THE LATTICE OF IDEALS
OF $M_R(R^2)$, $R$ A COMMUTATIVE PIR

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(Received 20 June 1990; revised 20 November 1990)

Communicated by P. Schultz

Abstract

In this paper we characterize the ideals of the centralizer near-ring $N = M_R(R^2)$, where $R$ is a commutative principle ideal ring. The characterization is used to determine the radicals $J_\nu(N)$ and the quotient structures $N/J_\nu(N)$, $\nu = 0, 1, 2$.


1. Introduction

Let $R$ be a ring with identity and let $G$ be a unitary (right) $R$-module. Then $M_R(G) = \{f: G \to G \mid f(ar) = f(a) \cdot r, \ a \in G, \ r \in R\}$ is a near-ring under function addition and composition, called the centralizer near-ring determined by the pair $(R, G)$. When $G$ is the free $R$-module on a finite number of (say $n$) generators, then $M_R(R^n)$ contains the ring $M_n(R)$ of $n \times n$ matrices over $R$, and in this case the known structure of $M_n(R)$ can be used to obtain structural results for $M_R(R^n)$. An investigation of these relationships was initiated in [5]. (As in [5] we restrict our attention to the case $n = 2$, which shows all the salient features, for ease of exposition.)

When $R$ is an integral domain, it was shown in [5] that $M_R(R^2)$ is a simple near-ring. Moreover, when $R$ is a principal ideal domain, there is a lattice isomorphism between the ideals of $R$ and the lattice of two-sided
invariant subgroups of $M_R(R^2)$. In this work we turn to the case in which $R$ is a commutative principal ideal ring and investigate the lattice of ideals of $M_R(R^2)$. Here the situation is quite different from that of the principal ideal domain.

Let $R$ be a commutative principal ideal ring with identity. It is well-known ([1], [8]) that $R$ is the direct sum of principal ideal domains (PID) and special principal ideal rings (PIR). A special PIR is a principal ideal ring which has a unique prime ideal and this ideal is nilpotent. Thus a special PIR is a local ring with nilpotent radical $J = \langle \theta \rangle$ (the principal ideal generated by $\theta$). If $m$ is the index of nilpotency of $\langle \theta \rangle$, then every non-zero element in a special PIR, $R$, can be written in the form $a\theta^l$ where $a$ is a unit in $R$, $0 \leq l < m$, $l$ is unique and $a$ is unique modulo $\theta^{m-l}$. Furthermore every ideal of $R$ is of the form $\langle \theta^j \rangle$, $0 \leq j \leq m$. We mention that special PIR's are chain rings. (See [3] and the references there for information and examples of finite chain rings.)

Our work also has geometric connections. Specifically, let $R$ be a principal ideal ring and let $\mathcal{C}$ be a cover (see [2]) of $R^2$ by cyclic submodules. Then for each $f \in M_R(R^2)$ and each $\mathcal{C}_\alpha \in \mathcal{C}$, there exists $\mathcal{C}_\beta \in \mathcal{C}$ such that $f(\mathcal{C}_\alpha) \subseteq \mathcal{C}_\beta$. Hence $M_R(R^2)$ is a set of operators for the geometry $\langle R^2, \mathcal{C} \rangle$ and we obtain a generalized translation space with operators as investigated in [4].

Throughout the remainder of this paper all rings $R$ will be commutative principal ideal rings, unless specified to the contrary, with identity and all $R$-modules will be unitary. We let $N = M_R(R^2)$ denote the centralizer near-ring and all near-rings will be right near-rings. For details about near-rings we refer the reader to the books by Meldrum [6] or Pilz [7]. Also, for any set $S$, let $S^* = S \setminus \{0\}$.

The objective of this investigation is to determine the ideals of $N = M_R(R^2)$. After developing some general results in the next section we establish the characterization of the ideals of $N$ in Section 3. As mentioned above, the situation here differs from the PID situation. In fact, we find for a special PIR, $R$, a very nice bijection between the ideals of $R$ and the ideals of $M_R(R^2)$. In the final section we use our results to determine the radicals $J_\nu(N)$, $\nu = 0, 1, 2$, and we find the quotient structure $N/J_\nu(N)$.

2. General results

We start out with an arbitrary (not necessarily commutative principal ideal) ring $S$ with identity and suppose $S = S_1 \oplus \cdots \oplus S_t$ is the direct...
sum of the ideals $S_1, S_2, \ldots, S_t$. Then $1 = e_1 + e_2 + \cdots + e_t$ where \{ $e_i$ \} is a set of orthogonal idempotents, $e_i$ the identity of $S_i$. Note further that $S^2 = S_1^2 \oplus \cdots \oplus S_t^2$, and let $\left( \alpha_{i'} \right) \in S^2$, $\left( \beta_{i'} \right) = \left( \alpha_{i'}^1 \right) + \cdots + \left( \alpha_{i'}^t \right)$, $\left( \gamma_{i'} \right) \in S_1^2$. For $f \in MS(S^2)$, $f(\gamma_{i'}) = f\left( \left( \alpha_{i'}^1 \right) + \cdots + \left( \alpha_{i'}^t \right) \right) = \left( a_{i_1}^1 \right) + \cdots + \left( a_{i_t}^t \right)$, $\left( a_{i_t}^t \right) \in S_t^2$. But $f(\gamma_{i'})$ implies $f(\gamma_{i'}) = \left( a_{i_t}^t \right)$, so we obtain $f(\gamma_{i'}) = f(\gamma_{i'}) + \cdots + f(\gamma_{i'})$. And $f(S^2) \subseteq S^2$.

If $M_1 = MS(S_1^2)$, then $\varphi: M \rightarrow M_1 \oplus \cdots \oplus M_t$ defined by $\varphi(f) = (f_1, \ldots, f_t)$, where $f_i = f\mid S_i^2$, is a near-ring homomorphism. Moreover, $\varphi$ is onto. For, if $(g_1, \ldots, g_t) \in M_1 \oplus \cdots \oplus M_t$, define $g: S^2 \rightarrow S^2$ by $g(\gamma_{i'}) = g_1(\gamma_{i'}^1) + \cdots + g_t(\gamma_{i'}^t)$, where $\gamma_{i'} = \gamma_{i'}^1 + \cdots + \gamma_{i'}^t$. Then $g \in M$ and $\varphi(g) = (g_1, \ldots, g_t)$. Next, suppose $f \in M$ and $\varphi(f) = 0$. This means that $f(S_i^2) = 0$, $i = 1, 2, \ldots, t$, so $f \equiv 0$, and hence $\varphi$ is an isomorphism.

Since $S_i \subseteq S$, we have $MS(S_i^2) \subseteq MS(S_i^2)$. On the other hand, for $s \in S$, $S = s_1 + \cdots + s_t$, $s_i \in S_i$, and for $(\alpha_{i'}) \in S^2$, $(a_{i'})s = (a_{i'})(s_1 + \cdots + s_t) = (a_{i'})s_i$. Thus if $f \in MS(S_i^2)$, then $f(\alpha_{i'})s = f(\alpha_{i'})s_i = f(a_{i'})s_i = f(a_{i'})s$, i.e., $f \in MS(S_i^2)$. We have established the following result.

**Theorem 2.1.** Let $S = S_1 \oplus \cdots \oplus S_t$ be a direct sum of ideals $S_1, \ldots, S_t$. Then $MS(S^2) \cong MS(S_1^2) \oplus \cdots \oplus MS(S_t^2)$.

Let $K = K_1 \oplus \cdots \oplus K_t$ be a direct sum of near-rings with identities $e_i$, and let $B$ denote an ideal of $K$. Note that $B \cap K_i$ is an ideal of $K_i$, and for $b \in B$, $b = (b_1, \ldots, b_t)$, we have $be_i = b_1e_i = b_i$, which implies $b_i \in B \cap K_i$. Thus $B = (B \cap K_1) \oplus \cdots \oplus (B \cap K_t)$, and so, from the previous theorem, to determine the ideals of $MS(S^2)$ it suffices to determine the ideals of the individual components.

If $R$ is a commutative PIR, then, as stated above, $R$ is the direct sum of principal ideal domains (PID) and special PIR's, say $R = R_1 \oplus \cdots \oplus R_t$. From Theorem 2.1, $N = M_R(R^2) \cong M_{R_1}(R_1^2) \oplus \cdots \oplus M_{R_t}(R_t^2)$, so we are going to determine the ideals of $M_{R_i}(R_i^2)$. We know, however, if $R_i$ is a PID then $M_{R_i}(R_i^2)$ is simple, so the only ideals are $M_{R_i}(R_i^2)$ and \{0\}. (See [5, Theorem II.12].) It remains to determine the ideals of $M_{R_i}(R_i^2)$ when $R_i$ is a special PIR.

To this end, let $R$ be a special PIR with unique maximal ideal $J = \langle \theta \rangle$, and let $m$ be the index of nilpotency of $J$, i.e., $\theta^m = 0$ and $\theta^{m-1} \neq 0$. [3]
We know that the ideals of $R$ are of the form $\langle \theta^k \rangle$, $k = 0, 1, 2, \ldots, m$. We denote $\langle \theta^k \rangle$ by $A_k$ and remark that $A_k = \{ (a_1, a_2) : a_1, a_2 \in A_k \}$ is an $R$-submodule of $R^2$ with the property $f(A_k) \subseteq A_k$ for each $f \in N = M_R(R^2)$, because $f(\langle \theta^k \rangle) = f(\theta^k)$ for all $r, s \in R$. But then $\{ (0) \} : A_k$ is an ideal of $N$. For $r, s \in R$ and $f \in \{ (0) \} : A_k$, we have $(0) = f(\theta^k) = f(\theta^k)$, so $f(\theta^k) = (\theta^m - k)^2 = A_{m-k}$. Therefore $\{ (0) \} : A_k \subseteq (A_{m-k} : R^2)$. Since the reverse inclusion is straightforward, we have the next result.

**Proposition 2.2.** If $R$ is a special PIR with $J = \langle \theta \rangle$ and index of nilpotency $m$, and if $A_k = \langle \theta^k \rangle$, then $\{ (0) \} : A_k = (A_{m-k} : R^2)$, $k = 0, 1, 2, \ldots, m$.

We know that if $I$ is an ideal of $N$, then there exists a unique ideal $A_k$ of $R$ with $I \cap M_2(R) = M_2(A_k)$. In particular from [5], if $f \in I$, say $f(x) = (a)$, then $f \circ \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$. This in turn implies $f(R^2) \subseteq A_k$, so we have $I \subseteq (A_{k'} : R^2)$.

**Proposition 2.3.** If $R$ is a special PIR with $J = \langle \theta \rangle$ and index of nilpotency $m$, then for each non-trivial ideal $I$ of $N = M_R(R^2)$ there is a unique integer $k$, $0 < k < m$, such that $I \subseteq (A_k^2 : R^2)$ for $l \leq k$, and $I \not\subseteq (A_l^2 : R^2)$ for $l > k$.

In the next section we develop the machinery to show that $I = (A^2_k : R^2)$.

**Theorem 2.4.** Let $R$ be a commutative principal ideal ring with $R = R_1 \oplus \cdots \oplus R_t$, where $R_i$ is a PID or a special PIR. Then $N = M_R(R^2) = M_{R_1}(R^2_1) \oplus \cdots \oplus M_{R_t}(R^2_t)$, and if $I$ is an ideal of $N$, then $I = I_1 \oplus \cdots \oplus I_t$, where $I_i$ is an ideal of $M_{R_i}(R^2_i)$. If $R_i$ is a PID, then $I_i = \{0\}$ or $I_i = M_{R_i}(R^2_i)$. If $R_i$ is a special PIR with $J = \langle \theta \rangle$ and index of nilpotency $m$, then $I_i = (A_k^2 : R^2_i) = \{ (0) \} : A_{m-k}$ for some $k$, $0 \leq k \leq m$, where $A_k = \langle \theta^k \rangle$.

**3. Ideals in $M_R(R^2)$, $R$ a special PIR**

Unless otherwise stated, in this section $R$ will denote a special PIR with unique maximal ideal $J = \langle \theta \rangle$ and index of nilpotency $m$. Let $I$ be an
ideal of \( N = M_R(R^2) \) with \( I \subseteq (A_k^2; R^2) \) as given in Proposition 2.3. From the fact that \( A_k^2(A_k) \subseteq I \) our plan is to show that an arbitrary function in \( (A_k^2; R^2) \) can be constructed from functions in \( I \). This will then give the desired equality. To aid in the construction of functions in \( N \) we recall from [5] that \( x, y \in (R^2)^* \) are connected if there exist \( x = a_0, a_1, \ldots, a_s = y \) in \( (R^2)^* \) such that \( a_i R \cap a_{i+1} R \not= \{(0)\}, \ i = 0, 1, 2, \ldots, s-1 \). This defines an equivalence relation on \( (R^2)^* \) and the equivalence classes are called connected components. We first determine the connected components of \( (R^2)^* \).

Let \( F \) be a set of representatives for the classes \( R/J \), where we choose 0 for the class \( J \). Thus for \( \alpha \in F^* \), \( \alpha \) is a unit in \( R \). We know for each \( r \in R \) there is a unique \( \alpha_0 \in F \) such that \( r = \alpha_0 + r_0 \theta \), \( r_0 \in R \). But \( r_0 = \alpha_1 + r_1 \theta \), with \( \alpha_1 \in F \), \( r_1 \in R \), implies \( r = \alpha_0 + \alpha_1 \theta + r_1 \theta^2 \). Continuing, we find that every element \( r \in R \) has a unique “base \( \theta \)” representation, \( r = \alpha_0 + \alpha_1 \theta + \cdots + \alpha_{m-1} \theta^{m-1} \), \( \alpha_i \in F \), \( i = 0, 1, 2, \ldots, m-1 \).

In the sequel, for ease of exposition we let \( \# \) denote a symbol not in \( F \), and we let \( \hat{F} = F \cup \{\#\} \).

**Lemma 3.1.** Let \( M_\# = \langle \theta^{m-1} \rangle \) and let \( M_\alpha = \langle \alpha \theta^{m-1} \rangle \), \( \alpha \in F \). The submodules \( M_\beta \), \( \beta \in \hat{F} \), are the minimal submodules of \( R^2 \).

**Proof.** Let \( H \) be an \( R \)-submodule of \( R^2 \), \( \{(0)\} \not\subset H \subseteq M_\beta \), \( \beta \in F \), and let \( \langle (0) \rangle \neq x \in H \). Then \( x = (\beta \theta^{m-1})s \) for some \( s \in R \), and since \( x \neq 0 \), we have \( s \not\in J \), so \( s \) is a unit in \( R \). But then \( xs^{-1} \in H \), hence \( M_\beta \subseteq H \). In the same manner if \( \beta = \# \), then \( H = M_\# \).

To show that the \( M_\beta \), \( \beta \in \hat{F} \), are the only minimal submodules, we show that every non-zero submodule \( L \) of \( R^2 \) must contain some \( M_\beta \), \( \beta \in \hat{F} \).

Let \( y = (u_1, u_2) \) be a non-zero element in \( L \), where \( u_1, u_2 \) are units in \( R \). Suppose \( l_1 \geq l_2 \). Then \( l_1 \geq 0 \). Then \( yu_2^{-1} \theta^{m-l_2-1} = (u_1, u_2^{-1}) \theta^{l_1-l_2+m-1} \).

**Lemma 3.2.** For \( x, y \in (R^2)^* \), the following are equivalent:
(i) \( x \) and \( y \) are connected;
(ii) \( xR \) and \( yR \) contain the same minimal submodule \( M \);
(iii) there exist positive integers $l_1$, $l_2$ such that $x\theta^{l_1} \in M^*$ and $y\theta^{l_2} \in M^*$ for some minimal submodule $M$.

**Proof.** (i) $\Rightarrow$ (ii). Suppose $x$ and $y$ are connected. As we showed in the previous proof, $xR$ and $yR$ contain minimal submodules, say $xR \supseteq M' = cR$ and $yR \supseteq M'' = dR$. Thus there exist $r, s \in R^*$ such that $c = xr$ and $d = ys$. Since $x$ and $y$ are connected, so are $c$ and $d$, say $cr_1 = b_1s_1 \neq 0$, $b_1r_2 = b_2s_2 \neq 0$, ..., $b_{l-1}r_l = ds_l \neq 0$. Since $cr_1 \in (M')^*$, it follows that $cr_1R = cR$, so there exists $r' \in R$ such that $cr_1r' = c$, hence $c = cr_1r' = b_1s_1r'$. Now $c$ has the form $\binom{a}{b} \theta^{m-1}$, so if $b_1 = \binom{u_1\theta^{l_1}}{u_{l_1}\theta^{l_1}}$ and $s_1r' = v_1\theta^{l_1}$, then $b_1 \theta^{l_3} = cv_1^{-1} \in (cR)^*$. If $r_2 = v_2\theta^{l_3}$, then $0 \neq b_1r_2 = b_1v_2\theta^{l_3} + (l_4 - l_3)$, and since $b_1 \theta^{l_3} \in cR$, a minimal submodule, it follows from Lemma 3.1 that $l_4 \leq l_3$, otherwise $b_1r_2 = 0$. Therefore $r_2 \theta^{l_3-l_4} = v_2\theta^{l_3}$, which in turn implies $b_1r_2 \theta^{l_3-l_4} = b_1v_2\theta^{l_3} \in (cR)^*$. Hence $b_2s_2 \theta^{l_3-l_4} \in (cR)^*$, so there exists $r'' \in R$ such that $b_2r'' = c$. Continuing in this manner we get $r$ such that $d\hat{r} = c$ for some $\hat{r} \in R$. But this means $M' = M''$.

(ii) $\Rightarrow$ (iii). If $xR \supseteq M$ and $yR \supseteq M$, then there exist $r, s \in R$ such that $xr, ys \in M^*$, say $r = u\theta^{l_1}$, $s = v\theta^{l_2}$, $u, v$ units. But then $x\theta^{l_1}$ and $y\theta^{l_2}$ are non-zero in $M$.

(iii) $\Rightarrow$ (i). From $x\theta^{l_1} \in M^*$ we have $\{(0)\} \neq M \cap xR = M$. Hence $M \subseteq xR$, and similarly, $M \subseteq yR$. Therefore, for some $r, s \in R^*$, $xr = ys \neq 0$, i.e., $x$ and $y$ are connected.

From this lemma we have that every minimal submodule $M$ determines a connected component $C$, where $C = (\bigcup \{xR \mid xR \supseteq M\}) \backslash \{(0)\}$.

Consider the minimal submodule $M_\alpha$, for some $\alpha \in F$. We consider the submodules $H(\alpha, \alpha_1, \ldots, \alpha_{m-1}) = \langle \alpha + \alpha_1 \theta + \cdots + \alpha_{m-1} \theta^{m-1} \rangle$, where $\alpha_1, \ldots, \alpha_{m-1}$ range over $F$. We note that $H(\alpha, \alpha_1, \ldots, \alpha_{m-1}) \cap H(\beta, \beta_1, \ldots, \beta_{m-1}) = \{(0)\}$ if and only if $\alpha \neq \beta$. For if $\alpha = \beta$, then $(\alpha + \alpha_1 \theta + \cdots + \alpha_{m-1} \theta^{m-1}) \theta^{m-1} = (\alpha \theta^{m-1}) = (\beta + \beta_1 \theta + \cdots + \beta_{m-1} \theta^{m-1}) \theta^{m-1}$, so

$$H(\alpha, \alpha_1, \ldots, \alpha_{m-1}) \cap H(\beta, \beta_1, \ldots, \beta_{m-1}) \supseteq M_{\alpha}.$$

Conversely, suppose $(\alpha + \alpha_1 \theta + \cdots + \alpha_{m-1} \theta^{m-1}) r = (\beta + \beta_1 \theta + \cdots + \beta_{m-1} \theta^{m-1}) s$ for some non-zero $r, s \in R$. Then if $r = a\theta^{l_1}, s = b\theta^{l_2}$, we get $l_1 = l_2$ and $(\alpha \theta^{m-1}) = (\beta \theta^{m-1})$. Hence $\alpha = \beta$, since $\alpha, \beta \in F$. In the same way
we see that \( H(#, \alpha_1, \ldots, \alpha_{m-1}) = \langle \alpha_1, \theta + \cdots + \alpha_{m-1} \theta^{m-1} \rangle \) contains \( M_\# \) and that \( H(#, \alpha_1, \ldots, \alpha_{m-1}) \cap H(\beta, \beta_1, \ldots, \beta_{m-1}) = \{ (0) \} \) for all \( \beta \in F \).

Let \( a \) be an arbitrary non-zero element of \( R^2 \), say \( a = \left( \begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix} \right) \). If \( l_1 \geq l_2 \), then \( a = \left( \begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix} \right) \theta^{l_2} = \left( a_1 a_2^{-1} \theta^{l_1 - l_2} \right) a_2 \theta^{l_2} \) implies \( a \) is in some \( H(\alpha, \alpha_1, \ldots, \alpha_{m-1}) \), \( \alpha \in F \). If \( l_1 < l_2 \), then

\[
a = \left( \begin{smallmatrix} a_1 \\ a_2 \theta^{l_2-l_1} \end{smallmatrix} \right) \theta^{l_1} = \left( a_2 a_1^{-1} \theta^{l_2-l_1} \right) a_1 \theta^{l_1}
\]

implies \( a \) is in some \( H(\#, \alpha_1, \ldots, \alpha_{m-1}) \). Thus we see that the collection of submodules \( \{ H(\beta, \alpha_1, \ldots, \alpha_{m-1}) \mid \beta \in F, \alpha_1, \ldots, \alpha_{m-1} \in F \} \) is a cover for \( R^2 \) (see [2]) and we call the submodules \( H(\beta, \alpha_1, \ldots, \alpha_{m-1}) \) covering submodules.

Therefore, to define a function \( f \) in \( N \) it suffices to define \( f \) on the generators of the covering submodules, use the homogeneous property \( f(xr) = f(x)r \to extend f \) to all of \( R^2 \) and then verify that \( f \) is well-defined. That is, if \( x \) and \( y \) are generators of covering submodules and \( 0 \neq xr = ys \) for \( r, s \in R \), then one must show that \( f(x)r = f(y)s \). Suppose \( r = a_1 \theta^{l_1}, s = a_2 \theta^{l_2} \) and \( x = (x_1), y = (y_1). \) (A similar argument works for \( x = (x_1), y = (y_1). \)) Thus we have \( x_1 a_1 \theta^{l_1} = y_1 a_2 \theta^{l_2} \) and \( a_1 \theta^{l_1} = a_2 \theta^{l_2} \).

Thus \( l_1 = l_2 \), and so \( a_2 = a_1 + r \theta^{m-1} \) for some \( r \in R \). Thus \( xr = ys \) implies \( x \theta^{l_1} = y \theta^{l_1} \). Consequently, to show that \( f \) is well-defined, it suffices to show that \( x \theta^{l_1} = y \theta^{l_1} \) implies \( f(x) \theta^{l_1} = f(y) \theta^{l_1} \), where \( x \) and \( y \) are generators of covering submodules.

For convenience in manipulating functions in \( N \) we give the next result.

**Lemma 3.3.** If \( f \in N \), then for any \( j \), \( 1 \leq j \leq m-1 \), \( f(\alpha^{\theta^{j+1}}) = f(\alpha^{\theta^{j}}) + \sigma_j^{1} \theta^{j+1} + \cdots + \sigma_{m-1}^{1} \theta^{m-1} \)\) and \( f(\alpha^{\theta^{j+1}}) = f(\alpha^{\theta^{j}}) + \sigma_j^{1} \theta^{j+1} + \cdots + \sigma_{m-1}^{1} \theta^{m-1} \), where \( \sigma_j^{1}, \ldots, \sigma_{m-1}^{1} \in R^2 \).

**Proof.** We note that \( f(\alpha^{\theta^{j+1}}) = f(\alpha^{\theta^{j}}) + \sigma_j^{1} \theta^{j+1} + \cdots + \sigma_{m-1}^{1} \theta^{m-1} \theta \) implies \( f(\alpha^{\theta^{j+1}}) = f(\alpha^{\theta^{j}}) + \sigma_j^{1} \theta^{j+1} \) for some \( \sigma_{m-1}^{1} \in R^2 \). The result now follows by induction. The second equality follows similarly.
Some additional notation will now be introduced. Let $x$ be a generator of a covering submodule. We denote by $m_{\theta^k}f(x)$ the multiplier of $\theta^k$ in $f(x)$. If $x = (a_1 \theta + \cdots + a_{m-1} \theta^{m-1})$ and $j+1 \geq k$, then from the above lemma, $f(x) = f(\alpha_1 \theta + \cdots + \alpha_{m-1} \theta^{m-1}) + \sigma_{j+1} \theta^{j+1} + \cdots + \sigma_{m-1} \theta^{m-1}$ and so $m_{\theta^k}f(x) = m_{\theta^k}(f(\alpha_1 \theta + \cdots + \alpha_{m-1} \theta^{m-1}) + \sigma_{j+1} \theta^{j+1} + \cdots + \sigma_{m-1} \theta^{m-1}).$

As at the beginning of this section, let $I \subseteq (A_k^2: R^2)$. We consider two cases, $F$ finite and $F$ infinite.

First, suppose $F$ is finite, and let $f \in (A_k^2: R^2)$. Since $F$ is finite, there are only a finite number of connected components, namely $\mathcal{C}_\beta$ where $\beta \in \hat{F}$, determined by $M_\beta$. We show how to find a function in $I$ which agrees with $f$ on a single component and is zero off this component. Then by adding we get $f \in I$. We work first with the component $\mathcal{C}_\beta$. We know the generators of the covering submodules for this component have the form $(a_1 \theta + a_2 \theta^2 + \cdots + a_{m-1} \theta^{m-1})$, $a_1, a_2, \ldots, a_{m-1} \in F$.

For the fixed $k$ above (determined by $I \subseteq (A_k^2: R^2)$) we partition these generators of the covering submodules of $\mathcal{C}_\beta$ into sets determined by the $(k-1)$-tuples $(a_1, a_2, \ldots, a_{k-1})$, $a_1, a_2, \ldots, a_{k-1} \in F$, where we take $k \geq 2$. (The case $k = 1$ will be handled separately.) That is, given $(a_1, \ldots, a_{k-1})$, in one set we have all generators $(\beta_1 \theta + \cdots + \beta_{m-1} \theta^{m-1})$ where $(\beta_1, \ldots, \beta_{k-1}) = (a_1, \ldots, a_{k-1})$. Define $p_{k-1}: R^2 \to R^2$ by $p_{k-1}(\beta_1 \theta + \cdots + \beta_{m-1} \theta^{m-1}) = (\beta_1 \theta + \cdots + \beta_{m-1} \theta^{m-1})$ if $(\beta_1, \ldots, \beta_{k-1}) \neq (a_1, \ldots, a_{k-1})$, extend using the homogeneous property, and define $p_{k-1}(x) = (0, 0)$ if $x \notin \mathcal{C}_\beta$. We show that $p_{k-1}$ is well-defined. Let $\alpha = \alpha_1 \theta + \cdots + \alpha_{m-1} \theta^{m-1}$, $\beta = \beta_1 \theta + \cdots + \beta_{m-1} \theta^{m-1}$ and suppose $(\alpha / \beta) \theta^l = (\beta / \beta) \theta^l$. This means $(a_1, \ldots, a_{m-l-1}) = (\beta_1, \ldots, \beta_{m-l-1})$. If $l \leq m-k-1$, then $m-l-1 \geq k$ and so $(\alpha / \beta)$ and $(\beta / \beta)$ are in the same set of the partition, thus $p_{k-1}(\alpha / \beta) \theta^l = (\alpha_1 \theta^{k+l} \cdots + \alpha_{m-1} \theta^{m-1+l}) = p_{k-1}(\beta / \beta) \theta^l$. If $l > m-k-1$, then $l \geq m-k$ and so $p_{k-1}(\alpha / \beta) \theta^l = (0, 0) = p_{k-1}(\beta / \beta) \theta^l$. Thus $p_{k-1} \in M_R(R^2)$. Also, since $[0 \ 0] \in I$, $f = [0 \ \theta^k \ 0] p_{k-1} \in I$.

Define $h: R^2 \to R^2$ by $h(\alpha_1 \theta + \cdots + \alpha_{m-1} \theta^{m-1}) = (\alpha_1 \theta^{k-1} + \cdots + \alpha_{m-1} \theta^{m-1})$, extend, and define $h(x) = (0, 0)$ if $x \notin \mathcal{C}_\beta$. As above one shows that $h$ is well-defined, i.e., $h \in M_R(R^2)$. Thus for each $g \in M_R(R^2)$, $\hat{q} = g(\hat{f} + h) - g(f)$. The lattice of ideals 375
For \( x \notin \mathbb{C}_n \), we have \( \hat{q}(x) = (0)_1 \), because \( p_{k-1}(x) = (0)_1 \) if \( x \notin \mathbb{C}_n \). Further, \( \hat{q}(\frac{1}{\beta}) = g(\hat{f}(\frac{1}{\beta}) + h(\frac{1}{\beta})) - gh(\frac{1}{\beta}) \). If \( (\beta_1, \ldots, \beta_{k-1}) \neq (\alpha_1, \ldots, \alpha_{k-1}) \), then \( \hat{f}(\frac{1}{\beta}) = (0)_1 \) and in this case \( \hat{q}(\frac{1}{\beta}) = (0)_1 \). Thus we focus on \( (\frac{1}{\beta}) \) where \( (\beta_1, \ldots, \beta_{k-1}) = (\alpha_1, \ldots, \alpha_{k-1}) \). Here, \( \hat{q}(\frac{1}{\beta}) = g((\theta^k + (\beta_1 \theta^k + \ldots + \theta m-1 \theta^m - 1)) - g(0)_1)(\theta^k + \ldots + \theta m-1 \theta^m - 1) \). We wish to define \( g \) so that \( \hat{q} \) agrees with \( f \) on all generators \((\beta_1, \ldots, \beta_{k-1}) = (\alpha_1, \ldots, \alpha_{k-1}) \). First define \( g(\mathbb{C}_n) = \{(0)_1\} \). Then define

\[
g \left( \beta_0 + \beta_1 \theta + \ldots + \beta_{m-1} \theta^{m-1} \right) = \cdots = g \left( \beta_0 + \beta_1 \theta + \ldots + \beta_{m-k-1} \theta^{m-k-1} \right) = m \theta^k f \left( \frac{1}{\alpha_1 \theta + \ldots + \alpha_{k-1} \theta^{k-1} + \beta_0 \theta^k + \ldots + \beta_{m-k-1} \theta^{m-1}} \right).
\]

We show that \( g \) is well-defined. Let \( \beta = \beta_0 + \beta_1 \theta + \ldots + \beta_{m-k-1} \theta^{m-k-1} \) and \( \gamma = \gamma_0 + \gamma_1 \theta + \ldots + \gamma_{m-k-1} \theta^{m-k-1} \), and suppose \((\beta^l) \theta^l = (\gamma^l) \theta^l \). Then

\[
(\beta_0, \beta_1, \ldots, \beta_{m-l-1}) = (\gamma_0, \gamma_1, \ldots, \gamma_{m-l-1}).
\]

If \( l \leq k \), then \( m - l - 1 \geq m - k - 1 \) and

\[
g \left( \frac{1}{\beta} \right) = m \theta^k f \left( \frac{1}{\alpha_1 \theta + \ldots + \alpha_{k-1} \theta^{k-1} + \beta_0 \theta^k + \ldots + \beta_{m-k-1} \theta^{m-1}} \right) = g \left( \frac{1}{\gamma} \right).
\]

If \( l \geq k + 1 \), then

\[
g \left( \frac{1}{\beta} \right) = m \theta^k \left[ f \left( \frac{1}{\alpha_1 \theta + \ldots + \alpha_{k-1} \theta^{k-1} + \beta_0 \theta^k + \ldots + \beta_{m-k-1} \theta^{m-k-1}} \right) \right] + \rho_1 \theta^m + \ldots + \rho_{k+1} \theta^{m-k-1},
\]

where \( \rho_{k+1}, \ldots, \rho_l \in \mathbb{R}^2 \). A similar expression holds for \( g \left( \frac{1}{\gamma} \right) \). But then \( g \left( \frac{1}{\beta} \right) \theta^l = g \left( \frac{1}{\gamma} \right) \theta^l \) as desired.
Thus,
\[
\hat{q} \left( \frac{1}{\beta} \right) = g \left( \frac{\beta_k \theta^k + \cdots + \beta_{m-1} \theta^{m-1}}{1} \right) - g \left( \frac{1}{0} \right) (\beta_k \theta^k + \cdots + \beta_{m-1} \theta^{m-1})
\]
\[
= \frac{g \left( \beta_k + \cdots + \beta_{m-1} \theta^{m-1-k} \right)}{1} \theta^k
\]
\[
= m_{\theta^k} f (\alpha_1 \theta + \cdots + \alpha_{k-1} \theta^{k-1} + \beta_k \theta^k + \cdots + \beta_{m-1} \theta^{m-1}) \theta^k
\]
\[
= f \left( \frac{1}{\beta} \right).
\]
Therefore \( \hat{q} \) agrees with \( f \) on those generators \( (\beta_1, \ldots, \beta_{k-1}) = (\alpha_1, \ldots, \alpha_{k-1}) \), and is zero on all other generators of covering submodules. Since there are \( |F|^{k-1} \) such functions, by adding we obtain a function \( q_\# \) which agrees with \( f \) on \( \mathbb{E}_\# \) and is 0 off \( \mathbb{E}_\# \).

For \( k = 1 \) the situation is somewhat easier. There is no need to partition the generators of the covering modules of \( \mathbb{E}_\# \). For this case we use \( [0,0] e_\# \) and the \( h \) defined above, where \( e_\mu \) is the idempotent determined by \( \mathbb{E}_\mu \), i.e., \( e_\mu (x) = x \) if \( x \in \mathbb{E}_\mu \) and \( e_\mu (x) = (0,0) \) if \( x \notin \mathbb{E}_\mu \), \( \mu \in \hat{F} \). Thus for each \( g \in M(R^2) \), \( \hat{q} = g ([0,0] e_\# + h) - gh \in I \). For \( x \notin \mathbb{E}_\# \), \( \hat{q}(x) = (0,0) \).

Further, \( \hat{q} \left( \frac{1}{\beta} \right) = g \left( \frac{\theta}{0} \right) + g \left( \frac{0}{\theta} \right) ) - g \left( \frac{1}{0} \right) - g \left( \frac{1}{\theta} \right) \theta = g (\beta_1 \theta + \beta_2 \theta^2 + \cdots + \beta_{m-1} \theta^{m-1}) - g \left( \frac{1}{0} \right) \theta.

Define \( g (\mathbb{E}_\#) = \{(0,0)\} \) and
\[
g \left( \alpha_0 + \alpha_1 \theta + \cdots + \alpha_{m-1} \theta^{m-1} \right) = g \left( \alpha_0 + \alpha_1 \theta + \cdots + \alpha_{m-2} \theta^{m-2} \right)
\]
\[
= m_{\theta^k} f \left( \alpha_0 \theta + \alpha_1 \theta^2 + \cdots + \alpha_{m-2} \theta^{m-1} \right)
\]
As above one verifies that \( g \in M_R(R^2) \) and that \( \hat{q} \) agrees with \( f \) on \( \mathbb{E}_\# \).

In a similar manner one constructs \( q_\alpha, \alpha \in F \), which agrees with \( f \) on \( \mathbb{E}_\alpha \) and is 0 off \( \mathbb{E}_\alpha \). Then \( f = \sum_{\beta \in \hat{F}} q_\beta \in I \), and so the proof of Theorem 2.4 is complete when \( F \) is finite.

Alternatively, one could use the following approach in the finite case. For \( \alpha \in F \), define \( p_\alpha (\alpha_0 + \alpha_1 \theta + \cdots + \alpha_{m-1} \theta^{m-1}) = (\alpha_0, \alpha_1 \theta + \cdots + \alpha_{m-1} \theta^{m-1}) \) and \( p_\alpha (x) = (0,0) \) for \( x \notin \mathbb{E}_\alpha \). For each \( g' \in N \), \( q' = [g' \left( \left[ \frac{\theta^k}{0} \right] + h \right) - g' h] p_\alpha \in I \). For \( x \notin \mathbb{E}_\alpha \), \( q'(x) = (0,0) \), and \( q' (\alpha_0 + \alpha_1 \theta + \cdots + \alpha_{m-1} \theta^{m-1}) = g' (\alpha_0 \theta + \alpha_1 \theta^2 + \cdots + \alpha_{m-2} \theta^{m-1}) \).
Define \( g'(\mathcal{C}_\alpha) = \{(0)\} \) and
\[
g'(\beta_0 + \beta_1 \theta + \cdots + \beta_{m-1} \theta^{m-1}) = m_{\theta_1} f',\]
where we have partitioned the generators \((\alpha + \alpha_1 \theta + \cdots + \alpha_{m-1} \theta^{m-1})\) of the covering submodules in \( \mathcal{C}_\alpha \) by using the \( k \)-tuples \((\alpha, \alpha_1, \ldots, \alpha_{k-1})\). One shows that \( g' \) is well-defined and continuing obtains a function which agrees with \( f \) on \( \mathcal{C}_\alpha \) and is zero off \( \mathcal{C}_\alpha \).

Suppose now \( F \) is infinite, and let \( \delta: F^k \to F \) be a bijection. We again start with \( \mathcal{C}_\alpha \), where as above we let \( \alpha = \alpha_1 \theta + \cdots + \alpha_{m-1} \theta^{m-1} \). Define \( h': R^2 \to R^2 \) by
\[
h'(\begin{pmatrix} 1 \\ \alpha \end{pmatrix}) = (\begin{pmatrix} 1 \\ \alpha \end{pmatrix}) + (\delta(\alpha_1, \ldots, \alpha_k) \theta + \alpha_{k+1} \theta^{k+1} + \cdots + \alpha_{m-1} \theta^{m-1})\]
and \( h'(x) = (0) \), \( x \notin \mathcal{C}_\alpha \). As above one shows that \( h' \in M_R(R^2) \). Thus for each \( g \in N \), \( t_\# = g(\epsilon_\#, h') - gh' \in I \). For \( x \notin \mathcal{C}_\alpha \), \( t_\#(x) = (0) \). For \( x = (\begin{pmatrix} 1 \\ \alpha \end{pmatrix}) \), \( t_\#(x) = g((0) + (\delta(\alpha_1, \ldots, \alpha_k) \theta + \alpha_{k+1} \theta^{k+1} + \cdots + \alpha_{m-1} \theta^{m-1})) - gh'(x) \). Define \( g'(\mathcal{C}_\alpha) = \{(0)\} \) and
\[
g'(\beta_0 + \beta_1 \theta + \cdots + \beta_{m-1} \theta^{m-1}) = \cdots = g'(\beta_0 + \cdots + \beta_{m-1-k} \theta^{m-1-k}) = m_{\theta_1} f',\]
where \( \delta(\mu_1, \ldots, \mu_k) = \beta_0 \).

If \( \gamma = \gamma_0 + \gamma_1 \theta + \cdots + \gamma_{m-1-k} \theta^{m-1-k} + \cdots + \gamma_{m-1} \theta^{m-1} \) and \( (\gamma)\theta^l = (\begin{pmatrix} 1 \\ \gamma \end{pmatrix}) \theta^l \), then \( (\gamma_0, \gamma_1, \ldots, \gamma_{m-1-k}) = (\beta_0, \beta_1, \ldots, \beta_{m-1-k}) \). If \( l \leq k \), then \( m - l - 1 \geq m - k - 1 \) and so \( g(\gamma)\theta^l = g(\begin{pmatrix} 1 \\ \gamma \end{pmatrix})\theta^l \). If \( l \geq k + 1 \), then
\[
g(\gamma) = m_{\theta_1} f_{(\mu_1, \ldots, \mu_k) \theta + \beta_0 \theta^{k+1} + \cdots + \beta_{m-1-k} \theta^{m-1-k}} + \sigma_l \theta^{m-l} + \cdots + \sigma_{k+1} \theta^{m-k-1}
\]
where \( \sigma_{k+1}, \ldots, \sigma_l \in R^2 \) and \( \delta(\mu_1, \ldots, \mu_k) = \gamma_0 \). Since \( \gamma_0 = \beta_0 \), \( (\mu_1, \ldots, \mu_k) = (\mu_1, \ldots, \mu_k) \) and \( g(\begin{pmatrix} \gamma \\ 1 \end{pmatrix})\theta^l = g(\begin{pmatrix} \gamma \\ 1 \end{pmatrix})\theta^l \). Hence \( g \in N \).
Further,

\[ t_\ast \left( \frac{1}{\alpha} \right) = g \left( \frac{\delta_k(\alpha_1, \ldots, \alpha_k)\theta^k + \alpha_{k+1}\theta^{k+1} + \cdots + \alpha_{m-1}\theta^{m-1}}{\theta^k} \right) - gh' \left( \frac{1}{\alpha} \right) \]

\[ = g \left( \frac{\delta_k(\alpha_1, \ldots, \alpha_k) + \alpha_{k+1}\theta + \cdots + \alpha_{m-1}\theta^{m-1-k}}{1} \right) \theta^k - 0 \]

\[ = m_{\theta^k} f \left( \alpha_1 \theta + \cdots + \alpha_k \theta^k + \alpha_{k+1} \theta^{k+1} + \cdots + \alpha_{m-1} \theta^{m-1} \right) \theta^k \]

\[ = f \left( \frac{1}{\alpha} \right). \]

Thus \( t_\ast \) agrees with \( f \) on \( \mathcal{E}_\ast \) and is zero off \( \mathcal{E}_\ast \).

We next show that there is a function \( i_\ast \) in \( I \) which agrees with \( f \) off \( \mathcal{E}_\ast \) and is zero on \( \mathcal{E}_\ast \). This will imply that \( f = t_\ast + i_\ast \in I \). To this end let \( \delta_{k+1}: F_{k+1} \to F \) be a bijection, let \( \alpha = \alpha_0 + \alpha_1\theta + \cdots + \alpha_{m-1}\theta^{m-1} \) and define \( h'': R^2 \to R^2 \) by \( h''(\mathcal{E}_\ast) = \{ (0, 0) \} \) while \( h''(\mathcal{E}_\ast) = \left( \delta_{k+1}(\alpha_0, \ldots, \alpha_k)\theta^k + \alpha_{k+1}\theta^{k+1} + \cdots + \alpha_{m-1}\theta^{m-1} \right) \). One finds that \( h'' \in N \). Let \( \hat{E}_\ast = \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] \) (id. \( -e_\ast \)). Then \( \hat{E}_\ast(\mathcal{E}_\ast) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \) and \( \hat{E}_\ast(\mathcal{E}_\ast) = \{ (0, 0) \} \). Since \( \hat{E}_\ast \in I \), for each \( g \in N \), \( i_\ast = g(\hat{E}_\ast + h'') - gh'' \) is in \( I \). For \( x \in \mathcal{E}_\ast \), \( i_\ast(x) = (0, 0) \) and for

\[ x = \left( \begin{array}{c} \alpha \\ 1 \end{array} \right), \quad i_\ast \left( \begin{array}{c} \alpha \\ 1 \end{array} \right) = g \left( \left( \begin{array}{c} 0 \\ \theta^k \end{array} \right) + h'' \left( \begin{array}{c} \alpha \\ 1 \end{array} \right) \right) - gh'' \left( \begin{array}{c} \alpha \\ 1 \end{array} \right) \]

\[ = g \left( \delta_{k+1}(\alpha_0, \ldots, \alpha_k) + \alpha_{k+1}\theta + \cdots + \alpha_{m-1}\theta^{m-1-k} \right) \theta^k \]

\[ - g \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \left( \delta_{k+1}(\alpha_0, \ldots, \alpha_k)\theta^k + \alpha_{k+1}\theta^{k+1} + \cdots + \alpha_{m-1}\theta^{m-1} \right). \]

Again we define \( g(\mathcal{E}_\ast) = \{ (0, 0) \} \) and

\[ g \left( \begin{array}{c} \gamma \\ 1 \end{array} \right) = g \left( \gamma_0 + \gamma_1\theta + \cdots + \gamma_{m-1}\theta^{m-1} \right) \]

\[ = \cdots = g \left( \gamma_0 + \gamma_1\theta + \cdots + \gamma_{m-1-k}\theta^{m-1-k} \right) \]

\[ = m_{\theta^k} f \left( c_0 + c_1\theta + \cdots + c_k\theta^k + \gamma_1\theta^{k+1} + \cdots + \gamma_{m-1-k}\theta^{m-1} \right), \]
where $\delta_{k+1}(c_0, c_1, \ldots, c_k) = \gamma_0$. As above, $g \in N$ and $i_\#(a) = f(a)$. Thus $f = t_\# + i_\# I$, and the proof of Theorem 2.4 is complete.

4. Applications

In this final section we apply the above characterization of the ideals of $N$ to determine the radicals $J_\nu(N)$ of $N$ and the quotient structures $N/J_\nu(N)$, $\nu = 0, 1, 2$.

From Theorem 2.1 and [7, Theorem 5.20], $J_\nu(N) = J_\nu(M_{R_1}(R_1^2)) \oplus \cdots \oplus J_\nu(M_{R_i}(R_i^2))$. If $R_i$ is a PID, then $J_0(M_{R_i}(R_i^2)) = \{0\}$. If $R_i$ is a PID, not a field, then $J_1(M_{R_i}(R_i^2)) = J_2(M_{R_i}(R_i^2)) = M_{R_i}(R_i^2)$, and if $R_i$ is a field, then $J_1(M_{R_i}(R_i^2)) = J_2(M_{R_i}(R_i^2)) = \{0\}$. If $R_i$ is a special PIR, then from the previous section we know that $M_{R_i}(R_i^2)$ has a unique maximal ideal $(A_i^2:R_i^2) = (((0)_0):A_{m-1}^2)$. Moreover, $A_{m-1}^2$ is a type 2, $M_{R_i}(R_i^2)$-module, for if $(x\theta^m_{m-1}) \in A_{m-1}^2$ then $x$ and $y$ are units in $R$ (or zero), and so if $x \neq 0$ (say) then $[x^{-1}_x0 \ x\theta^m_{m-1}] = (\theta^m_{m-1})$ for an arbitrary $(\theta^m_{m-1})$ in $A_{m-1}^2$. Therefore $J_2(N) \neq N$, so we have $J_0(M_{R_i}(R_i^2)) \subseteq J_1(M_{R_i}(R_i^2)) \subseteq J_2(M_{R_i}(R_i^2)) \subseteq (A_i^2:R_i^2)$. On the other hand it is straightforward to verify that $(A_i^2:R_i^2)$ is a nil ideal, so by [7, Theorem 5.37], $J_0(M_{R_i}(R_i^2)) \supseteq (A_i^2:R_i^2)$. This proves the following result.

**Theorem 4.1.** If $R$ is a special PIR with $J(R) = \langle \theta \rangle$, then $J_\nu(M_{R}(R^2)) = (\langle \theta \rangle^2:R^2)$, $\nu = 0, 1, 2$.

Since $N/J_\nu(N) \cong M_{R_1}(R_1^2)/J_\nu(M_{R_1}(R_1^2)) \oplus \cdots \oplus M_{R_i}(R_i^2)/J_\nu(M_{R_i}(R_i^2))$, it remains to determine $M_{R_i}(R_i^2)/J_\nu(M_{R_i}(R_i^2))$ when $R_i$ is a special PIR. This characterization is provided in the following result.

**Theorem 4.2.** Let $R$ be a special PIR with $J(R) = \langle \theta \rangle$ and index of nilpotency $m$. Then $M_{R}(R^2)/J_\nu(M_{R}(R^2)) \cong M_{R/J(R)}(R/J(R))^2$, $\nu = 0, 1, 2$.

**Proof.** We know that every element of $(R/J(R))^2$ has a unique representative $(\alpha + J(R), \beta + J(R))$, where $\alpha, \beta \in F$. We define $\psi: M_{R}(R^2) \rightarrow M_{R/J(R)}(R/J(R))^2$ as follows: for $f \in M_{R}(R^2)$, $\psi(f)(\alpha + J(R), \beta + J(R)) = f(\alpha + J(R), \beta + J(R))$. If $(\alpha + J(R), \beta + J(R)) = (\gamma + J(R), \delta + J(R))$, then $\alpha = \gamma$ and $\beta = \delta$, so $\psi(f)$ is well-defined. Furthermore
\[ \psi(f) \in M_{R/J(R)}(R/J(R))^2, \text{ since } \psi(f)[(\alpha+J(R))(\gamma+J(R))] = f(\alpha) + J(R)^2 = f(\beta)\gamma + J(R)^2 = \psi(f)(\alpha+J(R))(\gamma+J(R)). \]

It is clear that \( \psi(f + g) = \psi(f) + \psi(g) \). Further, \( \psi(fg)(\alpha+J(R)) = f(g(\beta)) + J(R)^2 \), while \( \psi(fg)(\alpha+J(R)) = \psi(f)(g(\beta)) + J(R)^2 \). If \( g(\beta) = \gamma + J(R) \), then \( \psi(f)(g(\beta)) + J(R)^2 = f(\beta) + J(R)^2 \). But, as in Lemma 3.3, one finds \( f(\alpha + \beta + \ldots + \alpha_{m-1}) = f(\alpha) + \sigma \theta, \sigma \in R^2 \), so \( f(\alpha) + J(R)^2 = f(\beta + \beta + \ldots + \beta_{m-1}) + J(R)^2 = f(g(\beta) + J(R)^2), \) i.e., \( \psi(fg) = \psi(f)\psi(g) \).

We complete the proof by showing that \( \psi \) is onto and \( \ker \psi = J(M_{R}(R^2)) \).

To show that \( \psi \) is onto, let \( g \in M_{R/J(R)}(R/J(R))^2 \). For \( (\alpha + \beta + \ldots + \alpha_{m-1}) \) define \( f: R^2 \to R^2 \) by \( f(\beta) = (\alpha') \) where \( g(\alpha') + J(R)^2 = f(\beta) + J(R)^2 \). If \( g(\beta) = \gamma + J(R) \), then \( \psi(f)(g(\beta)) + J(R)^2 = f(\beta) + J(R)^2 \). Moreover, \( \psi(f)(\beta + J(R)) = f(\beta) + J(R)^2 = g(\beta) + J(R)^2 \), and hence \( \psi(f) = g \).

Finally, \( \ker \psi = \{ f \in M_{R}(R^2) \mid f(\beta) \in J(R)^2, \text{ for all } \alpha, \beta \in F \} = \{ f \in M_{R}(R^2) \mid f(\beta) \in J(R)^2 \text{ for all } x, y \in R \} = (J(R^2) : R^2) = (\langle \beta \rangle : R^2) = J_{\nu}(M_{R}(R^2)) \).

Acknowledgement

This paper was written while the second author was visiting the Department of Mathematics at Texas A&M University in 1989–1990. He wishes to express his gratitude for financial assistance by the CSIR of South Africa and for the hospitality bestowed upon him by Texas A&M University.

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