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# THE RECOVERY OF THE NON-DIAGONAL TILE IN A TILED TRIANGULAR MATRIX RING

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We show that it can happen that the tile in the non-diagonal position of a $2\times 2$ upper triangular tiled matrix ring cannot be recovered up to isomorphism even if the base ring is finite.

In this note we provide an example regarding the impossibility of the recovery (up to isomorphism) of one of the algebraic structures involved in tiled triangular matrix rings even if the base ring is finite, and we mention a non-trivial class of finite rings, as base rings, for which the recovery is indeed possible. This ties in with recent work on the recovery of some of the algebraic structures involved in various classes of matrix rings.

We first sketch the background. Questions concerning the representability of certain rings as full matrix rings have led to studying ways of finding the corresponding base ring and determining whether or not the base ring is unique up to isomorphism. It is well-known that if $R$ and $S$ are commutative rings, then the full matrix rings $M_n(R)$ and $M_n(S)$ are isomorphic, since $R$ and $S$ are isomorphic to the centres of $M_n(R)$ and $M_n(S)$ respectively. However, it was shown by Smith **[15]** that there are non-isomorphic simple Noetherian integral domains $R$ and $S$ (one of which is the first Weyl algebra) for which $M_n(R) \approx M_n(S)$. Therefore, even for naturally-occurring Noetherian non-commutative rings $R$ and $S$ it is possible that $M_n(R) \approx M_n(S)$ but $R \neq S$. See also **[10]** and **[16]**. Thus it is not possible to recover up to isomorphism the base ring $R$ from the complete matrix ring $M_n(R)$.

A ring more closely related to commutative rings is the ring $H := Z[i, j, k]$ of quaternions with integer coefficients. This ring was the inspiration of a series of papers by Chatters. Let $H_p$ denote the localization of $H$ at some odd prime $p$, and

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consider the following tiled subrings of $M_2(H)$ and $M_2(H_p)$ respectively:

$$T := \begin{bmatrix} H & pH \\ H & H \end{bmatrix}, \; T' := \begin{bmatrix} H_p & pH_p \\ H_p & H_p \end{bmatrix}.$$

Chatters showed in **[2]** that $T'$ is isomorphic to a full $2 \times 2$ matrix ring over a suitable ring, and he asked whether $T$ is isomorphic to a full $2 \times 2$ matrix ring $M_2(W)$ too, and if so, what is $W$? The first question was answered in the affirmative independently by Chatters **[3]** and Robson **[14]**, thus making $T$ into a "hidden" matrix ring. The ring $W$ can be chosen as the idealizer $I(J) := \{r \in H;\ rJ \subseteq J\}$ of a right ideal $J$ of $H$, but $T$ does not determine the isomorphism-type of $W$. In fact, Chatters showed in [3, Theorem 3.10] that there at least as many pairwise non-isomorphic rings $W$ with $T \approx M_2(W)$ as there are representations of $p$ as a sum of four squares. Furthermore, in [5, Proposition 5.7] Chatters showed that if $W$ is any ring such that $T \approx M_2(W)$, then $W$ is indeed isomorphic to one of the mentioned idealizers $I(J)$. See also **[11]** for a thorough study of certain tiled subrings of full matrix rings, which, despite appearing otherwise, are themselves full matrix rings.

In **[4]** Chatters showed that even in the prime Noetherian case it can happen $M_n(R) \approx M_n(S)$, with $R \neq S$. Chatters constructed an uncountable family of pairwise non-isomorphic rings $R_i$ such that the corresponding matrix rings $M_2(R_i)$ are isomorphic to one another. We conclude from [3], [4], and **[17]** that examples are known of non-isomorphic orders $R$ and $S$ in a finite-dimensional central simple algebra such that $M_2(R) \approx M_2(S)$. The examples in [3] and [4] are not maximal orders and those in [17] are maximal orders, albeit relatively complicated. This led Chatters in **[6]** to constructing many pairwise non-isomorphic maximal $\mathbf{Z}$-orders which have isomorphic full matrix rings. To be precise: given an $n \geq 2$, Chatters constructed $n$ pairwise non-isomorphic maximal $\mathbf{Z}$-orders with isomorphic full $n \times n$ matrix rings.

On a "positive" note it was shown in **[8]** that the underlying Boolean matrix matrix $B$ of a structural matrix ring $M(B, R)$ over a semiprime Noetherian ring $R$ can be recovered up to conjugation. To be more precise, [8, Theorem 2.4] shows that the underlying Boolean matrices $B_1$ and $B_2$ of two isomorphic matrix rings $M(B_1, R)$ and $M(B_2, R)$ over a semiprime left Noetherian ring $R$ are conjugated, that is one of them can be obtained from the other by a permutation of the rows and columns, which is equivalent to saying that the directed graphs associated with $B_1$ and $B_2$ are isomorphic. In [8, Corollary 2.5] it was shown that semiprimeness can be dropped in [8, Theorem 2.4] in case the underlying ring $R$ is commutative, and at the end of [8] it was conjectured that semiprimeness can be dropped in general in [8, Theorem 2.4]. In



**[9]** it was shown that semiprimeness can be dropped in [8, Theorem 2.4] if the underlying Boolean matrix is complete blocked triangular. Moreover, it was shown in this case that the underlying Boolean matrices are equal, that is the underlying Boolean matrix of a complete blocked triangular matrix ring over a Noetherian ring is unique. Complete blocked triangular matrix rings over division rings feature, for example, in the representation of left Artinian CI-prime rings in **[12]**.

Abrams, Haefner and Del Río showed in **[1]** that the conjecture mentioned above is indeed true. They call a ring $R$ with the property that the integer $\max(\{n \mid$ there exist nonzero right ideals $K_1, K_2, \cdots, K_n$ with $R = K_1 \oplus \cdots \oplus K_n\})$ exists, a ring with *finite summand length*. These rings are abundant and include, for example, rings with Goldie dimension and hence Noetherian rings. In [1, Theorem 1.12] they then proved the following much stronger version of the mentioned conjecture in [8]: if $R$ is a ring with finite summand length, and $P$ and $P'$ are finite preordered sets such that the incidence rings $I(P, R)$ and $I(P', R)$ are isomorphic, then $P$ and $P'$ are isomorphic as preordered sets.

Although the above summary of results on the recovery of some of the underlying algebraic structures in matrix rings is not complete, it gives the reader a flavour of the type of problems. So much for background.

We note now that a $2 \times 2$ upper triangular matrix ring $\begin{bmatrix} R & R \\ 0 & R \end{bmatrix}$ is a special case of a complete blocked triangular matrix ring in the sense that each block has size 1. Hence, considering a holistic picture of the known results, the question of the possible recovery of the tile in the non-diagonal position of a $2 \times 2$ upper triangular tiled matrix ring over a left Noetherian ring arises. To be more precise: if $R$ is a left Noetherian ring and $T_1$ and $T_2$ are two-sided ideals of $R$ such that the tiled triangular matrix rings.

$$\begin{bmatrix} R & T_1 \\ 0 & R \end{bmatrix} \text{and} \begin{bmatrix} R & T_2 \\ 0 & R \end{bmatrix}$$

are isomorphic, does it necessarily follow that $T_1$ and $T_2$ are isomorphic as $R$-bimodules, i.e. can the tile in the non-diagonal position be recovered?

We note that if no kind of finiteness condition is imposed on the base ring $R$, then there is no hope of recovery of the tile. In fact, in [8, Example 1.1] a ring $R$ was constructed such that

$$\begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} \approx \begin{bmatrix} R & R \\ 0 & R \end{bmatrix}.$$

Moreover, we now show that it can happen that even if $R$ is finite, then the tile cannot be recovered.

EXAMPLE. Consider the structural matrix ring

$$M(B,R) = \begin{bmatrix} R & 0 & R \\ 0 & R & R \\ 0 & 0 & R \end{bmatrix} \text{ associated with the Boolean matrix } B := \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and any ring $R$. This structural matrix ring will now act as the base ring in a tiled triangular matrix ring. Consider the two-sided ideals

$$T_1 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & R \\ 0 & 0 & 0 \end{bmatrix} \text{ and } T_2 := \begin{bmatrix} 0 & 0 & R \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

of $M(B,R)$ and the corresponding tiled triangular matrix rings.

$$\begin{bmatrix} M(B,R) & T_1 \\ 0 & M(B,R) \end{bmatrix} \text{ and } \begin{bmatrix} M(B,R) & T_2 \\ 0 & M(B,R) \end{bmatrix}.$$

These two tiled triangular matrix rings are isomorphic to the structural matrix rings $M(B_1,R)$ and $M(B_2,R)$ respectively, where

$$B_1 := \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } B_2 := \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$



The permutation $(12) \in S_6$, with $S_6$ the cyclic group on six elements, shows that the directed graphs associated with $B_1$ and $B_2$ are isomorphic, and so $M(B_1,R) \approx M(B_2,R)$. Hence

$$\begin{bmatrix} M(B,R) & T_1 \\ 0 & M(B,R) \end{bmatrix} \approx \begin{bmatrix} M(B,R) & T_2 \\ 0 & M(B,R) \end{bmatrix}.$$

However, it is straightforward to verify that $T_1$ and $T_2$ are not isomorphic as left $M(B,R)$-modules. Indeed, if $\psi : T_1 \to T_2$ is a left $M(B,R)$-isomorphism and

$$\varphi\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) =: \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for some $x \in R$, then

$$\begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \varphi\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\varphi\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $x \neq 0$, it follows that $T_1$ and $T_2$ are not isomorphic as left $M(B,R)$-modules, which concludes the example.

Two crucial features of the above example is the fact that the base ring

$$M(B,R) = \begin{bmatrix} R & 0 & R \\ 0 & R & R \\ 0 & 0 & R \end{bmatrix}$$

in the tiled triangular matrix rings

$$\begin{bmatrix} M(B,R) & T_1 \\ 0 & M(B,R) \end{bmatrix} \text{and} \begin{bmatrix} M(B,R) & T_2 \\ 0 & M(B,R) \end{bmatrix}$$

is non-commutative, and, moreover, that it has two two-sided ideals (which are non-isomorphic as $M(B,R)$-bimodules) which have the same cardinality. We conclude that, contrary to the mentioned example, any class of finite commutative rings with the property that all the ideals in a ring $A$ in such a class have different cardinalities, will be such that the tile $T$, with $T$ a two-sided ideal of $A$, in the non-diagonal position of the $2 \times 2$ tiled triangular matrix ring

$$\begin{bmatrix} A & T \\ 0 & A \end{bmatrix}$$

can indeed be recovered.

A non-trival class of such rings is the class of finite commutative chain rings or finite *special* principal ideal rings (PIR's). Such a ring $A$ has a unique prime ideal $(\theta)$ and this ideal is nilpotent. (Therefore a special PIR is a local ring). If $k$ is the nilpotency index of $(\theta)$ (which is the nilpotency index of $\theta$), then every nonzero element in $A$ can be written in the form $x\theta^l$, where $x$ is invertible in $R$, $0 \le l < k$, $l$ is unique and $x$ is unique modulo $\theta^{k-l}$. Furthermore, every ideal of $R$ is of the form $(\theta^j)$, $0 \le j \le k$. Concrete examples of these rings are the rings $Z_{p^n}$ of integers modulo $p^n$ ($p$ a prime), the Galois fields $GF(p^n)$ and the Galois rings $GR(p^n, r) = Z_{p^n}[x]/(g(x))$ of characteristic $p^n$ *and rank* $r$, where $g(x)$ is monic of degree $r$ and irreducible modulo the prime $p$. (See, for example, [7] and [13] for further examples of finite commutative chain rings.)

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