, is not even a PI on the 2×2 matrix algebra $\mathbb{M}_2(K)$. Notice that an

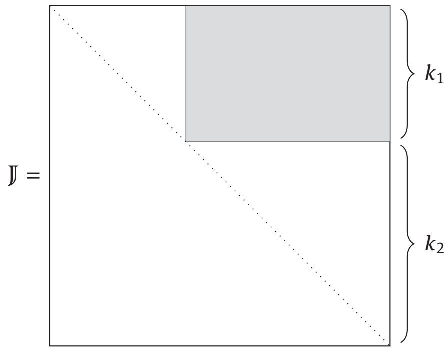

Figure 1: The strictly upper triangular block associated with the rectangular array B defined above.

algebra or a ring is commutative if and only if it is Lie nilpotent of index 1, i.e., if and only if $[x_1, x_2] = 0$ for all elements x_1 and x_2 in the algebra or ring.

A sharp upper bound for the dimension over K of any Lie nilpotent subalgebra of $\mathbb{M}_n(K)$ of index m is given in [35]. The importance of Lie nilpotency is buttressed by the fact that the (countably) infinite-dimensional Grassmann algebra G has “only one identity” in the sense that the polynomial identity

$$[[x_1, x_2], x_3] = 0$$

generates the T-ideal of the polynomial identities satisfied by G. The latter is a highly non-trivial result by Krakowski and Regev; see [26].

It is extremely important to note that the mentioned Grassmann algebra G plays a fundamental role in Kemer’s monumental structure theory of T-ideals, as well as in his solution of the famous Specht problem about the finite generation of the polynomial identities of associative algebras over a field K of characteristic zero. A remarkable consequence of the mentioned structure theory is that any T-ideal contains all polynomial identities of $\mathbb{M}_n(G)$ for some n.

In general, the dimension of a subalgebra of $\mathbb{M}_n(K)$ cannot be arbitrary. For instance, the dimension of any proper (unital) subalgebra of $\mathbb{M}_n(K)$, with K being a field of characteristic zero, is less than or equal to $n^2 - n + 1$ (see [7]). It seems to be a very challenging problem to describe the integers between 1 and n^2 which can appear as the dimension of a certain subalgebra of $\mathbb{M}_n(K)$. Note that the mentioned results by Schur and Jacobson produce the integer $\lfloor \frac{n^2}{4} \rfloor + 1$.

There is no doubt that commutativity is extremely important, and Lie nilpotency is the most natural generalization of it. In the light of the foregoing setting, it is also worth drawing the reader’s attention to, for example, the PI’s studied in certain subalgebras $\mathbb{M}_n(K)$ in [37].

In [22] Gil Canto and Nasr-Isfahani constructed and investigated a maximal (with respect to inclusion) commutative subalgebra of a Leavitt path algebra $L_R(E)$ over a commutative unital ring R (for a given graph E), called the *commutative core* of $L_R(E)$, and denoted by $\mathcal{M}_R(E)$ (see [22, Proposition 4.5 and Theorem 4.13]).

Recall that the Leavitt path algebra $L_K(\mathcal{E})$ for a field K and the graph

$$\mathcal{E} \equiv \bullet_{v_1} \xrightarrow{e_1} \bullet_{v_2} \xrightarrow{e_2} \bullet_{v_3} \cdots\cdots \bullet_{v_{n-1}} \xrightarrow{e_{n-1}} \bullet_{v_n} \tag{1.3}$$

is isomorphic to $\mathbb{M}_n(K)$, via φ (say), where, for all $1 \leq i < j \leq n$ (and recalling that $E_{(i,j)}$ denotes the standard matrix unit),

$$\varphi(v_i) = E_{(i,i)}, \quad \varphi(e_i \cdots e_{j-1}) = E_{(i,j)}, \quad \varphi(e_{j-1}^* \cdots e_i^*) = E_{(j,i)}. \tag{1.4}$$

Therefore, in the light of the preceding deliberation, it would be interesting to see what the algebra $\mathcal{M}_K(\mathcal{E})$ looks like (see the construction by Gil Canto and Nasr-Isfahani in [22]) if we consider $L_K(\mathcal{E})$. It turns out that we obtain the commutative subalgebra of $L_K(\mathcal{E})$ generated by all vertices $v_1, \ldots, v_n$, which, translated to matrix language, yields the commutative subalgebra of $\mathbb{M}_n(K)$ generated by the matrix units $E_{(1,1)}, E_{(2,2)}, \ldots, E_{(n,n)}$. Thus, in the case of a full matrix algebra, it can be said that $\mathcal{M}_K(E)$ is some kind of trivial example of a maximal commutative subalgebra.

In the present paper, inspired by the above facts, we construct a class of maximal commutative subalgebras of a prime Leavitt path algebra $L_K(E)$ such that in the case of the matrix algebra $\mathbb{M}_n(K)$, seen as an isomorphic copy of $L_K(\mathcal{E})$, one of the elements of this class is the commutative subalgebra $\mathcal{A} = KI_n + \mathbb{J}$ mentioned in (1.2).

An important part of [22] is the proof of the maximality of the commutative core $\mathcal{M}_R(E)$ amongst the commutative subalgebras of a Leavitt path algebra $L_R(E)$ whose coefficients are in a commutative unital ring R. In Section 10 of the present paper, we provide a new proof of this result, which is more elementary; indeed, it uses only information about the structure of $L_R(E)$.

2 Preliminaries, notation and terminology

We recall some basic definitions, notation and terminology.

A (directed) graph $E = (E^0, E^1, s, r)$ comprises two countable sets, namely E^0 and E^1, and two functions $s, r : E^1 \to E^0$. The elements of E^0 and E^1 are called *vertices* and *edges*, respectively. For every edge $e \in E^1$, $s(e)$ and $r(e)$ are called the *source* of e and the *range* of e, respectively. A vertex which emits no edges is called a *sink*.

If $s^{-1}(v)$ is a finite set for every vertex $v \in E^0$, then the graph E is called *row-finite*.

A *path* π in a graph E is a sequence of edges $\pi = e_1 e_2 \cdots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. In this case, n is called the *length* of π, and it is denoted by $|\pi|$. We define $s(\pi) = s(e_1)$ and $r(\pi) = r(e_n)$, and π^0 denotes the set of all vertices which are the source or the range of an edge appearing in π, i.e., $\pi^0 = \{s(e_1), s(e_2), \dots, s(e_n), r(e_n)\}$. All elements of E^0 are viewed as paths of length 0. The set of all paths in E is denoted by $\mathrm{Path}(E)$.

A path $\pi = e_1 \cdots e_n$ is called *closed* if $s(e_1) = r(e_n)$. A closed path $\pi = e_1 \cdots e_n$ is called *simple* in case $s(e_i) \neq s(e_1)$ for all $2 \leq i \leq n$. Such a simple closed path π is said to be based at $v = s(e_1)$. A simple closed path $\pi = e_1 \cdots e_n$ is called a *cycle* in case there are no repeats in the list of vertices $s(e_1), s(e_2), \dots, s(e_n)$.

A graph E is called *acyclic* in case there are no cycles in E. An *exit* of a cycle $\pi = e_1 \cdots e_n$ is an edge f such that $s(f) = s(e_i)$ for some i and $f \neq e_i$. If no such f exists for a cycle π, then we say that π is a cycle *without exits*.

If there is a path from a vertex u to a vertex v, then we write $u \geq v$. A graph E is called *downward directed* if, for all $u, v \in E^0$, there exists a vertex $w \in E^0$ such that $u \geq w$ and $v \geq w$.

Following [4], a graph E with finitely many edges and vertices is called a *comet* if E has precisely one cycle, say π, and this cycle is without exits, and $v \geq u$ for every $v \in E$ and every $u \in \pi^0$.

For every edge $e \in E^1$, the so-called *ghost edge* e^* is such that $s(e^*) := r(e)$ and $r(e^*) := s(e)$, and for a path $\pi = e_1 e_2 \cdots e_n$ we denote the so-called *ghost path* $e_n^* e_{n-1}^* \cdots e_1^*$ by π^*.

With $\delta_{v,w}$ denoting the Kronecker delta function, we recall the following definition (see [2]).

Definition 2.1. Let E be an arbitrary graph, and let K be a field. The Leavitt path (K-)algebra associated with E, denoted by $L_K(E)$, is the K-algebra generated by the set E^0 of vertices and the set $\{e, e^* : e \in E^1\}$ of edges and ghost edges, satisfying the following relations:

(V) $vw = \delta_{v,w} v$ for all $v, w \in E^0$.

(E1) $s(e)e = er(e) = e$ for all $e \in E^1$.

(E2) $e^* s(e) = r(e)e^* = e^*$ for all $e \in E^1$.

(CK1) $e^* f = \delta_{e,f} r(f)$ for all $e, f \in E^1$.

(CK2) $v = \sum_{\{e \in E^1 : s(e)=v\}} ee^*$ for every vertex v which is not a sink and emits a finite number of edges.

Note that one can also consider a Leavitt path algebra with coefficients in an arbitrary commutative unital ring (see Definition 10.1), but we restrict ourselves to coefficients from a field in all sections apart from Section 10.

It is well known that if E is a graph with finitely many vertices, i.e., if $|E^0| < \infty$, then the algebra $L_K(E)$ has an identity (namely $\sum_{v \in E^0} v$) for every field K.

It was shown in [3] that $L_K(E)$ is a $\mathbb{Z}$-graded K-algebra, spanned as a K-vector space by

$$\{\alpha\beta^* : \alpha, \beta \in \mathrm{Path}(E)\}.$$

In particular, for every $n \in \mathbb{Z}$, the degree n component $L_K(E)_n$ is spanned by all elements of the form $\{\alpha\beta^* : |\alpha| - |\beta| = n\}$. The set of homogeneous elements is $\bigcup_{n\in\mathbb{Z}} L_K(E)_n$, and an element of $L_K(E)_n$ is said to be homogeneous of degree n.

For vertices $u, v \in E^0$, consider the following (possibly empty) set of nonzero monomials:

$$\mathfrak{M}(u, v) = \{\alpha\beta^* : \alpha, \beta \in \mathrm{Path}(E),\ \alpha\beta^* \neq 0,\ s(\alpha) = u,\ r(\beta^*) = v\}. \tag{2.1}$$

Notice that if α and β are paths such that $\alpha^*\beta \neq 0$, then either $\alpha = \beta\alpha'$ for some path α' (in the case where $|\alpha| \geq |\beta|$) or $\beta = \alpha\beta'$ for some path β' (in the case where $|\alpha| < |\beta|$). In the first case, we have $\alpha^*\beta = (\alpha')^*$, and in the second case we have $\alpha^*\beta = \beta'$. The process leading to $(\alpha')^*$ or β' will be called a *reduction* of $\alpha^*\beta$.

For the general notation, terminology and results in Leavitt path algebras, we refer the reader to, for example, [1, 3, 14].

3 Formulation of the main result about a class of maximal commutative subalgebras of prime Leavitt path algebras

In this section, we introduce a class of commutative subalgebras of a Leavitt path algebra $L_K(E)$ over a field K.

For positive integers k_1 and k_2 such that $k_1 + k_2 = n$, consider the subalgebra $\mathcal{A}$ (see (1.2)) of $\mathbb{M}_n(K)$, and its isomorphic copy $\mathcal{C}$ in $L_K(\mathcal{E})$ (see (1.3)). It follows readily that the algebra $\mathcal{C}$ is generated by all paths α in $L_K(\mathcal{E})$ such that $s(\alpha) \in \{v_1, \dots, v_{k_1}\}$ and $r(\alpha) \in \{v_{k_1+1}, \dots, v_n\}$.

Motivated by this observation, we state the following definition.

Definition 3.1. Let $E = (E^0, E^1, s, r)$ be a graph such that $|E^0| > 1$, and let (E^0_s, E^0_r) be a pair of nonempty subsets of E^0 such that $E^0_s \cap E^0_r = \emptyset$ and $E^0 = E^0_s \cup E^0_r$. For this partition of E^0, let $L_K(E^0_s, E^0_r)$ be the subalgebra of $L_K(E)$ generated by all monomials $\alpha\beta^*$ such that $\alpha, \beta \in \mathrm{Path}(E)$, $s(\alpha) \in E^0_s$ and $r(\beta^*) \in E^0_r$.

Note that if $|E^0| = 1$ and $E^1 = \emptyset$, then $L_K(E)$ is isomorphic to K. If $|E^0| = 1$ and $|E^1| = 1$, then $L_K(E)$ is commutative. Therefore, in these situations, it is pointless to consider commutative subalgebras of $L_K(E)$. For $|E^0| = 1$ and $|E^1| > 1$, the algebra $L_K(E)$ is simple, and so it is potentially of interest to us, but unfortunately the above construction does not apply in this case.

Therefore, we assume henceforth that $|E^0| > 1$. (The observant reader will notice that some of the subsequent results hold without this assumption, but for the critical parts of the paper relying on the mentioned partition we do need and thus use this assumption.)

It follows readily that for any partition (E^0_s, E^0_r) of E the product of any two elements in the subalgebra $L_K(E^0_s, E^0_r)$ of $L_K(E)$ is 0, and so $L_K(E^0_s, E^0_r)$ is trivially commutative. The observant reader will note that $L_K(E^0_s, E^0_r)$ mimics the (strictly) upper triangular block J in (1.1).

We will prove the maximality (with respect to inclusion) of a class of commutative subalgebras of a prime Leavitt path algebra $L_K(E)$ related to the $L_K(E^0_s, E^0_r)$'s.

It is well known (see [5, Theorem 2.4]) that for an arbitrary graph E and a field K the algebra $L_K(E)$ is prime if and only if E is downward directed.

Throughout the paper, the center of a Leavitt path algebra $L_K(E)$ is denoted by $Z(L_K(E))$. It is worth mentioning that the center of a Leavitt path algebra $L_K(E)$ was investigated and described in [12, 16].

The main result of this paper is the following theorem.

Theorem 3.2. *Let E be an arbitrary graph with $|E^0| > 1$, and let K be a field, such that $L_K(E)$ is a prime algebra. Let (E^0_s, E^0_r) be a partition of E^0.*

(1) *If E has finitely many vertices, and*
 (i) *E is acyclic, or*
 (ii) *E has at least two cycles, or*
 (iii) *E has precisely one cycle, and this cycle has an exit,*

then the algebra

$$K \cdot 1 + L_K(E_s^0, E_r^0)$$

is a maximal commutative subalgebra of $L_K(E)$.

(2) *If E has finitely many vertices and precisely one cycle, and this cycle is without exits, then the algebra*

$$Z(L_K(E)) + L_K(E_s^0, E_r^0)$$

is a maximal commutative subalgebra of $L_K(E)$.

(3) *If E has infinitely many vertices, then the algebra*

$$L_K(E_s^0, E_r^0)$$

is a maximal commutative subalgebra of $L_K(E)$.

4 General results in Leavitt path algebras

In this section, we want to collect some general results that will be invoked freely in the sequel.

Lemma 4.1. *Let K be a field, E a graph and σ a non-trivial simple closed path based at a vertex v. Suppose that, for $\alpha, \beta \in \mathrm{Path}(E)$ and $k \in K$,*

$$(\sigma^*)^n (k\alpha\beta^*)\sigma^n \neq 0$$

in the algebra $L_K(E)$ for all $n \geq 0$.

(a) *If $|\alpha| > |\beta|$, then there are positive integers m and t such that*

$$(\sigma^*)^n (k\alpha\beta^*)\sigma^n = k\sigma^t$$

for all $n \geq m$, and $|\sigma^t| = |\alpha| - |\beta|$.

(b) *If $|\alpha| < |\beta|$, then there are positive integers m and t such that*

$$(\sigma^*)^n (\alpha\beta^*)\sigma^n = k(\sigma^*)^t$$

for all $n \geq m$, and $|(\sigma^*)^t| = |\beta| - |\alpha|$.

(c) *If $|\alpha| = |\beta|$, then there is a positive integer m such that*

$$(\sigma^*)^n (k\alpha\beta^*)\sigma^n = kv$$

for all $n \geq m$.

Proof. (a) Let m be the smallest positive integer such that $|\sigma^m| > |\alpha|$. Then, after reduction, $(\sigma^*)^m(\alpha\beta^*)\sigma^m$ is a path, denoted by δ, which is not a vertex. Moreover, $s(\delta) = v = r(\delta)$. Since $(\sigma^*)^{n-m}\delta \neq 0$ for all $n \geq m$, we deduce that $\delta = \sigma^t$ for some $t > 0$. Hence,

$$(\sigma^*)^n (k\alpha\beta^*)\sigma^n = k(\sigma^*)^{n-m}\delta\sigma^{n-m} = k(\sigma^*)^{n-m}\sigma^t\sigma^{n-m} = k\sigma^t.$$

(b) The assertion can be proved in a similar way to (a).

(c) Consider $k\alpha\beta^*\sigma$, which, after reduction, satisfies the assumption in (a). Therefore, we conclude from (a) that there is a positive integer m such that $(\sigma^*)^n(k\alpha\beta^*\sigma)\sigma^n = k\sigma$ for all $n \geq m$. Consequently, it follows readily that

$$(\sigma^*)^{n+1}(k\alpha\beta^*)\sigma^{n+1} = kv. \qquad \square$$

Remark 4.2. Henceforth, if we consider an element $\mathfrak{a} \in L_K(E)$, then we simultaneously fix a presentation $\mathfrak{a} = \sum_{j \in J} k_j \alpha_j \beta_j^*$, where J is a finite set of indices, and $\alpha_j, \beta_j \in \mathrm{Path}(E)$ and $k_j \in K$ for all $j \in J$. We also assume that the presentation we work with is chosen such that the cardinality of J is as small as possible.

Moreover, we also assume that $e_n f_1^* \neq s(e_n)$ for every monomial $\alpha_j\beta_j^* = e_1 e_2 \cdots e_n f_1^* f_2^* \cdots f_m^*$ appearing in the considered presentation of $\mathfrak{a}$, where $e_1, \ldots, e_n, f_1, \ldots, f_m \in E^1$ and $n, m \geq 1$.

In such a case, we say that $\mathfrak{a}$ is in a *reduced form* (or, $\mathfrak{a}$ is *reduced*). Notice, then, that for every subset $I \subseteq J$ the element $\sum_{j\in I} k_j\alpha_j\beta_j^*$ is also reduced; in particular, the elements $v\mathfrak{a}w$, $v\mathfrak{a}$, $\mathfrak{a}w \in L_K(E)$ are reduced for all $v, w \in E^0$.

Lemma 4.3. *Let E be a graph, and let K be a field. If $\mathfrak{a}$ is a nonzero homogeneous element in $L_K(E)$, and in a reduced form $\mathfrak{a}$ has the presentation*

$$\mathfrak{a} = \sum_{j=1}^{n} k_j\alpha_j\alpha_j^*$$

for some paths α_j, with $|\alpha_j| > 0$ and $0 \neq k_j \in K$, $j = 1, \dots, n$, then there is a path γ such that $r(\gamma^*) = s(\alpha_i)$ for some i, and

$$\gamma^*\alpha_j = 0 = \alpha_j^*\gamma$$

for all j.

Proof. The proof is by induction on n. If $n = 1$, then there is a nonzero $k \in K$ and an $\alpha \in \mathrm{Path}(E)$ such that $\mathfrak{a} = k\alpha\alpha^*$. Let $\alpha = e_1 \cdots e_\ell$ for some edges $e_1, \dots, e_\ell$ and some positive integer ℓ. In the light of

$$\mathfrak{a} = ke_1 \cdots e_\ell e_\ell^* \cdots e_1^*$$

being in reduced form, there is an edge $f \neq e_\ell$ such that $s(f) = s(e_\ell)$; otherwise, by (CK2), $e_\ell e_\ell^* = s(e_\ell)$, implying that $\mathfrak{a}$ is not in reduced form. Then, taking $\gamma = e_1 \cdots e_{\ell-1}f$, we are done.

Consider the case $n > 1$. Let i be a positive integer such that α_i has the smallest length among all α_j's for $j \in \{1, \dots, n\}$, and let $\alpha_i = e_1 \cdots e_\ell$ for some edges $e_1, \dots, e_\ell$ and some $\ell \geq 1$. This time, using the assumption that $\mathfrak{a}$ is in a reduced form, we deduce that there is an edge f such that $f \neq e_\ell$, $s(f) = s(e_\ell)$ and for $\delta = e_1 \cdots e_{\ell-1}f$,

$$\delta\delta^* \neq \alpha_j\alpha_j^* \tag{4.1}$$

for every $j \in \{1, \dots, n\}$. Obviously, $\delta^*\alpha_i = 0 = \alpha_i^*\delta$.

If $\delta^*\alpha_j = 0$ for every $j \in \{1, \dots, n\}$, then we set $\gamma = \delta$, and we are done.

Assume now that $\delta^*\alpha_j \neq 0$ for some j. We may assume, without loss of generality, that there is an integer $s \geq 1$ such that, $\delta^*\alpha_j = 0$ for every $j \in \{1, \dots, s\}$, and $\delta^*\alpha_j \neq 0$ for every $j \in \{s+1, \dots, n\}$. Then $\alpha_j = \delta\alpha_j'$ for every $j \geq s+1$ for some path α_j', and by (4.1), we have $|\alpha_j'| > 0$.

Since $\mathfrak{a}$ is in a reduced form and $\mathfrak{a} = \sum_{j\leq s} \alpha_j\alpha_j^* + \delta\delta^*\mathfrak{a}\delta\delta^*$, it follows that $\delta^*\mathfrak{a}\delta \neq 0$. Notice that $\delta^*\mathfrak{a}\delta$ has a reduced presentation $\sum_{j=s+1}^{n} \alpha_j'\alpha_j'^*$, and

$$s(\alpha_{s+1}') = s(\alpha_{s+2}') = \cdots = s(\alpha_n') = r(\delta).$$

Therefore, by the induction hypothesis, there is a path σ such that $r(\sigma^*) = s(\alpha_{s+1}') = s(\delta^*)$ and $\sigma^*\alpha_j' = 0 = \alpha_j'^*\sigma$ for every $j \in \{s+1, \dots, n\}$.

Now, consider $0 \neq \gamma = \delta\sigma$. Since $r(\gamma^*) = s(e_1) = s(\alpha_i)$ and $\gamma^*\alpha_j = 0 = \alpha_j^*\gamma$ for every $j \in \{1, \dots, n\}$, the proof is complete. □

Definition 4.4. For a given graph E and a field K, let $\mathfrak{a}$ be a homogeneous element in $L_K(E)$ which is in a fixed reduced form $\mathfrak{a} = \sum_{j\in J} k_j\alpha_j\beta_j^*$, with $0 \neq k_j \in K$ for all $j \in J$. Then, for every $n \geq 0$, we set

$$M_n(\mathfrak{a}) := \{\alpha_j\beta_j^* \in \mathrm{supp}(\mathfrak{a}) : |\alpha_j| = n\}.$$

Recall that $\mathrm{supp}(\mathfrak{a})$ comprises all $\alpha_j\beta_j^*$'s in the representation of $\mathfrak{a}$ above.

The following result is motivated by and has strong links with the reduction theorem (see [10] or [13]), the latter being well known in the literature.

Proposition 4.5. *Let E be a graph, K a field and $0 \neq \mathfrak{a} \in L_K(E)$ as in Definition 4.4. If n_0 is the smallest integer such that $M_{n_0}(\mathfrak{a})$ is a nonempty subset of $\mathrm{supp}(\mathfrak{a})$, then, for every $i \in J$ such that $\alpha_i\beta_i^* \in M_{n_0}(\mathfrak{a})$, there are paths γ and $\overline{\gamma}$ such that*

$$\gamma^*\Big(\bigcup_{n\geq n_0} M_n(\mathfrak{a})\Big)\overline{\gamma} = \gamma^*\{\alpha_i\beta_i^*\}\overline{\gamma} = \{s(\gamma^*)\} = \{r(\overline{\gamma})\}.$$

Moreover, $\gamma = \alpha_i\sigma$ and $\overline{\gamma} = \beta_i\sigma$ for some $\sigma \in \mathrm{Path}(E)$.

Proof. Consider all non-negative integers, say $n_0, n_1, \dots, n_t$, with $n_0 < n_1 < \dots < n_t$, such that $M_{n_\ell}(\mathfrak{a}) \neq \emptyset$ for $\ell = 0, 1, \dots, t$. Obviously, $\operatorname{supp}(\mathfrak{a}) = \bigcup_{\ell=0}^{t} M_{n_\ell}(\mathfrak{a})$.

Fix $i \in J$ such that $\alpha_i\beta_i^* \in M_{n_0}(\mathfrak{a})$. Letting $\gamma_0 = \alpha_i$ and $\overline{\gamma}_0 = \beta_i$, we have

$$\gamma_0^* M_{n_0}(\mathfrak{a})\overline{\gamma}_0 = \gamma_0^*\{\alpha_i\beta_i^*\}\overline{\gamma}_0 = \{s(\gamma_0^*)\} = \{r(\overline{\gamma}_0)\}.$$

Note that if $\gamma_0^* M_{n_\ell}(\mathfrak{a})\overline{\gamma}_0 = \{0\}$ for every $\ell > 0$, then, taking $\gamma = \gamma_0$, $\overline{\gamma} = \overline{\gamma}_0$ and $\sigma = r(\alpha_i) = r(\beta_i)$, we are done.

Suppose otherwise that there is an $n_{\ell_1} > n_0$ such that

$$\gamma_0^* M_{n_{\ell_1}}(\mathfrak{a})\overline{\gamma}_0 \neq \{0\}.$$

We may assume that n_{ℓ_1} is the smallest positive integer having these properties. Fix an element $j_1 \in J$ such that $\alpha_{j_1}\beta_{j_1}^* \in M_{n_{\ell_1}}(\mathfrak{a})$ and $\gamma_0^*\alpha_{j_1}\beta_{j_1}^*\overline{\gamma}_0 \neq 0$. Then

$$\alpha_{j_1}\beta_{j_1}^* = \alpha_i\mu_1 e_1 f_1^* \omega_1^* \beta_i^*$$

for some paths μ_1 and ω_1 and edges e_1 and f_1. Moreover, $|\alpha_i\mu_1 e_1| = n_{\ell_1}$ and $|\mu_1 e_1| = |f_1^*\omega_1^*|$. Using μ_1 (which may be a vertex) appearing in the above presentation of $\alpha_{j_1}\beta_{j_1}^*$, we define the (possibly empty) set

$$N_1 = \{\alpha_j\beta_j^* \in M_{n_{\ell_1}}(\mathfrak{a}) : \mu_1^*\alpha_i^*\alpha_j\beta_j^*\beta_i\mu_1 \neq 0\}.$$

We claim that there is an edge h_1 such that

$$\alpha_i\mu_1 h_1 h_1^*\mu_1^*\beta_i^* \notin N_1 \quad \text{and} \quad \alpha_i\mu_1 h_1 h_1^*\mu_1^*\beta_i^* \neq 0.$$

Indeed, if $N_1 = \emptyset$, then we take $h_1 = e_1$. Suppose now that $N_1 \neq \emptyset$. For every $\alpha_j\beta_j^* \in N_1$, we have

$$\alpha_j\beta_j^* = \alpha_i\mu_1 e_2 f_2^*\mu_1^*\beta_i^*$$

for some edges e_2 and f_2. Given the relations defining $L_K(E)$ and the assumption that the element $\mathfrak{a}$ is in a reduced form, we deduce that there is an edge h_1 such that

$$\alpha_i\mu_1 h_1 h_1^*\mu_1^*\beta_i^* \notin N_1 \quad \text{and} \quad \alpha_i\mu_1 h_1 h_1^*\mu_1^*\beta_i^* \neq 0,$$

as required.

Let

$$\gamma_1 = \alpha_i\mu_1 h_1 \quad \text{and} \quad \overline{\gamma}_1 = \beta_i\mu_1 h_1.$$

Then

$$\gamma_1^*(M_{n_0}(\mathfrak{a}) \cup M_{n_1}(\mathfrak{a}) \cup \dots \cup M_{n_{\ell_1}}(\mathfrak{a}))\overline{\gamma}_1 = \gamma_1^*\{\alpha_i\beta_i^*\}\overline{\gamma}_1 = \{s(\gamma_1^*)\} = \{r(\overline{\gamma_1})\}.$$

As above, we note that if $\gamma_1^* M_{n_\ell}(\mathfrak{a})\overline{\gamma}_1 = 0$ for all $\ell > \ell_1$, then, taking $\gamma = \gamma_1$, $\overline{\gamma} = \overline{\gamma}_1$ and $\sigma = \mu_1 h_1$, we are done.

Otherwise, consider the smallest integer n_{ℓ_2} such that $n_{\ell_2} > n_{\ell_1}$ and $\gamma_1^* M_{n_{\ell_2}}(\mathfrak{a})\overline{\gamma}_1 \neq 0$. Then in the same way that we found $\gamma_1, \overline{\gamma}_1, \mu_1$ and h_1, we obtain $\gamma_2, \overline{\gamma}_2, \mu_2$ and h_2 such that

$$\gamma_2^*(M_{n_0}(a) \cup M_{n_1}(a) \cup \dots \cup M_{n_{\ell_2}}(a))\overline{\gamma}_2 = \gamma_2^*\{\alpha_i\beta_i^*\}\overline{\gamma}_2 = \{s(\gamma_2^*)\} = \{r(\overline{\gamma_2})\}.$$

If $\gamma_2^* M_{n_\ell}(\mathfrak{a})\overline{\gamma}_2 = 0$ for all $\ell > \ell_2$, then, with $\gamma = \gamma_2$, $\overline{\gamma} = \overline{\gamma}_2$ and $\sigma = \mu_2 h_2$, we obtain the desired result. Otherwise, we continue the process in an obvious way.

Evidently, after finitely many steps we will find γ and $\overline{\gamma}$ such that

$$\gamma^*\Big(\bigcup_{\ell=0}^{t} M_{n_\ell}(\mathfrak{a})\Big)\overline{\gamma} = \gamma^*\{\alpha_i\beta_i^*\}\overline{\gamma} = \{s(\gamma^*)\} = \{r(\overline{\gamma})\},$$

and all other properties that we need for γ and $\overline{\gamma}$ are satisfied. □

Corollary 4.6. *For E, K and $\mathfrak{a}$ as in Proposition 4.5, there are paths γ and $\overline{\gamma}$ such that $\gamma^*\mathfrak{a}\overline{\gamma} = ks(\gamma^*) = kr(\overline{\gamma})$ for some $0 \neq k \in K$.*

By considering (2.1) and the definition of a downward directed graph, the proof of the following result is immediate.

Lemma 4.7. *If K is a field, E is a downward directed graph and $u, v \in E^0$, then $\mathfrak{M}(u, v) \neq \emptyset$.*

If we take Lemma 4.7 into account, then the following result also follows from Proposition 4.5.

Corollary 4.8. *Let K be a field, E a downward directed graph and $\mathfrak{a}$ as in Proposition 4.5. If $v \in E^0$ is such that $\mathfrak{a}v \neq 0$ (resp. $v\mathfrak{a} \neq 0$), then, for every $w \in E^0$, there is a monomial $\alpha\beta^* \in \mathfrak{M}(v, w)$ (resp. $\alpha\beta^* \in \mathfrak{M}(w, v)$) such that $\mathfrak{a}\alpha\beta^* \neq 0$ (resp. $\alpha\beta^*\mathfrak{a} \neq 0$).*

Proof. It will be clear that it suffices to prove the first version of the claim, since the second can be justified in a similar way. Let γ and $\overline{\gamma}$ be as in Corollary 4.6, but in this case for the element $\mathfrak{a}v$. Then, for $u = s(\gamma^*) = r(\overline{\gamma})$, we have $\gamma^*\mathfrak{a}v\overline{\gamma} = u$. By Lemma 4.7, for every $w \in E^0$ there is a nonzero monomial $\sigma\delta^*$ in $\mathfrak{M}(u, w)$. Then $\gamma^*\mathfrak{a}v\overline{\gamma}\sigma\delta^* \neq 0$, which implies that $\mathfrak{a}v\overline{\gamma}\sigma\delta^* \neq 0$. Taking $\alpha = \overline{\gamma}\sigma$ and $\beta = \delta$ gets us home. □

5 On elements which commute with $L_K(E^0_s, E^0_r)$

It is obvious that

$$\mathfrak{a} = \sum_{u,v \in E^0} u\mathfrak{a}v \tag{5.1}$$

for every $\mathfrak{a} \in L_K(E)$. An element $\mathfrak{a} \in L_K(E)$ is said to be *diagonal* if

$$\mathfrak{a} = \sum_{v \in E^0} v\mathfrak{a}v.$$

In this short section, we show that our considerations can be reduced to the diagonal elements of $L_K(E)$.

Lemma 5.1. *Let E be a downward directed graph, K a field and $\mathfrak{a}$ a homogeneous element in $L_K(E)$, as in Definition 4.4. If $\mathfrak{a}$ commutes with all elements in $L_K(E^0_s, E^0_r)$, then the following assertions hold:*
(a) $u\mathfrak{a}v = 0$ for all $u \in E^0$ and all $v \in E^0_s$ such that $u \neq v$.
(b) $u\mathfrak{a}v = 0$ for all $u, v \in E^0_r$ such that $u \neq v$.

Proof. (a) Suppose that there is a $u \in E^0$ and a $v \in E^0_s$, with $u \neq v$, such that $u\mathfrak{a}v \neq 0$. By (2.1) and Corollary 4.8, for any vertex $w \in E^0_r$, there is a nonzero monomial $\alpha\beta^* \in \mathfrak{M}(v, w)$ such that $u\mathfrak{a}\alpha\beta^* \neq 0$. Notice that, as $\alpha\beta^* \in L_K(E^0_s, E^0_r)$ (see Definition (3.1)), we have $u\mathfrak{a}\alpha\beta^* = u\alpha\beta^*\mathfrak{a} = 0$ because $uv = 0$, a contradiction.

(b) Suppose, again for a contradiction, that there are distinct vertices $u, v \in E^0_r$ such that $u\mathfrak{a}v \neq 0$. Take any $w \in E^0_s$. Again, by Corollary 4.8, there is a monomial $\alpha\beta^* \in \mathfrak{M}(w, u)$ such that $\alpha\beta^*\mathfrak{a}v \neq 0$. As in (a), we conclude that $\alpha\beta^*\mathfrak{a}v = \mathfrak{a}\alpha\beta^*v = 0$, again a contradiction. □

Corollary 5.2. *Let E be a downward directed graph, K a field and $\mathfrak{a}$ a homogeneous element in $L_K(E)$, as in Definition 4.4. If $\mathfrak{a}$ commutes with all elements in $L_K(E^0_s, E^0_r)$ and $u\mathfrak{a}v = 0$ for all $u \in E^0_s$ and all $v \in E^0_r$, then $\mathfrak{a}$ is diagonal.*

Proof. Invoking (5.1), we have

$$\mathfrak{a} = \sum_{u,v \in E^0_s} u\mathfrak{a}v + \sum_{u \in E^0_s,\, v \in E^0_r} u\mathfrak{a}v + \sum_{u \in E^0_r,\, v \in E^0_s} u\mathfrak{a}v + \sum_{u,v \in E^0_r} u\mathfrak{a}v,$$

and so the result follows from Lemma 5.1. □

Corollary 5.3. *With the notation and assumption as in Lemma 5.1,*

$$v\mathfrak{a}v\alpha\beta^* = \mathfrak{a}\alpha\beta^* = \alpha\beta^*\mathfrak{a} = \alpha\beta^*w\mathfrak{a}w$$

for all $v \in E^0_s$, $w \in E^0_r$ and $\alpha\beta^* \in \mathfrak{M}(v, w)$, with $\alpha, \beta \in \mathrm{Path}(E)$.

Corollary 5.3 will be used in the sequel, frequently without further mentioning. Now, we will prove the following lemma.

Lemma 5.4. *Sticking to the assumptions of Corollary 5.2, if* $\mathfrak{a} \neq 0$, *then* $v\mathfrak{a}v \neq 0$ *for every vertex* $v \in E^0$.

Proof. Suppose that $\mathfrak{a} \neq 0$ and $v\mathfrak{a}v = 0$ for every $v \in E_r^0$. By Corollary 5.2, $\mathfrak{a}$ is diagonal, and so there is a $u \in E_s^0$ such that $u\mathfrak{a}u \neq 0$. Hence, $u\mathfrak{a} \neq 0$. Let $v \in E_r^0$. By Corollary 4.8, there is a nonzero monomial $\alpha\beta^* \in \mathfrak{M}(u, v)$ such that $u\mathfrak{a}\alpha\beta^* \neq 0$, and so $\mathfrak{a}\alpha\beta^* \neq 0$. As $\alpha\beta^* \in L_K(E_s^0, E_r^0)$, we have $\alpha\beta^*\mathfrak{a} = \mathfrak{a}\alpha\beta^*$. Since $r(\beta^*) = v$ and $v \in E_r^0$, it follows that $\alpha\beta^*\mathfrak{a} = 0$, a contradiction. Hence, there is a $v \in E_r^0$ such that $v\mathfrak{a}v \neq 0$.

Next, suppose that $u\mathfrak{a}u = 0$ for some $u \in E_s^0$. Then, again by Corollary 4.8, for a nonzero monomial $\alpha\beta^* \in \mathfrak{M}(u, v)$ such that $\alpha\beta^*\mathfrak{a}v \neq 0$ we have $\alpha\beta^*\mathfrak{a}v = \mathfrak{a}\alpha\beta^* = 0$, a contradiction. Therefore, $u\mathfrak{a}u \neq 0$ for every $u \in E_s^0$.

Repeating almost the same arguments as above, one obtains that $v\mathfrak{a}v \neq 0$ for every vertex $v \in E_r^0$. □

We are now in a position to prove Theorem 3.2 (3).

Proof of Theorem 3.2 (3). Let E be a graph such that the set E^0 is infinite. Suppose, for a contradiction, that $\mathfrak{a}$ commutes with all elements of $L_K(E_s^0, E_r^0)$ and $\mathfrak{a} \notin L_K(E_s^0, E_r^0)$ (in particular, $\mathfrak{a} \neq 0$). We may assume that $u\mathfrak{a}w = 0$ for every $u \in E_s^0$, $w \in E_r^0$. Then Lemma 5.4 implies that $v\mathfrak{a}v \neq 0$ for every $v \in E^0$. Thus E^0 must be finite, a contradiction. The proof is complete. □

6 On elements which commute with $L_K(E_s^0, E_r^0)$ in the case where E has finitely many vertices

Throughout this section, we assume that $|E^0| < \infty$ (and, as mentioned earlier, $|E^0| > 1$).

Remark 6.1. (1) Let K be a field, and let E be a graph such that $L_K(E)$ is a prime algebra. It should be clear that, in order to prove Theorem 3.2 (1) and Theorem 3.2 (2), it suffices to show that if $\mathfrak{a} \in L_K(E)$ is a nonzero homogeneous element satisfying the properties

(P1) $u\mathfrak{a}v = 0$ for all $u \in E_s^0$ and all $v \in E_r^0$;

(P2) $\mathfrak{a}g = g\mathfrak{a}$ for all $g \in L_K(E_s^0, E_r^0)$,

then:

(I) $\mathfrak{a} \in K \cdot 1$ if E is a graph with finitely many vertices such that
   - (i) E is acyclic, or
   - (ii) E has at least two cycles, or
   - (iii) E has precisely one cycle, and this cycle has an exit;

(II) $\mathfrak{a} \in Z(L_K(E))$ if E is a graph with finitely many vertices and precisely one cycle, and this cycle is without exits.

(2) Consider a reduced representation for $\mathfrak{a}$ (see Remark 4.2):

$$\mathfrak{a} = \sum_{j \in J} k_j \alpha_j \beta_j^*.$$

Suppose that there are nonempty subsets J_1 and J_2 of J such that $J_1 \cap J_2 = \emptyset$ and $J_1 \cup J_2 = J$. Moreover, assume that $\sum_{j \in J_1} k_j \alpha_j \beta_j^*$ is a central element of $L_K(E)$. Then the element $\mathfrak{a}' = \sum_{j \in J_2} k_j \alpha_j \beta_j^*$ obviously satisfies (P1) and (P2), and to show that $\mathfrak{a}$ satisfies, in the appropriate case, (I) or (II), is equivalent to showing that $\mathfrak{a}'$ satisfies (I) or (II), respectively.

Therefore, throughout Sections 6–8, if we assume that $\mathfrak{a}$ satisfies (P1) and (P2), but, depending on the situation, neither (I) nor (II), then we may assume that there is no partition of J (as described in the preceding paragraph) for any reduced representation of $\mathfrak{a}$.

Henceforth, throughout Sections 6–8, we keep a reduced presentation

$$\mathfrak{a} = \sum_{j \in J} k_j \alpha_j \beta_j^* \tag{6.1}$$

and the assumptions on $\mathfrak{a}$ in Remark 6.1.

Lemma 6.2. *If E is a graph with $|E^0| < \infty$, and the element $\mathfrak{a}$ is of degree zero, then for every $v \in E^0$,*

$$v\mathfrak{a}v = \sum_{j\in I} k_j\alpha_j\alpha_j^*$$

for some $I \subseteq J$ (see (6.1)).

Proof. Notice that $v\mathfrak{a}v = \sum_{j\in I} k_j\alpha_j\beta_j^*$ for some $I \subseteq J$. Since $\mathfrak{a}$ has degree zero, it follows that $|\alpha_j| = |\beta_j|$ for every $j \in I$. For this reduced presentation of $v\mathfrak{a}v$, consider the sets $M_{n_\ell}(v\mathfrak{a}v)$, $\ell = 0, 1, \dots, m$ (see Definition 4.4), where $n_0 < n_1 < \cdots < n_m$ are all non-negative integers such that $M_{n_\ell}(v\mathfrak{a}v) \neq \emptyset$, $\ell = 0, 1, \dots, m$.

Suppose, for a contradiction, that there is a q, $0 \le q \le m$, such that $\alpha_i\beta_i^* \in M_{n_q}(v\mathfrak{a}v)$ for some $i \in I$, with $\alpha_i \neq \beta_i$. Let q be the smallest element in $\{0, 1, \dots, m\}$ with this property. Then, for every $\ell < q$, we have $\beta_j = \alpha_j$ for every monomial $\alpha_j\beta_j^* \in M_{n_\ell}(v\mathfrak{a}v)$.

For every ℓ, let

$$(v\mathfrak{a}v)_\ell := \sum_{\alpha_j\beta_j^* \in M_{n_\ell}(v\mathfrak{a}v)} k_j\alpha_j\beta_j^*.$$

Then, obviously,

$$v\mathfrak{a}v = \sum_{\ell\ge 0}(v\mathfrak{a}v)_\ell.$$

Using Remark 4.2 and Proposition 4.5, we get paths γ and $\overline{\gamma}$ such that

$$\gamma^*\Big(\sum_{\ell\ge q}(v\mathfrak{a}v)_\ell\Big)\overline{\gamma} = k_i\gamma^*(\alpha_i\beta_i^*)\overline{\gamma} \neq 0,$$

with $|\gamma| \ge |\alpha_i|$ and $|\overline{\gamma}| \ge |\beta_i|$. By Proposition 4.5 and the fact that $\mathfrak{a}$ has zero degree, we have $\gamma = \alpha_i\sigma$ and $\overline{\gamma} = \beta_i\sigma$ (for some σ), and so γ and $\overline{\gamma}$ have the same length, but, of course, $\gamma \neq \overline{\gamma}$. This implies that

$$\gamma^*(\alpha_j\alpha_j^*)\overline{\gamma} = 0$$

for every $\alpha_j\alpha_j^* \in M_{n_\ell}(v\mathfrak{a}v)$ with $\ell < q$. Therefore, invoking Corollary 5.2, we have

$$\gamma^*\mathfrak{a}\overline{\gamma} = \gamma^*\Big(\sum_{u\in E^0} u\mathfrak{a}u\Big)\overline{\gamma},$$

and so

$$\gamma^*\mathfrak{a}\overline{\gamma} = \gamma^*(v\mathfrak{a}v)\overline{\gamma} = \gamma^*\Big(\sum_{\ell\ge 0}(v\mathfrak{a}v)_\ell\Big)\overline{\gamma} = \gamma^*\Big(\sum_{\ell\ge q}(v\mathfrak{a}v)_\ell\Big)\overline{\gamma} = k_i\gamma^*(\alpha_i\beta_i^*)\overline{\gamma} \neq 0$$

and

$$k_i\gamma^*(\alpha_i\beta_i^*)\overline{\gamma} = k_i s(\gamma^*) = k_i r(\overline{\gamma}).$$

If $v \in E_s^0$, then we fix a $w \in E_r^0$ and consider any nonzero monomial $\tau\delta^* \in \mathfrak{M}(r(\overline{\gamma}), w)$. Then $\gamma^* v\mathfrak{a}v\overline{\gamma}\tau\delta^* \neq 0$. On the other hand, as $v\overline{\gamma}\tau\delta^*$ is in $L_K(E_s^0, E_r^0)$, we get

$$\gamma^* v\mathfrak{a}v\overline{\gamma}\tau\delta^* = \gamma^*\overline{\gamma}\tau\delta^*\mathfrak{a} = 0$$

because $\gamma \neq \overline{\gamma}$ and $|\gamma| = |\overline{\gamma}|$, a contradiction.

If $v \in E_r^0$, then we use similar arguments which also result in a contradiction. The proof is complete. □

Lemma 6.3. *Let E be a graph with $|E^0| < \infty$. If there is a $v \in E^0$ such that $v\mathfrak{a}v = kv$ for some $0 \neq k \in K$, then, for every $w \in E^0$, $\alpha_i = w = \beta_i^*$ for some $i \in J$ (see (6.1)).*

Proof. As $v\mathfrak{a}v = kv$, it follows that $\mathfrak{a}$ is of degree zero. Hence, by Lemma 6.2, $\mathfrak{a}$ takes the form $\mathfrak{a} = \sum_{j\in J} k_j\alpha_j\alpha_j^*$ (see (6.1)), and so it suffices to show that, for every $w \in E^0$, $w = \alpha_i$ for some $i \in J$.

Suppose firstly that $v \in E_s^0$. Fix $x \in E_r^0$, and suppose that $x \neq \alpha_j$ for every $j \in J$. By Lemma 4.3 and Lemma 6.2 there is a ghost path γ^* such that $r(\gamma^*) = x$ and $\gamma^* x\mathfrak{a}x = 0$. Let $\alpha\beta^* \in \mathfrak{M}(v, s(\gamma^*))$. Then $\alpha\beta^*\gamma^*$ is a nonzero element of $L_K(E_s^0, E_r^0)$ and we have

$$0 = \alpha\beta^*\gamma^* x\mathfrak{a}x = v\mathfrak{a}v\alpha\beta^*\gamma^* = kv\alpha\beta^*\gamma^* \neq 0,$$

a contradiction. Hence, we have proved that, for every $x \in E_r^0$, $x = \alpha_i$ for some $i \in J$.

Now we consider any $u \in E_s^0$ and $x \in E_r^0$. From what we have already proved, $x\mathfrak{a}x = k_x x + \sum_{j\in I} k_j \alpha_j \alpha_j^*$ for some $I \subseteq J$, with $|\alpha_j| > 0$ (if $I \neq \emptyset$) and $0 \neq k_x \in K$. Notice that there is a ghost path γ^* such that $r(\gamma^*) = x$ and $\gamma^* x\mathfrak{a}x = k_x\gamma^*$. Indeed, if $I = \emptyset$, then we take $\gamma^* = x$; otherwise we take γ^*, with $\gamma^*(\sum_{j\in I} k_j\alpha_j\alpha_j^*) = 0$ and $r(\gamma^*) = x$ (see Lemma 4.3).

If $u \neq \alpha_j$ for every $j \in J$, then there is a path δ such that $s(\delta) = u$ and $u\mathfrak{a}u\delta = 0$. Let $\alpha\beta^*$ be a nonzero monomial in $\mathfrak{M}(r(\delta), s(\gamma^*))$. Then $u\delta\alpha\beta^*\gamma^*$ is a nonzero element of $L_K(E_s^0, E_r^0)$, and

$$0 = u\mathfrak{a}u\delta\alpha\beta^*\gamma^* = u\delta\alpha\beta^*\gamma^* x\mathfrak{a}x = k_x u\delta\alpha\beta^*\gamma^* \neq 0,$$

a contradiction. Therefore, for every $u \in E_s^0$, we have $u = \alpha_i$ for some $i \in J$.

By the above consideration, if $v \in E_s^0$, then for every $w \in E^0$ we have $w = \alpha_i$ for some $i \in J$.

Now, suppose that $v \in E_r^0$. Using similar arguments to the ones in the previous case, we find that also in this case, for every $w \in E^0$, we have $w = \alpha_i$ for some $i \in J$. The proof is complete. □

7 Final steps of the proof of Theorem 3.2 (1)

We will invoke the next technical result, the proof of which is straightforward, in Lemma 7.2.

Lemma 7.1. *Let E be a graph, and let K be a field. If $uav \neq 0$ for vertices $u, v \in E^0$ and $a \in L_K(E)$, then $\alpha a\beta^* \neq 0$ for all paths α and β such that $r(\alpha) = u$ and $s(\beta^*) = v$.*

As agreed, we assume that $\mathfrak{a}$ is as in (6.1), and we assume the mentioned conditions on $\mathfrak{a}$.

Lemma 7.2. *Let E be a downward directed graph with finitely many vertices, and let K be a field. If*
(i) *E is acyclic, or*
(ii) *E has at least two cycles, or*
(iii) *E has precisely one cycle, and this cycle has an exit,*
then $v\mathfrak{a}v = kv$ for some nonzero $k \in K$ and some vertex $v \in E^0$.

Proof. If E is as in (i) or (iii), then there is a vertex $v \in E^0$ which is a sink. Then $v\mathfrak{a}v = kv$ for some $k \in K$, and $k \neq 0$ by Lemma 5.4.

Next, let E be as in (ii). Since E is downward directed, there is a cycle π based at a vertex w with an exit e such that $s(e) = w$. Let $x = r(e)$.

Suppose that there are vertices $u \in E_s^0$ and $v \in E_r^0$ such that $u \geq w$ and $x \geq v$. Then there are paths $\sigma_1 \in \mathfrak{M}(u, w)$ and $\sigma_2 \in \mathfrak{M}(x, v)$ such that $\tau_\ell = \sigma_1\pi^\ell e\sigma_2 \neq 0$, $s(\tau_\ell) = u$ and $r(\tau_\ell) = v$ for every positive integer ℓ. Notice that $\tau_\ell \in L_K(E_s^0, E_r^0)$ for every ℓ. Hence, for every ℓ, we have $\tau_\ell\mathfrak{a} = \mathfrak{a}\tau_\ell$. Multiplying both sides of the last equation by τ_ℓ^* from the left, we get $\tau_\ell^*\mathfrak{a}\tau_\ell = v\mathfrak{a}v \neq 0$, which yields

$$\sigma_2^* e^*(\pi^\ell)^*\sigma_1^*\mathfrak{a}\sigma_1\pi^\ell e\sigma_2 \neq 0.$$

If the degree of $\mathfrak{a}$ is not zero, then we conclude from Lemma 4.1 that there are positive integers m and t and a nonzero $k \in K$ such that, for every $\ell > m$, $(\pi^\ell)^*\sigma_1^*\mathfrak{a}\sigma_1\pi^\ell$ is equal to either $k\pi^t$ or $k(\pi^*)^t$. In each of these cases, since e is an exit for π, we have $e^*(\pi^\ell)^*\sigma_1^*\mathfrak{a}\sigma_1\pi^\ell e = 0$, a contradiction. Consequently, $\mathfrak{a}$ has degree zero. But this time, by Lemma 4.1 (c), there is a positive integer m such that, for every $\ell \geq m$,

$$v\mathfrak{a}v = \tau_\ell^*\mathfrak{a}\tau_\ell = \sigma_2^* e^*(\pi^\ell)^*\sigma_1^*\mathfrak{a}\sigma_1\pi^\ell e\sigma_2 = kv$$

for some nonzero $k \in K$.

Similarly, we get what is required if, regarding the above notation, we have $u \in E_r^0$, $v \in E_s^0$ and still $u \geq w$ and $x \geq v$ (notice that in this case, $(\tau_\ell)^* \in L_K(E_s^0, E_r^0)$ for every ℓ).

Assume now that for all $u, v' \in E^0$ such that $u \geq w$ and $x \geq v'$ we have $u, v' \in E_s^0$; in particular, $w \in E_s^0$. Fix $z \in E_r^0$. Since E is downward directed, we can find a vertex $v \in E^0$ and paths $\delta \in \mathfrak{M}(x, v)$ and $\gamma \in \mathfrak{M}(z, v)$. By the assumption we made at the beginning of this paragraph, we have $v \in E_s^0$. Then, for every positive integer ℓ, both $\pi^\ell e\delta\gamma^*$ and γ^* are elements of $L_K(E_s^0, E_r^0)$, and both are nonzero. Therefore, using Lemma 7.1, we have

$$\mathfrak{a}\pi^\ell e\delta\gamma^* = \pi^\ell e\delta\gamma^*\mathfrak{a} = \pi^\ell e\delta\mathfrak{a}\gamma^* \neq 0.$$

Hence, $\mathfrak{a}\pi^\ell e\delta = \pi^\ell e\delta\mathfrak{a}$. Multiplying both sides of the last equation by $\delta^* e^*(\pi^\ell)^*$ from the left, we get $\delta^* e^*(\pi^\ell)^*\mathfrak{a}\pi^\ell e\delta = v\mathfrak{a}v \neq 0$.

By the same arguments as in the first paragraph of this proof, we get a contradiction if we assume that $\mathfrak{a}$ is not of degree zero. On the other hand, assuming that the degree of $\mathfrak{a}$ is equal to zero, we get $v\mathfrak{a}v = kv$ for some nonzero $k \in K$.

Similar reasoning leads us to the required conclusion if we assume that for all $u, v' \in E^0$ such that $u \geq w$ and $x \geq v'$ we have $u, v' \in E^0_r$. □

Proof of Theorem 3.2 (1). In the light of Remark 6.1, suppose that $\mathfrak{a}$ satisfies (P1) and (P2), but not (I). Combining (6.1) with Lemma 6.2 and Lemma 7.2, we have $\mathfrak{a} = \sum_{j\in J} k_j\alpha_j\alpha_j^*$. Moreover, by Lemma 6.3, for every $w \in E^0$, there is an $i \in J$ such that $w = \alpha_i$. Therefore, the presentation of $\mathfrak{a}$ in (6.1) takes the form

$$\mathfrak{a} = \sum_{w\in E^0} k_w w + \sum_{j\in J'} k_j\alpha_j\alpha_j^*$$

for some nonzero elements $k_w, k_j \in K$, and $J' \subseteq J$ such that $|\alpha_j| > 0$ for every $j \in J'$. Notice that $|J| = |J'| + |E^0|$.

Fix a vertex $v \in E^0$. Then

$$\sum_{w\in E^0} k_w w = \sum_{w\in E^0} kw + \sum_{w\in E^0\setminus\{v\}} k'_w w = k + \sum_{w\in E^0\setminus\{v\}} k'_w w$$

for some k and k'_w in K. Consider

$$\mathfrak{b} = \sum_{w\in E^0\setminus\{v\}} k'_w w + \sum_{j\in J'} k_j\alpha_j\alpha_j^*.$$

Since $\mathfrak{a} = k + \mathfrak{b}$, and since it is a reduced representation of $\mathfrak{a}$, we get a contradiction with the assumption made in Remark 6.1 (2). The proof is complete. □

8 Remaining case

In order to give a complete proof of Theorem 3.2, in this section we consider the remaining case, where, by assumption, E is a downward directed graph with precisely one cycle π, say, and π is without exists.

Proof of Theorem 3.2 (2). First assume that E is not row-finite, and consider the element $\mathfrak{a}$ with a reduced presentation as in (6.1). Let w be a vertex such that the set $s^{-1}(w)$ is infinite, and let $e \in s^{-1}(w)$ be an edge such that neither e nor e^* appears in a monomial belonging to $\operatorname{supp}(\mathfrak{a})$. Let $z = r(e)$.

Fix a vertex $v \in \pi^0$. By Lemma 4.7, for every vertex u, there is a path $\alpha_{uv} \in \mathfrak{M}(u, v)$, i.e., α_{uv} is of the form $\gamma\mu^*$ for certain $\gamma, \mu \in \operatorname{Path}(E)$ with $s(\gamma) = u$ and $r(\mu^*) = v$ (see the definition of $\mathfrak{M}(u, v)$ in (2.1)). We will use this notation related to paths throughout this proof.

Suppose that $v, w \in E^0_s$, and consider any $u \in E^0_r$. Then the monomials α_{uv}^* and $e\alpha_{zv}\alpha_{uv}^*$ are in $L_K(E^0_s, E^0_r)$. By Lemma 5.4, $v\mathfrak{a}v \neq 0$, and so, since $v\mathfrak{a}v = \alpha_{zv}^* e^* e\alpha_{zv}\mathfrak{a}\alpha_{uv}^*\alpha_{uv}$, we have

$$\mathfrak{a}e\alpha_{zv}\alpha_{uv}^* = e\alpha_{zv}\alpha_{uv}^*\mathfrak{a} = e\alpha_{zv}\mathfrak{a}\alpha_{uv}^* \neq 0. \tag{8.1}$$

Ignoring the second monomial in (8.1) and multiplying the first and third monomials in (8.1) by α_{uv} from the right, we get

$$\mathfrak{a}e\alpha_{zv} = e\alpha_{zv}\mathfrak{a},$$

and so, again by Lemma 5.4,

$$\alpha_{zv}^* e^*\mathfrak{a}e\alpha_{zv} = v\mathfrak{a}v \neq 0. \tag{8.2}$$

Notice that $e^*\alpha_j\beta_j^* e = 0$ for every $j \in J$ such that $|\alpha_j| + |\beta_j| > 0$. Therefore, $v = \alpha_i = \beta_i^*$ for some $i \in J$, and by (8.2), we have $v\mathfrak{a}v = k_v v$ for some nonzero $k_v \in K$.

Inserting now, here, the part of the proof of Theorem 3.2 (1), from “by Lemma 6.3” in the second line of the proof of Theorem 3.2 (1) onward, we conclude that $\mathfrak{a} \in K \subseteq Z(L_K(E))$.

The following possibilities can be treated in a similar way:

- $v \in E_r^0$ and $w \in E_s^0$.
- $v, w \in E_r^0$.
- $v \in E_s^0$ and $w \in E_r^0$.

Now, we consider the case where the graph E is, by assumption, row-finite. Following the proof of [4, Theorem 3.3], we fix a vertex $v \in \pi^0$. Then $u \geq v$ for every $u \in E^0$. Consider the (necessarily finite) set

$$P = \{p_i : 1 \leq i \leq n\}$$

of all paths which end in v and which do not contain the cycle π. It follows readily that the set

$$B = \{p_i \pi^\ell p_j^*\}_{i,j \in \{1,\dots,n\},\, \ell \geq 1} \cup \{p_i (\pi^*)^\ell p_j^*\}_{i,j \in \{1,\dots,n\},\, \ell \geq 1} \cup \{p_i p_j^*\}_{i,j \in \{1,\dots,n\}}$$

is a basis for $L_K(E)$.

Moreover, $L_K(E) \cong \mathbb{M}_n(K[x, x^{-1}])$. Indeed, recalling that $E_{(i,j)}$ denotes the standard matrix unit, the isomorphism φ in question acts on the basis B as follows:

$$\varphi(p_i \pi^\ell p_j^*) = x^\ell E_{(i,j)}, \quad \varphi(p_i (\pi^*)^\ell p_j^*) = x^{-\ell} E_{(i,j)} \quad \varphi(p_i p_j^*) = E_{(i,j)}.$$

By [16, Theorem 3.3], the center $Z(L_K(E))$ of $L_K(E)$ is spanned by the set

$$B_Z = \Big\{ \sum_{i=1}^n p_i \pi^\ell p_i^* \Big\}_{\ell \geq 1} \cup \Big\{ \sum_{i=1}^n p_i (\pi^*)^\ell p_i^* \Big\}_{\ell \geq 1} \cup \Big\{ \sum_{i=1}^n p_i p_i^* \Big\}.$$

Evidently, $L_K(E_s^0, E_r^0)$ is spanned by all monomials in B which have their sources in E_s^0 and their ranges in E_r^0. We may assume that $j \in \{1, \dots, n-1\}$ is such that $s(p_i) \in E_s^0$ for every i such that $1 \leq i \leq j$, and $s(p_i) \in E_r^0$ for every i such that $j < i \leq n$.

Let $F = \mathrm{Frac}(K[x, x^{-1}])$ be the field of fractions of the integral domain $K[x, x^{-1}]$. Then $\mathbb{M}_n(K[x, x^{-1}])$ is a subalgebra of $\mathbb{M}_n(F)$, and recall (see (1.3)) that $\mathbb{M}_n(F) \cong L_F(\mathcal{E})$.

Let $A = Z(L_K(E)) + L_K(E_s^0, E_r^0)$. Composing the two mentioned isomorphisms and slightly abusing notation, we can see that the image of A in $L_F(\mathcal{E})$ is equal to $K[x, x^{-1}] + L_{K[x,x^{-1}]}(\mathcal{E}_s^0, \mathcal{E}_r^0)$, where $\mathcal{E}_s^0 = \{v_1, \dots, v_j\}$ and $\mathcal{E}_r^0 = \{v_{j+1}, \dots, v_n\}$.

By Theorem 3.2 (1), $F \cdot 1 + L_F(\mathcal{E}_s^0, \mathcal{E}_r^0)$ is a maximal commutative subalgebra of $L_F(\mathcal{E})$. By considering any element $r \in F \cdot 1 + L_F(\mathcal{E}_s^0, \mathcal{E}_r^0)$, it follows readily that $r = ab^{-1}$ for some elements

$$a \in K[x, x^{-1}] + L_{K[x,x^{-1}]}(\mathcal{E}_s^0, \mathcal{E}_r^0) \quad \text{and} \quad b \in K[x, x^{-1}].$$

Since $F \cdot 1 = Z(L_F(\mathcal{E}))$, it follows that $K[x, x^{-1}] + L_{K[x,x^{-1}]}(\mathcal{E}_s^0, \mathcal{E}_r^0)$ is a maximal commutative subalgebra of $L_{K[x,x^{-1}]}(\mathcal{E})$. As $L_{K[x,x^{-1}]}(\mathcal{E})$ is the image of $\mathbb{M}_n(K[x, x^{-1}])$ in $L_F(\mathcal{E})$, we conclude that A is a maximal commutative subalgebra of $L_K(E)$. □

9 Further comments and considerations

Remark 9.1. (1) Obviously, $L_K(E_s^0, E_r^0) \cap Z(L_K(E)) = \{0\}$ for any partition (E_s^0, E_r^0) of E^0 (where $|E^0| > 1$). It is also clear that $Z(L_K(E))$ is a subset of every maximal commutative subalgebra of $L_K(E)$. Therefore, by Theorem 3.2 (1) and the first part of the proof of Theorem 3.2 (2), if $L_K(E)$ is a prime Leavitt path algebra, then

$$Z(L_K(E)) = \begin{cases} K \cdot 1 & \text{if } |E| < \infty \text{ and } E \text{ is not a comet,} \\ \{0\} & \text{if } |E^0| = \infty. \end{cases}$$

(2) By (1) above and the well-known fact that simplicity of a Leavitt path algebra $L_K(E)$ implies that every cycle in E has an exit, it follows that if $L_K(E)$ is a simple Leavitt path algebra, then

$$Z(L_K(E)) = \begin{cases} K \cdot 1 & \text{if } |E| < \infty, \\ \{0\} & \text{if } |E^0| = \infty. \end{cases}$$

Remark 9.2. Considering $L_K(E^0_s, E^0_r)$ for a partition (E^0_s, E^0_r) of E^0, and the commutative core $\mathcal{M}_K(E)$ of $L_K(E)$ constructed by Gil Canto and Nasr-Isfahani (see [22]), we have $\mathcal{M}_K(E) \cap L_K(E^0_s, E^0_r) = \{0\}$.

Now we would like to propose some consideration showing an idea how to transfer the fact that, for every field K, $\lfloor \frac{n^2}{4} \rfloor + 1$ is a sharp upper bound for the dimension of a commutative subalgebra of $\mathbb{M}_n(K)$ (see Section 1) to the territory of prime Leavitt path algebras.

Definition 9.3. Let E be a graph with $|E^0| < \infty$, K a field and A a subalgebra of $L_K(E)$. Then the largest number d such that there are d distinct pairs $(v, w) \in E^0 \times E^0$ with $\{0\} \neq vAw \subseteq A$ is called the E-dimension of A and denoted by $E\dim(A)$.

Proposition 9.4. *Let E be a graph with $|E^0| < \infty$, and let K be a field. If A is a subalgebra of $L_K(E)$, then $E\dim(A) \leq \dim_K A$.*

Proof. Let A be a subalgebra of $L_K(E)$, and let

$$(v_1, w_1), (v_2, w_2), \ldots, (v_m, w_m)$$

be distinct pairs in $E^0 \times E^0$ such that $\{0\} \neq v_i A w_i \subseteq A$, $i = 1, \ldots, m$. Then taking, for each i, an element $a_i \in A$ such that $v_i a_i w_i \neq 0$, we get linearly independent elements $v_1 a_1 w_1, \ldots, v_m a_m w_m$, from which the result follows. □

It follows from (1.3) and (1.4) that, for every algebra A belonging to the class $\mathrm{Comm}(L_K(\mathcal{E}))$ of all commutative subalgebras of $L_K(\mathcal{E})$, we have

$$\mathcal{E}\dim(A) \leq \dim_K A \leq \Big\lfloor \frac{|\mathcal{E}^0|^2}{4} \Big\rfloor + 1.$$

On the other hand, by using the paragraph preceding Definition 3.1, it is evident that there are algebras $A \in \mathrm{Comm}(L_K(\mathcal{E}))$ such that

$$\mathcal{E}\dim(A) = \Big\lfloor \frac{|E^0|^2}{4} \Big\rfloor.$$

It shows the relationship between the $\mathcal{E}$-dimension and vector space dimension in the context of the Leavitt path algebra $L_K(\mathcal{E})$.

It is obvious that if $L_K(E)$ is a prime Leavitt path algebra, where E is a graph with finitely many vertices and at least one cycle, and if A is a commutative subalgebra of $L_K(E)$, then $\dim_K A$ can be infinite. However, we can still consider the notion of the E-dimension of A; in fact, we have the following result.

Proposition 9.5. *Let K be a field, and E a graph with finitely many vertices. If $L_K(E)$ is a prime algebra, then*

$$E\dim(A) \leq \Big\lfloor \frac{|E^0|^2}{4} \Big\rfloor$$

for every commutative subalgebra A of $L_K(E)$. Moreover, there are commutative subalgebras A of $L_K(E)$ such that

$$E\dim(A) = \Big\lfloor \frac{|E^0|^2}{4} \Big\rfloor.$$

Proof. Suppose, for a contradiction, that for some commutative subalgebra A of $L_K(E)$ we can find more than $\lfloor \frac{|E^0|}{4} \rfloor$ pairs $(v_i, w_i) \in E^0 \times E^0$ such that $\{0\} \neq v_i A w_i \subseteq A$ for each such pair. Then there is a vertex u such that, for some $i \neq j$, we have $u = v_i$, $u = w_j$ and either $w_i \neq u$ or $v_j \neq u$. Take any nonzero element $v_j a u \in v_j A u \subseteq A$. We may assume that a is a monomial, say $a = \alpha\beta^*$, for some $\alpha, \beta \in \mathrm{Path}(E)$. Let $z = r(\alpha)$, and consider $\delta\sigma^* \in \mathfrak{M}(z, w_i)$. Then $\beta\delta\sigma^* \neq 0$ and $a\beta\delta\sigma^* \neq 0$. Notice that $s(\beta) = r(\beta^*) = u$, which implies that $\beta\delta\sigma^* \in A$. Hence,

$$\beta\delta\sigma^* a = a\beta\delta\sigma^* \neq 0.$$

The last fact tells us that $v_j = u$ and $w_i = u$, a contradiction.

For the second part of the proposition, take

$$q = \begin{cases} \dfrac{|E^0|}{2} & \text{if } |E^0| \text{ is even,} \\ \dfrac{|E^0| - 1}{2} & \text{if } |E^0| \text{ is odd.} \end{cases}$$

If $v_1, \ldots, v_{|E^0|}$ denote the distinct vertices of E, then let $E_s^0 = \{v_1, \ldots, v_q\}$ and $E_r^0 = \{v_{q+1}, \ldots, v_{|E^0|}\}$. Clearly,

$$E\dim(K \cdot 1 + L_K(E_s^0, E_r^0)) = \left\lfloor \frac{|E^0|^2}{4} \right\rfloor. \qquad \square$$

10 On the commutative core of Leavitt path algebras

The main object of our interest in the present section is the construction presented in [36] by Tomforde, which is a generalization of Leavitt path algebras.

Definition 10.1. Let E be a graph, and R a commutative unital ring. Then the Leavitt path algebra with coefficients in R, denoted by $L_R(E)$, is the universal R-algebra generated by the set E^0 of all vertices and the set $\{e, e^* : e \in E^1\}$ of all edges and ghost edges, satisfying conditions (V), (E1), (E2), (CK1) and (CK2).

By [22, Proposition 4.5], the *commutative core* $\mathcal{M}_R(E)$ of a Leavitt path algebra $L_R(E)$ over a commutative unital ring R is a commutative subalgebra of $L_R(E)$, and it is generated as an R-algebra by all elements of the form $\alpha\alpha^*$, $\alpha\lambda\alpha^*$ and $\alpha\lambda^*\alpha^*$, where α is a path and λ is a cycle without exits. Moreover, by [22, Theorem 4.13]), $\mathcal{M}_R(E)$ is a maximal commutative subalgebra of $L_R(E)$.

The proof of the commutativity of $\mathcal{M}_R(E)$ presented in [22] involves only the structure of the considered elements of $L_R(E)$. On the other hand, the proof of the maximality of $\mathcal{M}_R(E)$ requires more sophisticated tools and considerations.

In this section, we would like to prove the maximality of $\mathcal{M}_R(E)$ using arguments which are totally different from those in [22]. Our reasoning is more direct and refers only to the structure of $L_R(E)$.

Proposition 10.2 (see [22, Theorem 4.13]). *Let E be a graph, and R a commutative unital ring. Then $\mathcal{M}_R(E)$ is a maximal commutative subalgebra of $L_R(E)$, and $\mathcal{M}_R(E) = \Lambda$, where*

$$\Lambda = \{x \in L_R(E) : x\alpha\alpha^* = \alpha\alpha^* x \text{ for every } \alpha \in \operatorname{Path}(E)\}.$$

Proof. The fact that $\mathcal{M}_R(E)$ is commutative follows from [22, Proposition 4.5]. It should also be clear that $\mathcal{M}_R(E) \subseteq \Lambda$. Notice that the set Λ is closed with respect to multiplication.

Suppose, for a contradiction, that there is an $x \in \Lambda \setminus \mathcal{M}_R(E)$. We may assume, without loss of generality, that x is homogeneous. Considering x in a reduced form, we assume that n is the smallest non-negative integer such that $M_n(x) \neq \emptyset$. Let $\alpha\beta^* \in M_n(x)$. Clearly, we may assume that $\alpha\beta^* \notin \mathcal{M}_R(E)$.

By a version of Proposition 4.5, adapted to the currently discussed construction, there are paths γ and $\overline{\gamma}$ such that

$$\gamma^* x\overline{\gamma} = \gamma^*\alpha\beta^*\overline{\gamma} = s(\gamma^*) = r(\overline{\gamma}). \tag{10.1}$$

Moreover, $\gamma = \alpha\sigma$ and $\overline{\gamma} = \beta\sigma$ for some path σ.

For $\gamma\gamma^*$ and $\overline{\gamma}\overline{\gamma}^*$, by the definition of Λ and by (10.1),

$$x\gamma\gamma^*\overline{\gamma}\overline{\gamma}^* = \gamma\gamma^* x\overline{\gamma}\overline{\gamma}^* = \gamma\overline{\gamma}^* = \alpha\sigma\sigma^*\beta^* \neq 0,$$

which implies that $\gamma^*\overline{\gamma} \neq 0$. Hence, $\sigma^*\alpha^*\beta\sigma \neq 0$, and so $\alpha^*\beta \neq 0$. If $|\alpha| = |\beta|$, then $\alpha\beta^* = \alpha\alpha^* \in \mathcal{M}_R(E)$, a contradiction. Therefore, either $|\alpha| > |\beta|$ or $|\alpha| < |\beta|$.

We first consider the case $|\alpha| > |\beta|$. In this situation, $\alpha = \beta\lambda$ for some path λ, and $\alpha\beta^* = \beta\lambda\beta^*$. Since $\alpha\beta^* \notin \mathcal{M}_R(E)$ and $\beta\lambda\beta^* \neq 0$, it follows that λ is a closed path which is neither of the form c^ℓ nor of the form $(c^*)^\ell$, where c is a cycle without exits and ℓ is a positive integer. Hence, assuming that

$$\lambda = e_1 e_2 \cdots e_m$$

for some edges $e_1, e_2, \ldots, e_m$, with $s(e_1) = r(e_m)$, there is a j, with $1 \leq j \leq m$, and an edge f such that $s(e_j) = s(f)$ and $e_j \neq f$.

Notice that, by (10.1) and the fact that Λ is closed under multiplication, we get, for $z = \beta\lambda\sigma\sigma^*\beta^*$, the following:

$$z = \beta\lambda\sigma\sigma^*\beta^* = \gamma\overline{\gamma}^* = \gamma\gamma^* x\overline{\gamma}\overline{\gamma}^* \in \Lambda \setminus \{0\}. \tag{10.2}$$

Thus, for $\beta\sigma\sigma^*\beta^*$, we have $\beta\sigma\sigma^*\beta^* z = z\beta\sigma\sigma^*\beta^*$, which implies that

$$\beta\sigma\sigma^*\lambda\sigma\sigma^*\beta^* = \beta\lambda\sigma\sigma^*\beta^* = z \neq 0.$$

Therefore, $\sigma^*\lambda\sigma \neq 0$, and it follows readily that

$$\sigma^*\lambda\sigma = e_i e_{i+1} \cdots e_m e_1 e_2 \cdots e_{i-1}$$

for some $i \in \{1, 2, \dots, m\}$.

Consider $\delta = e_i e_{i+1} \cdots e_{j-1} f$, which is of length smaller than or equal to n, and

$$y = \beta\sigma\delta\delta^*\sigma^*\beta^* = \beta\sigma\delta(\beta\sigma\delta)^*,$$

which, by its construction, is nonzero. Then, by (10.2), we have $zy = yz$. Notice that $zy = \beta\lambda\sigma\delta\delta^*\sigma^*\beta^* \neq 0$. On the other hand, in the product yz we have $\delta^*\sigma^*\lambda\sigma$ as a factor, and

$$\delta^*\sigma^*\lambda\sigma = (f^* e_{j-1}^* \cdots e_{i+1}^* e_i^*)(e_i e_{i+1} \cdots e_m e_1 e_2 \cdots e_{i-1}) = 0,$$

since $f \neq e_j$. Hence $yz = 0$, a contradiction.

In a similar way, we arrive at a contradiction if we consider the case $|\alpha| < |\beta|$.

Consequently, we deduce that $\Lambda = \mathcal{M}_R(E)$. The maximality of $\mathcal{M}_R(E)$ follows from the same argument as the one presented at the end of the proof of [22, Theorem 4.13]. □

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