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The J_2-Radical in Structural Matrix Near-Rings

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We show that the two obvious definitions for a structural matrix near-ring somewhat unexpectedly yield the same near-ring. The strictly maximal left ideals of a structural matrix near-ring are characterized and used to describe its J_2-radical.

1. INTRODUCTION

The maximal left ideal structure of and special radicals in an important class of subrings of matrix rings, the *structural matrix rings*, were studied in Van Wyk [7, 8]. A structural matrix ring is a subring of a full matrix ring which is a subring solely by virtue of the shape of the matrices it contains; prominent examples are the rings of upper and lower triangular matrices over an arbitrary ring. Matrix near-rings were defined by Meldrum and Van der Walt in [1]; more results appeared, e.g., in Meyer [2] and in Van der Walt [5, 6]. In Section 2 of this paper it is shown that the two possible definitions for the type of near-rings we are dealing with here, called structural matrix near-rings, somewhat unexpectedly yield the same near-ring. In Section 3 some of the results of [6–8] are extended to structural matrix near-rings, the strictly maximal left ideals of a structural matrix near ring are characterized, and its J_2-radical is described.

We briefly recall the pertinent definitions. Let $(R, +, \cdot)$ be a rigth near-ring with identity 1. The term "subnear-ring of R" will mean "subnear-ring of R with the same identity as R". R^n will denote the direct sum of n copies of $(R, +)$, and similarly for subgroups of $(R, +)$. The elements of R^n are thought of as column vectors and written in transposed form with pointed brackets, e.g. $\langle r_1, r_2, ..., r_n \rangle$. The symbols ι_j and π_j will denote the jth coordinate injection and projection functions respectively. The *elementary $n \times n$ matrices over R* are the functions f^r_{ij}: $R^n \to R^n$, where $f^r_{ij} := \iota_i \lambda(r) \pi_j$. Here $r \in R$ and $\lambda(r)$: $R \to R$ is the left multiplication $s \mapsto rs$, for all $s \in R$. The subnear-ring of $M(R^n)$ generated by the f^r_{ij} is the *near-ring of $n \times n$ matrices*

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over R, denoted by $\mathbb{M}_n(R)$, and the elements of $\mathbb{M}_n(R)$ are called matrices. The identity matrix $f_{11}^1 + f_{22}^1 + \cdots + f_{nn}^1$ will be denoted by I.

A matrix is, of course, a function from R^n into R^n, but we shall often need representations of matrices. For this reason we use the set $\mathbb{E}_n(R)$ of *matrix expressions*, i.e., the subset of the free semigroup over the alphabet of symbols $\{f_{ij}^r \mid r \in R,\ 1 \leqslant i, j \leqslant n\} \cup \{(\ ,\ ),\ +\}$, recursively defined by the following rules:

(1) $f_{ij}^r \in \mathbb{E}_n(R)$ for $1 \leqslant i, j \leqslant n$ and all $r \in R$.

(2) If A, $E \in \mathbb{E}_n(R)$, then $A + E \in \mathbb{E}_n(R)$.

(3) If A, $E \in \mathbb{E}_n(R)$, then $(A)(E) \in \mathbb{E}_n(R)$.

The *length* $\ell(E)$ of an expression E is the number of f_{ij}^r in E. The *weight* $\omega(X)$ of a matrix X is the length of an expression of minimal length representing X. The matrix represented by $E \in \mathbb{E}_n(R)$ is denoted by $\mu(E)$. Every matrix is represented by at least one expression; however, the same matrix may be represented by many different expressions. In spite of this we shall usually not distinguish between expressions and matrices, except when such a distinction becomes necessary to avoid ambiguity, e.g. in the last part of Section 2 and in Lemma 3.9 and Theorem 3.10. Also, we shall omit parentheses if the meaning is clear.

Notation and standard results not given here may be looked up in Pilz [3].

2. Structural Subnear-Rings of Matrix Near-Rings

There are two obvious ways in which one can define structural subnear-rings of matrix near-rings. In the first place one can imitate the definition of a structural matrix ring and define the structural matrix near-ring $\mathbb{M}(B, R)$ associated with the reflexive and transitive $n \times n$ Boolean matrix $B = [b_{ij}]$ and the near-ring R as the subnear-ring of $\mathbb{M}_n(R)$ generated by the set $\{f_{ij}^r \mid r \in R,\ b_{ij} = 1\}$. In the second place one can follow a more functional approach imitating the definition of a matrix near-ring as certain functions from R^n into R^n.

Because of the lack of one distributive law it is in general not easy to predict which properties of matrix rings carry over to matrix near-rings. For instance, in [5] it was shown that, for a two-sided ideal $\mathscr{I}$ of R, the two-sided ideals $\mathscr{I}^* := \{X \in \mathbb{M}_n(R) \mid Xu \in \mathscr{I}^n$ for all $u \in R^n\}$ and $\mathscr{I}^+ := \mathrm{id}\{f_{ij}^r \mid r \in \mathscr{I},\ 1 \leqslant i, j \leqslant n\}$ (the two-sided ideal generated by the given set) of $\mathbb{M}_n(R)$ may differ. However, in this section we show that the two possible definitions for a structural matrix near-ring yield the same near-ring.

Let $u, v \in R^n$ and consider any $n \times n$ Boolean matrix $B = [b_{ij}]$. We write

$$u \sim_i v \text{ iff } \pi_j u = \pi_j v \text{ for all } j \text{ such that } b_{ij} = 1.$$

Then $\sim_i$ is trivially seen to be an equivalence relation on R^n. Consider the subset

$$\mathscr{S}(B, R) := \{X \in \mathbb{M}_n(R) \mid (\forall i, 1 \leqslant i \leqslant n)(u \sim_i v \Rightarrow \pi_i Xu = \pi_i Xv)\}$$

of the matrix near-ring $\mathbb{M}_n(R)$ as defined in [1]. Obviously, $\mathscr{S}(B, R)$ is an additive subgroup of $\mathbb{M}_n(R)$. That it is a subnear-ring of $\mathbb{M}_n(R)$ iff B is reflexive and transitive will be shown in Theorem 2.5. In the definition of $\mathscr{S}(B, R)$ it may perhaps seem more natural to require that $u \sim_i v$ imply $Xu \sim_i Xv$. However, this does not have the desired effect, as is shown in

EXAMPLE 2.1. Let

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

and let $X = f_{23}^1$. One would like to have $X \in \mathscr{S}(B, R)$. But if $u = \langle 0, 1, 1 \rangle$ and $v = \langle 0, 1, 0 \rangle$, then $u \sim_1 v$ and $\pi_2 Xu = 1 \neq 0 = \pi_2 Xv$, and so $Xu \not\sim_1 Xv$, which would imply that $X \notin \mathscr{S}(B, R)$. Note that B is not transitive.

PROPOSITION 2.2. *Let $X \in \mathscr{S}(B, R)$, B transitive. If $u \sim_i v$, then $Xu \sim_i Xv$.*

Proof. Let $b_{ij} = 1$. We show that $\pi_j Xu = \pi_j Xv$. Let $b_{jk} = 1$. Then $b_{ik} = 1$ and so $\pi_k u = \pi_k v$. Hence $u \sim_j v$, and so $\pi_j Xu = \pi_j Xv$ as $X \in \mathscr{S}(B, R)$.

LEMMA 2.3. *Let B be any $n \times n$ Boolean matrix. Then $b_{ij} = 1$ iff $f_{ij}^r \in \mathscr{S}(B, R)$ for every $r \in R$.*

Proof. First, let $r \in R$ and $b_{ij} = 1$, and let $u, v \in R^n$ with $u \sim_k v$. Then

$$\pi_k(f_{ij}^r u) = \begin{cases} 0 & \text{if } k \neq i \\ r\pi_j u & \text{if } k = i \end{cases}$$

and

$$\pi_k(f_{ij}^r v) = \begin{cases} 0 & \text{if } k \neq i \\ r\pi_j v & \text{if } k = i, \end{cases}$$

and so $\pi_k(f_{ij}^r u) = \pi_k(f_{ij}^r v)$ as $\pi_j u = \pi_j v$. Hence, $f_{ij}^r \in \mathscr{S}(B, R)$. Conversely, let $f_{ij}^1 \in \mathscr{S}(B, R)$ and suppose $b_{ij} = 0$. Then $\iota_j(1) \sim_i 0$, and so $1 = \pi_i(f_{ij}^1(\iota_j(1))) = \pi_i(f_{ij}^1 0) = 0$, a contradiction. Hence, $b_{ij} = 1$.

COROLLARY 2.4. *$I \in \mathcal{S}(B, R)$ iff B is reflexive.*

Proof. If B is reflexive, then Lemma 2.3 and the fact that $\mathcal{S}(B, R)$ is an additive subgroup of $\mathbb{M}_n(R)$ imply that $I \in \mathcal{S}(B, R)$. Conversely, if $I \in \mathcal{S}(B, R)$, then an argument similar to the one used in the last part of the proof of Lemma 2.3 shows that $b_{ii} = 1$ for every i.

THEOREM 2.5. *$\mathcal{S}(B, R)$ is a subnear-ring of $\mathbb{M}_n(R)$ iff B is reflexive and transitive.*

Proof. Let B be reflexive and transitive, and let $X, Y \in \mathcal{S}(B, R)$ and $u, v \in R^n$ with $u \sim_i v$. Then by Proposition 2.2 $\pi_i(XY)u = \pi_i(X(Yu)) = \pi_i(X(Yv)) = \pi_i(XY)v$ as $Yu \sim_i Yv$, and so $XY \in \mathcal{S}(B, R)$. Hence by Corollary 2.4 $\mathcal{S}(B, R)$ is a subnear-ring of $\mathbb{M}_n(R)$. Conversely, let $\mathcal{S}(B, R)$ be a subnear-ring of $\mathbb{M}_n(R)$. Then by Corollary 2.4 B is reflexive. Furthermore, if $b_{ij} = 1 = b_{jk}$, then by Lemma 2.3 $f_{ij}^r, f_{jk}^1 \in \mathcal{S}(B, R)$ for every $r \in R$, and so $f_{ik}^r = f_{ij}^r f_{jk}^1 \in \mathcal{S}(B, R)$; i.e., $b_{ik} = 1$ and B is transitive.

Henceforth B will be reflexive and transitive, and we shall write $\mathbb{M}(B, R)$ instead of $\mathcal{S}(B, R)$ to stress that $\mathcal{S}(B, R)$ is indeed a matrix near-ring, called a *structural matrix near-ring*.

It follows from Lemma 2.3 that the subnear-ring of $\mathbb{M}_n(R)$ generated by the set $\{f_{ij}^r \mid r \in R, b_{ij} = 1\}$ is contained in $\mathbb{M}(B, R)$. In fact, equality holds, as will be shown in Theorem 2.8. We first need the concept of the depth of an expression.

We assign a unique number $d(E)$, the *depth* of E, to each expression $E \in \mathbb{E}_n(R)$ as follows:

(1) $d(f_{ij}^r) = 0$

(2) $d(A + E) = \max(d(A), d(E))$

(3) $d((A)(E)) = d(A) + d(E) + 1$.

LEMMA 2.6. *For every $E \in \mathbb{E}_n(R)$ there is an $E' \in \mathbb{E}_n(R)$ representing the same matrix such that $d(E) = d(E')$ and E' is a sum of f_{ij}^r's and $(f_{kl}^s)(A)$'s with $A \in \mathbb{E}_n(R)$, $A \neq I$.*

Proof. An easy (but tedious) proof by induction on $d(E)$.

Set $W = \{f_{ij}^r \mid r \in R, 1 \leqslant i, j \leqslant n\}$. We make the following distinction among the f_{ij}^r's in an expression:

(1) If $E = a_1 \in W$, then a_1 is an *incisor* in E.

(2) Let $A = a_1 a_2 \cdots a_m \in \mathbb{E}_n(R)$ and $E = a_1' a_2' \cdots a_{m'}' \in \mathbb{E}_n(R)$. If $a_k \in W$, then a_k is a *molar* in $(A)(E) = (a_1 a_2 \cdots a_m)(a_1' a_2' \cdots a_{m'}')$ and a_k is *influenced* by a_ℓ' in $(A)(E)$ (or a_ℓ' influences a_k in $(A)(E)$, $1 \leqslant k \leqslant m$,

$1 \leqslant \ell \leqslant m'$. If $a'_\ell \in W$, then a'_ℓ is an incisor (resp. a molar) in $(A)(E)$ if a'_ℓ is an incisor (resp. a molar) in E, $1 \leqslant \ell \leqslant m'$.

(3) Let A and E be as in (2). If $a_k \in W$, then a_k is an incisor (resp. a molar) in $A + E = a_1 a_2 \cdots a_m + a'_1 a'_2 \cdots a'_{m'}$ if a_k is an incisor (resp. a molar) in A, $1 \leqslant k \leqslant m$. If $a'_\ell \in W$, then a'_ℓ is an incisor (resp. a molar) in $A + E$ if a'_ℓ is an incisor (resp. a molar) in E, $1 \leqslant \ell \leqslant m'$.

Note that $f^r_{ij} f^s_{jk} f^t_{k\ell} = f^r_{ii} f^s_{ii} f^t_{i\ell}$ for every $r, s, t \in R$ and $1 \leqslant i, j, k, \ell \leqslant n$. This observation leads to

Proposition 2.7. *For every $E \in \mathbb{E}_n(R)$ there is an $E' \in \mathbb{E}_n(R)$ representing the same matrix such that $d(E') \leqslant d(E)$ and every molar in E' is an f^r_{ii} for some $r \in R$ and some i, $1 \leqslant i \leqslant n$.*

Proof. We use again induction on $d(E)$. If $d(E) = 0$, then E is a sum of incisors. Now suppose $d(E) = p > 0$ and the result holds for all matrix expressions with depth $< p$. By Lemma 2.6 there is an $E' \in \mathbb{E}_n(R)$ such that $\mu(E') = \mu(E)$, $d(E') = d(E)$, and E' is a sum of f^r_{ij}'s and $(f^s_{k\ell})(A)$'s with $A \in \mathbb{E}_n(R)$, $A \neq I$. Consider any such $(f^s_{k\ell})(A)$ with depth p. Again by Lemma 2.6 there is an $A' \in \mathbb{E}_n(R)$ such that $\mu(A') = \mu(A)$, $d(A') = d(A)$, and A' is a sum of $f^{r'}_{i'j'}$'s and $(f^{s'}_{k'\ell'})(A'')$'s with $A'' \in \mathbb{E}_n(R)$, $A'' \neq I$. Set $A' = a_1 a_2 \cdots a_m$. We can assume that there is at least one a_i in A', influencing no $a_{i'}$ in A', such that a_i is $f^t_{\ell q}$ for some $t \in R$ and some q, $1 \leqslant q \leqslant n$. [Otherwise $\mu((f^r_{k\ell})(A')) = \mu(f^{r0}_{kk})$ and we are home.] By [1, Lemma 3.1(5)] $\mu((f^r_{k\ell})(A')) = \mu((f^r_{k\ell})(C))$, where $C \in \mathbb{E}_n(R)$ is obtained from A' by deleting the "non-contributing" symbols. Then $d(C) \leqslant d(A')$. By [2, Lemma 1.15(e)] and the remark preceding this proposition, $\mu((f^r_{k\ell})(C)) = \mu((f^r_{kk})(D))$, where $D \in \mathbb{E}_n(R)$ is obtained from C by replacing every $f^t_{\ell q}$ in $C = c_1 c_2 \cdots c_{m'}$ (say), influencing no c_j in C, by f^t_{kq}. Then $d(D) = d(C) \leqslant d(A)$, and so by the induction hypothesis there is an $F \in \mathbb{E}_n(R)$ such that $\mu(F) = \mu(D)$, $d(F) \leqslant d(D)$, and every molar in F has the desired form.

We can now show that every matrix in $\mathbb{M}(B, R)$ has an expression representing it which consists only of those f^r_{ij}'s such that $b_{ij} = 1$.

Theorem 2.8. *$\mathbb{M}(B, R)$ is the subnear-ring of $\mathbb{M}_n(R)$ generated by the set $\{f^r_{ij} \mid r \in R, b_{ij} = 1\}$.*

Proof. Let $U \in \mathbb{M}(B, R)$. By Proposition 2.7 and [1, Lemma 3.1(2)] we can assume without loss of generality that there is a matrix expression E representing U such that $E = E_1 + E_2 + \cdots + E_n$, where for every i, $1 \leqslant i \leqslant n$:

(1) every $f^r_{k\ell}$ in E_i is such that $k = \ell = i$ if $f^r_{k\ell}$ is a molar and

(2) every $f^r_{k\ell}$ in E_i is such that $k = i$ if $f^r_{k\ell}$ is an incisor.

Let $1 \leqslant i \leqslant n$, and let U_i be the matrix represented by E_i. We show that every f_{ij}^r in E_i, with $i \neq j$ and $b_{ij} = 0$, can be replaced by f_{ii}^{r0} such that the resulting expression, say E_i', still represents U_i. To this end let $u = \langle u_1, u_2, ..., u_n \rangle \in R^n$, and let $v \in R^n$ be obtained from u by replacing every u_j, $1 \leqslant j \leqslant n$, by 0 if there is an f_{ij}^r in E_i with $b_{ij} = 0$. Then, as $U \in \mathbb{M}(B, R)$ and $u \sim_i v$, $\pi_i(U_i u) = \pi_i(U_1 u + U_2 u + \cdots + U_n u) = \pi_i(Uu) = \pi_i(Uv) = \pi_i(U_i v)$. Hence, if f_{ij}^r is an incisor in E_i and $b_{ij} = 0$, then by a very tedious verification, which is best illustrated by Example 2.9 below, $\pi_i(U_i u) = \pi_i(U_i' u)$, where U_i' is the matrix represented by the mentioned expression E_i'. Therefore $\pi_i U_i = \pi_i U_i'$, and so

$$\begin{aligned}
U &= \sum_{i=1}^{n} \iota_i \pi_i U = \sum_{i=1}^{n} \iota_i \pi_i (U_1 + U_2 + \cdots + U_n) \\
&= \sum_{i=1}^{n} \iota_i \pi_i U_i = \sum_{i=1}^{n} \iota_i \pi_i U_i' \\
&= \sum_{i=1}^{n} \iota_i \pi_i (U_1' + U_2' + \cdots + U_n') \\
&= U_1' + \cdots + U_n' \in \mathbb{M}(B, R).
\end{aligned}$$

EXAMPLE 2.9. Let

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and let $a_i, u_j \in R$, $1 \leqslant i \leqslant 8$, $1 \leqslant j \leqslant 3$. Then $u := \langle u_1, u_2, u_3 \rangle \sim_3 \langle 0, 0, u_3 \rangle =: v$, and so if $U_3 := (f_{33}^{a_1})(f_{31}^{a_2} + f_{33}^{a_3} + (f_{33}^{a_4})(f_{32}^{a_5} + f_{33}^{a_6} + f_{31}^{a_7}) + f_{32}^{a_8}) \in \mathbb{M}(B, R)$, then $\pi_3(U_3 u) = \pi_3(U_3 v)$; i.e.,

$$\begin{aligned}
&\pi_3((f_{33}^{a_1})(\langle 0, 0, a_2 u_1 + a_3 u_3 \rangle \\
&\qquad + (f_{33}^{a_4})(\langle 0, 0, a_5 u_2 + a_6 u_3 + a_7 u_1 \rangle) + \langle 0, 0, a_8 u_2 \rangle)) \\
&= \pi_3((f_{33}^{a_1})(\langle 0, 0, a_2 u_1 + a_3 u_3 \\
&\qquad + a_4(a_5 u_2 + a_6 u_3 + a_7 u_1) + a_8 u_2 \rangle)) \\
&= \langle 0, 0, a_1(a_2 u_1 + a_3 u_3 + a_4(a_5 u_2 + a_6 u_3 + a_7 u_1) + a_8 u_2) \rangle \\
&= \langle 0, 0, a_1(a_2 0 + a_3 u_3 + a_4(a_5 0 + a_6 u_3 + a_7 0) + a_8 0) \rangle \\
&= \pi_3((f_{33}^{a_1})(f_{31}^{a_2 0} + f_{33}^{a_3} + (f_{33}^{a_4})(f_{32}^{a_5 0} + f_{33}^{a_6} + f_{31}^{a_7 0}) \\
&\qquad + f_{32}^{a_8 0})\langle u_1, u_2, u_3 \rangle).
\end{aligned}$$

PROPOSITION 2.10. *If R is a ring with identity, then $\mathbb{M}(B, R)$ is (isomorphic to) the structural matrix ring $\mathscr{S}(B, R)$ (see [7]).*

3. STRICTLY MAXIMAL LEFT IDEALS AND THE J_2-RADICAL

Van der Walt [6] characterized the 2-primitive ideals of a matrix near-ring $\mathbb{M}_n(R)$ in terms of those of R and obtained the result $J_2(\mathbb{M}_n(R)) = (J_2(R))^*$. In the first part of this section we use the characterization of the J_2-radical of R as the intersection of the strictly maximal left ideals of R, i.e., the maximal left ideals of R which are also maximal R-submodules of ${}_R R$, to obtain an alternative proof of the mentioned result.

Stone [4] used the Morita equivalence of R and $\mathbb{M}_n(R)$, R a ring, to characterize the maximal left ideals of $\mathbb{M}_n(R)$ as the sets $(M^n:\alpha) = \{X \in \mathbb{M}_n(R) \mid X\alpha \in M^n\}$, for M a maximal left ideal of R and $\alpha \in R^n \backslash M^n$. Although R is not Morita equivalent to any other structural matrix ring, this result of Stone's was generalized to the case of structural matrix rings (see [7]). Meyer [2] showed that if M is a strictly maximal left ideal of a zero-symmetric near-ring R, then $(M^n:\alpha)$ is a strictly maximal left ideal of $\mathbb{M}_n(R)$, provided that $\alpha \in R^n \backslash M^n$. We show that these $(M^n:\alpha)$'s are indeed all the strictly maximal left ideals of $\mathbb{M}_n(R)$. In the second part of this section this characterization is used to characterize the strictly maximal left ideals of a structural matrix near-ring in general. Finally we describe the J_2-radical of a structural matrix near-ring and show that the obtained result is a generalization of both [6, Theorem 4.4] and [8, Theorem 2.7].

Throughout this section R will be a zero-symmetric near-ring. Van der Walt [6] generalized the concept of a monogenic module to that of a connected module, and showed that if G is a connected R-module, then G^n is a (connected) $\mathbb{M}_n(R)$-module. We need the following lemma as the first step in an induction process in Proposition 3.2.

LEMMA 3.1. *Let G be a connected R-module. Then $f_{ij}^r\langle g_1, g_2, ..., g_n\rangle = \iota_i(rg_j)$ for every $r \in R$, $g_1, ..., g_n \in G$, $1 \leq i, j \leq n$.*

Proof. By the definition of G^n as an $\mathbb{M}_n(R)$-module (see [6]), $f_{ij}^r\langle g_1, g_2, ..., g_n\rangle = (f_{ij}^r\langle r_1, r_2, ..., r_n\rangle) g = (\iota_i(rr_j)) g = \iota_i((rr_j) g) = \iota_i(r(r_j g)) = \iota_i(rg_j)$.

Van der Walt showed in [6, Theorem 3.5] that as a group any connected $\mathbb{M}_n(R)$-module Γ is isomorphic to G^n for an appropriate R-module G. In the last part of [6, Theorem 3.10] it was shown that if Γ is of type 2, then so is G. But then Γ and G^n are isomorphic as $\mathbb{M}_n(R)$-modules:

PROPOSITION 3.2. *Let Γ be an $\mathbb{M}_n(R)$-module of type 2. Then $\Gamma \cong_{\mathbb{M}_n(R)} G^n$ for an appropriate R-module G of type 2.*

Proof. Let $G := f_{11}^1 \Gamma = \{f_{11}^1 \gamma \mid \gamma \in \Gamma\}$, $r(f_{11}^1 \gamma) = f_{11}^r \gamma$, and define ϕ: $\Gamma \to G^n$ by $\gamma\phi = \langle f_{11}^1 \gamma, f_{12}^1 \gamma, ..., f_{1n}^1 \gamma \rangle$ (see [6, Theorem 3.5]). We only have to show that $(U\gamma)\phi = U(\gamma\phi)$ for every $U \in \mathbb{M}_n(R)$, $\gamma \in \Gamma$. We proceed by induction on the weight $\omega(U)$ of U. If $\omega(U) = 1$, say $U = f_{ij}^r$, then $(U\gamma)\phi = \iota_i(f_{1j}^r \gamma)$. By [6, Theorem 3.10] G is monogenic, and hence connected, and so by Lemma 3.1 $U(\gamma\phi) = \iota_i(r(f_{1j}^1 \gamma)) = \iota_i(r(f_{11}^1(f_{1j}^1 \gamma))) = \iota_i(f_{11}^r(f_{1j}^1 \gamma)) = \iota_i(f_{1j}^r \gamma)$. The rest of the induction process is straightforward.

Note that if G and H are connected R-modules and $G \cong_R H$, then $G^n \cong_{\mathbb{M}_n(R)} H^n$. This easily-proven result is needed in

THEOREM 3.3. *The set of $(M^n : \alpha)$, for M a strictly maximal left ideal of R and $\alpha \in R^n \backslash M^n$, is the set of all the strictly maximal left ideals of* $\mathbb{M}_n(R)$.

Proof. By [2, Proposition 1.30] we only have to show that every strictly maximal left ideal $\mathcal{M}$ of $\mathbb{M}_n(R)$ has the mentioned form. $\mathbb{M}_n(R)/\mathcal{M}$ is an $\mathbb{M}_n(R)$-simple $\mathbb{M}_n(R)$-module, and so of type 2, since it is monogenic. By Proposition 3.2 $\mathbb{M}_n(R)/\mathcal{M} \cong_{\mathbb{M}_n(R)} G^n$ for an appropriate R-module G of type 2. Since G is monogenic, $G \cong_R R/\mathrm{Ann}_R\, g$ for any generator g of G, and $\mathrm{Ann}_R\, g$ is a strictly maximal left ideal of R. By [2, Proposition 1.29] and the remark above, $\mathbb{M}_n(R)/\mathcal{M} \cong_{\mathbb{M}_n(R)} R^n/(\mathrm{Ann}_R\, g)^n$. Let the isomorphism map $I + \mathcal{M}$ to $\alpha + (\mathrm{Ann}_R\, g)^n$ for some $\alpha \in R^n \backslash (\mathrm{Ann}_R\, g)^n$. It is easily checked that $\mathcal{M} = ((\mathrm{Ann}_R\, g)^n : \alpha)$.

COROLLARY 3.4. $J_2(\mathbb{M}_n(R)) = (J_2(R))^*$.

Proof. Note that $\bigcap \{(M^n : \alpha) \mid M$ is a strictly maximal left ideal of R and $\alpha \in R^n \backslash M^n\} = \bigcap \{(M^n : R^n) \mid M$ is a strictly maximal left ideal of $R\}$, since by [1, Proposition 4.1] M^n is an ideal of the $\mathbb{M}_n(R)$-module R^n. Hence (see [3, Proposition 1.44]) $J_2(\mathbb{M}_n(R)) = (\bigcap \{M \mid M$ is a strictly maximal left ideal of $R\})^* = (J_2(R))^*$.

We now turn to structural matrix near-rings in general. Only special cases of the following $\mathbb{M}(B, R)$-ideals of the $\mathbb{M}(B, R)$-module R^n will be needed to characterize the strictly maximal left ideals of $\mathbb{M}(B, R)$:

PROPOSITION 3.5. *Let $A_1, A_2, ..., A_n$ be left ideals of R such that $b_{ij} = 1$ implies $A_i \supseteq A_j$. Then $\langle A_1, A_2, ..., A_n \rangle$ is an $\mathbb{M}(B, R)$-ideal of R^n.*

Proof. Since each $(A_i, +)$ is normal in $(R, +)$, $(\langle A_1, A_2, ..., A_n \rangle, +)$ is normal in $(R^n, +)$. Let $u = \langle a_1, a_2, ..., a_n \rangle \in \langle A_1, A_2, ..., A_n \rangle$, $v = \langle r_1, r_2, ..., r_n \rangle \in R^n$, and $X \in \mathbb{M}(B, R)$. We show that $X(u + v) = Xv + w$ for some $w \in \langle A_1, A_2, ..., A_n \rangle$. If the weight $\omega(X)$ of X is 1, i.e., by Theorem 2.8 $X = f_{ij}^r$ for some $r \in R$, where $b_{ij} = 1$, then $X(u + v) = \iota_i(r(a_j + r_j)) = \iota_i(rr_j + c)$ for some $c \in A_j \subseteq A_i$, since A_j is a left ideal of R.

Hence, $X(u+v)=Xv+w$, where $w=\iota_i(c)\in\langle A_1, A_2, ..., A_n\rangle$. The result follows by induction on $\omega(X)$.

Structural $\mathbb{M}(B, R)$-modules, i.e., certain $\mathbb{M}(B, R)$-submodules of R^n, were introduced in [7] in case R is a ring. We need the same concept here. For the ease of the reader we provide the pertinent definitions.

Recall that $B=[b_{ij}]$ is a reflexive and transitive $n\times n$ Boolean matrix. B determines and is determined by the binary relation $\leqslant_B$ on $\underline{n}:=\{1, 2, ..., n\}$ defined by $i\leqslant_B j :\Leftrightarrow b_{ij}=1$. The quasi-order relation $\leqslant_B$ gives rise in the usual way to an equivalence relation $\sim_B$ on $\underline{n}$ defined by $i\sim_B j :\Leftrightarrow i\leqslant_B j$ and $j\leqslant_B i$. The number of equivalence classes of $\underline{n}$ induced by $\sim_B$ will be denoted by b, and $z_1, z_2, ..., z_b$ will be representatives of the different equivalence classes, which we denote by $[z_a]$, $a\in\underline{b}$. We consider the following $\mathbb{M}(B, R)$-ideals of the *structural* $\mathbb{M}(B, R)$-*modules* $R^n(a, R)$, $a\in\underline{b}$:

COROLLARY 3.6. *Let L be a left ideal of R, and let $a\in\underline{b}$. Then*

$$R^n(a, L) := \{u=\langle u_1, u_2, ..., u_n\rangle\in R^n \mid u_k=0 \text{ if } z_a\leqslant_B k \text{ and } z_a\not\sim_B k,\ u_k\in L \text{ if } z_a\sim_B k\}$$

is an $\mathbb{M}(B, R)$-ideal of the structural $\mathbb{M}(B, R)$-module $R^n(a, R)$.

Proof. We show that $R^n(a, L)$ is an $\mathbb{M}(B, R)$-ideal of R^n, which as a special case implies that $R^n(a, R)$ is an $\mathbb{M}(B, R)$-ideal of R^n and hence an $\mathbb{M}(B, R)$-submodule of R^n, since $\mathbb{M}(B, R)$ is zero-symmetric (see [1, Corollary 3.2]). We use Proposition 3.5 to establish the result. We assert that

$$R^n(a, L)=\langle A_1, A_2, ..., A_n\rangle \quad \text{where} \quad A_k=\begin{cases}0, & \text{if } z_a\leqslant_B k \text{ and } z_a\not\sim_B k\\ L, & \text{if } z_a\sim_B k\\ R, & \text{otherwise.}\end{cases}$$

To prove this, let $b_{ij}=1$ and consider the following two possibilities:

(i) $z_a\leqslant_B i$ and $z_a\not\sim_B i$: In this case $b_{jz_a}=0$ and $b_{z_aj}=1$, otherwise the transitivity of B is contradicted. Hence, $A_j=0$.

(ii) $z_a\sim_B i$: In this case $b_{z_aj}=1$, and so $A_j=0$ or L. In every case $A_i\supseteq A_j$, which proves our assertion.

Note that $R^n(a, L)=R^n(a', L')$ for proper left ideals L and L' of R if and only if $a=a'$ and $L=L'$, because $\iota_{z_a}(1)\in R^n(a', L)\backslash R^n(a, L)$ if $b_{z_{a'}z_a}=0$.

Let L be a left ideal of R and let $u \in R^n(a, R)$ for some $a \in \underline{b}$. Consider the $\mathbb{M}(B, R)$-homomorphisms

$$\mathbb{M}(B, R) \xrightarrow{f} R^n(a, R) \xrightarrow{g} R^n(a, R)/R^n(a, L),$$

where $Uf = Uu$ and g is the canonical epimorphism reducing mod $R^n(a, L)$. Then $(R^n(a, L):u) := \ker(f \circ g)$ is an $\mathbb{M}(B, R)$-ideal of $\mathbb{M}(B, R)$, i.e., a left ideal of $\mathbb{M}(B, R)$, which is proper unless $u \in R^n(a, L)$, since $f^1_{kk} \notin (R^n(a, L):u)$ if $\pi_k u \notin L$. We shall show in Theorem 3.16 that these kernels, for L a strictly maximal left ideal of R, produce all the strictly maximal left ideals of $\mathbb{M}(B, R)$. From now on M will be a strictly maximal left ideal of R, $a \in \underline{b}$ and $\alpha \in R^n(a, R) \backslash R^n(a, M)$.

PROPOSITION 3.7. *$R^n(a, R)/R^n(a, M)$ is an $\mathbb{M}(B, R)$-simple $\mathbb{M}(B, R)$-module.*

Proof. Let $\alpha \in R^n(a, R) \backslash R^n(a, M)$; i.e., $\pi_k \alpha \notin M$ for some $k \sim_B z_a$. Then $M + R\pi_k \alpha = R$. We show that $R^n(a, M) + \mathbb{M}(B, R)\alpha = R^n(a, R)$. Let $\beta \in R^n(a, R) \backslash R^n(a, M)$; i.e., $\pi_{k_i} \beta \notin M$ for $k_i \sim_B z_a$, $i = 1, 2, ..., m$ (say), where $m \geqslant 1$. Note that $b_{k_ik} = 1$ and so by Lemma 2.3 and Theorem 2.8 $f^r_{k_ik} \in \mathbb{M}(B, R)$ for all $r \in R$, $i \in \underline{m}$. For every such i we have $\pi_{k_i} \beta = s_i + r_i \pi_k \alpha$ for some $s_i \in M$ and $r_i \in R$, and so $\iota_{k_i} s_i + f^{r_i}_{k_ik} \alpha = \iota_{k_i}(\pi_{k_i} \beta)$. Let $\delta \in R^n(a, M)$ be defined by

$$\pi_q \delta = \begin{cases} s_i, & \text{if} \quad q = k_i \text{ for some } i \in \underline{m} \\ \pi_q \beta, & \text{otherwise,} \end{cases}$$

and let $U = f^{r_1}_{k_1k} + f^{r_2}_{k_2k} + \cdots + f^{r_m}_{k_mk}$. Then $\delta + U\alpha = \beta$.

COROLLARY 3.8. *$(R^n(a, M):\alpha)$ is a strictly maximal left ideal of $\mathbb{M}(B, R)$.*

Proof. By Proposition 3.7 and the $\mathbb{M}(B, R)$-isomorphism $\mathbb{M}(B, R)/(R^n(a, L):\alpha) \cong R^n(a, R)/R^n(a, L)$.

In [7] it was shown that, for a ring R, the set

$$\mathscr{K}_a = \{C = [c_{rs}] \in \mathbb{M}(B, R) \mid c_{rs} = 0 \text{ if } r, s \in [z_a]\}$$

is a two-sided ideal of $\mathbb{M}(B, R)$, and furthermore, that $\mathbb{M}(B, R)/\mathscr{K}_a$ and $\mathbb{M}_{n_a}(R)$ are isomorphic as rings, where $n_a := |[z_a]|$. For a near-ring R we now set

$$\mathscr{K}_a := (R^n(a, 0):R^n(a, R)).$$

Then by [3, Proposition 1.42] $\mathscr{K}_a$ is a two-sided ideal of $\mathbb{M}(B, R)$ and it is easily seen to be a generalization of the previous definition.

In order to show that the $(R^n(a, M):\alpha)$'s are all the strictly maximal left ideals of $\mathbb{M}(B, R)$, we show first that $\mathbb{M}(B, R)/\mathscr{K}_a$ and $\mathbb{M}_{n_a}(R)$ are isomorphic as near-rings, and second that every strictly maximal left ideal of $\mathbb{M}(B, R)$ contains some $\mathscr{K}_a$. For the isomorphism we use the idea in the proof of [6, Lemma 4.2] to find an epimorphism from $\mathbb{M}(B, R)$ onto $\mathbb{M}_{n_a}(R)$ with kernel $\mathscr{K}_a$. Recall that for any near-ring S and any $E \in \mathbb{E}_n(S)$, the matrix represented by E is denoted by $\mu(E)$. Next, let $j_1 < j_2 < \cdots < j_{n_a}$ denote the different elements of $[z_a]$, and consider the subset $\mathbb{E}$ of $\mathbb{E}_n(R)$ such that every $E \in \mathbb{E}$ consists only of f^r_{ij}'s where $b_{ij} = 1$ (see Theorem 2.8). Now, θ: $\mathbb{E} \to \mathbb{E}_{n_a}(R)$, where $\theta(E)$ is the expression derived from E by replacing every $f^r_{j_k j_\ell}$ in E by $f^r_{k\ell}$ and everything else by $f^0_{z_a z_a}$. Then μ and θ are surjections and $\theta(A+E) = \theta(A) + \theta(E)$, $\theta(AE) = \theta(A)\,\theta(E)$ for all $A, E \in \mathbb{E}$. We define Φ: $\mathbb{M}(B, R) \to \mathbb{M}_{n_a}(R)$ by $\Phi(X) = \mu(\theta(E))$, where $E \in \mu^{-1}(X)$. That Φ is well defined follows directly from

LEMMA 3.9. *$\pi_m(\mu(\theta(E))\langle r_1, r_2, ..., r_{n_a}\rangle) = \pi_{j_m}(\mu(E)(\sum_{k=1}^{n_a} \iota_{j_k}(r_k) + \sum_{i(b_{z_a i}=0)} \iota_i(s_i)))$ for every $E \in \mathbb{E}$, $r_k \in R$ $(k = 1, 2, ..., n_a)$, $s_i \in R$, and $1 \leqslant m \leqslant n_a$.*

Proof. We use induction on the length $\ell(E)$ of E. Let $\ell(E) = 1$, and consider the two possibilities:

(i) $E = f^r_{j_k j_\ell}$; then $\mu(\theta(f^r_{j_k j_\ell}))\langle r_1, r_2, ..., r_{n_a}\rangle = f^r_{k\ell}\langle r_1, r_2, ..., r_{n_a}\rangle = \iota_k(rr_\ell)$, and $\mu(f^r_{j_k j_\ell})(\sum_{q=1}^{n_a} \iota_{j_q}(r_q) + \sum_{i(b_{z_a i}=0)} \iota_i(s_i)) = \iota_{j_k}(rr_\ell)$, and so we are finished.

(ii) $E = f^r_{vw}$, where $\{v, w\} \nsubseteq \{j_1, j_2, ..., j_{n_a}\}$; here every mentioned projection gives 0. Hence the result holds if $\ell(E) = 1$. The rest of the induction process is easy.

THEOREM 3.10. *$\mathbb{M}(B, R)/\mathscr{K}_a \cong \mathbb{M}_{n_a}(R)$.*

Proof. Φ is an epimorphism, and we assert that $\operatorname{Ker}\Phi = (R^n(a, 0): R^n(a, R))$. Let $U \in \mathbb{M}(B, R)$ and let $E \in \mu^{-1}(U)$. First, suppose $U \in (R^n(a, 0): R^n(a, R))$. If $r_1, r_2, ..., r_{n_a} \in R$, then $u := \iota_{j_1}(r_1) + \iota_{j_2}(r_2) + \cdots + \iota_{j_{n_a}}(r_{n_a}) \in R^n(a, R)$, and so $Uu \in R^n(a, 0)$. Hence, $\pi_{j_m}(Uu) = 0$ for $m = 1, 2, ..., n_a$, i.e., by Lemma 3.9 $\Phi(U) = 0$. Conversely, let $U \in \operatorname{Ker}\Phi$, and let $u = \langle u_1, u_2, ..., u_n\rangle \in R^n(a : R)$. Then $u = \sum_{k=1}^{n_a} \iota_{j_k}(u_{j_k}) + \sum_{i(b_{z_a i}=0)} \iota_i(u_i)$. We need to show that $\pi_k(Uu) = 0$ for every k such that $b_{z_a k} = 1$ (see the definition of $R^n(a, L)$). If $b_{k z_a} = 0$, then $\pi_k(Uu) = 0$ anyhow, since $Uu \in R^n(a, R)$, and if $b_{k z_a} = 1$, then $k = j_\ell$ for some ℓ, and so by Lemma 3.9 $\pi_k(Uu) = \pi_{j_\ell}(Uu) = \pi_\ell(\mu(\theta(E))\langle u_{j_1}, u_{j_2}, ..., u_{j_{n_a}}\rangle) = \pi_\ell(\Phi(U)\langle u_{j_1}, u_{j_2}, ..., u_{j_{n_a}}\rangle) = 0$. Therefore, $U \in (R^n(a, 0): R^n(a, R))$.

Before we can show that every strictly maximal left ideal of $\mathbb{M}(B, R)$ contains some $\mathscr{K}_a$, we prove a few technical results exploring the structure of the $\mathscr{K}_a$'s and the strictly maximal left ideals of $\mathbb{M}(B, R)$ in general.

The first result is obvious:

LEMMA 3.11. *Let $i \in \underline{n}$, and let $\mathscr{L}$ be an $\mathbb{M}(B, R)$-submodule of $\mathbb{M}(B, R)$ with $f_{ii}^1 \in \mathscr{L}$. Then $f_{ki}^r \in \mathscr{L}$ for all $r \in R$ and $k \in \underline{n}$ such that $k \leqslant_B i$.*

Henceforth $\mathscr{M}$ will be a strictly maximal left ideal of $\mathbb{M}(B, R)$.

LEMMA 3.12. *Let $f_{ii}^1 \notin \mathscr{M}$ for some $i \in [z_a]$, $a \in \underline{b}$. Then $f_{kk}^1 \in \mathscr{M}$ for every k such that $b_{z_a k} = 1$ and $b_{k z_a} = 0$.*

Proof. Let $\mathscr{N}$ be the $\mathbb{M}(B, R)$-submodule of $\mathbb{M}(B, R)$ generated by f_{ii}^1. Then $\mathscr{M} + \mathscr{N} = \mathbb{M}(B, R)$, since $\mathscr{M} + \mathscr{N}$ is an $\mathbb{M}(B, R)$-submodule of $\mathbb{M}(B, R)$, and so $f_{kk}^1 = U + V$ for some $U \in \mathscr{L}$, $V \in \mathscr{N}$, where V is a sum of f_{ji}^r's such that $j \leqslant_B i$ (see [2, Proposition 1.24]). Since $b_{ki} = 0$, it follows that $\pi_k f_{kk}^1 u = \pi_k U u$ for every $u \in R^n$, and so $U = \sum_{\ell(\ell \neq k)} \iota_\ell(\pi_\ell U) + \iota_k(\pi_k U)$ $= \sum_{\ell(\ell \neq k)} \iota_\ell(\pi_\ell U) + \iota_k(\pi_k f_{kk}^1)$. Hence, $f_{kk}^1 U = f_{kk}^1$, and so $f_{kk}^1 \in \mathscr{M}$.

PROPOSITION 3.13. *Let $f_{ij}^r \in \mathbb{M}(B, R)$, $r \neq 0$. Then $f_{ij}^r \in \mathscr{K}_a$ iff $\{i, j\} \not\subseteq [z_a]$.*

Proof. Let $f_{ij}^r \in \mathscr{K}_a$. If $i = j_k$ and $j = j_\ell$ for some $j_k, j_\ell \in [z_a]$, then $0 = \pi_{j_k}(f_{j_k j_\ell}^r(\iota_{j_\ell}(1))) = r$, since $\iota_{j_\ell}(1) \in R^n(a, R)$, a contradiction. Conversely, let $\{i, j\} \not\subseteq [z_a]$ and let $u = \langle u_1, u_2, ..., u_n \rangle \in R^n(a, R)$, i.e. $u_\ell = 0$ if $b_{z_a \ell} = 1$ and $b_{\ell z_a} = 0$. It is only necessary to consider the case $i \in [z_a]$, $j \notin [z_a]$ (otherwise $f_{ij}^r \in \mathscr{K}_a$, trivially). Then $b_{z_a j} = 1$ and $b_{j z_a} = 0$, and so $u_j = 0$. Hence $\pi_{j_k}(f_{ij}^r u) = 0$, and so $f_{ij}^r \in \mathscr{K}_a$.

COROLLARY 3.14. *Let $f_{ii}^1 \notin \mathscr{M}$ for some $i \in [z_a]$, $a \in \underline{b}$. Then $f_{ii}^1 \mathscr{K}_a \subseteq \mathscr{M}$.*

Proof. Let $U \in \mathscr{K}_a$. By [2, Lemma 1.41] there exists an expression E for $f_{ii}^1 U$ which contains only symbols of the type f_{ij}^r, $r \in R$, $j \in \underline{n}$, and $i \leqslant_B j$, apart from operators and parentheses. It follows from Proposition 3.13 that $j \notin [z_a]$ for every such incisor f_{ij}^r in E, since $f_{ii}^1 U \in \mathscr{K}_a$. Hence $b_{z_a j} = 1$ and $b_{j z_a} = 0$, and so by Lemma 3.12 $f_{jj}^1 \in \mathscr{M}$. Therefore by Lemma 3.11 $f_{ij}^r \in \mathscr{M}$. Since every incisor in E is in $\mathscr{M}$, it follows that the matrix represented by E is in $\mathscr{M}$.

The foregoing results now lead to

PROPOSITION 3.15. *Let $f_{ii}^1 \notin \mathscr{M}$ for some $i \in [z_a]$, $a \in \underline{b}$. Then $\mathscr{K}_a \subseteq \mathscr{M}$.*

Proof. Suppose $\mathscr{K}_a \not\subseteq \mathscr{M}$. Then $f_{ii}^1 = U + V$ for some $U \in \mathscr{M}$, $V \in \mathscr{K}_a$, since $\mathscr{M} + \mathscr{K}_a = \mathbb{M}(B, R)$. Hence by Corollary 3.14 and [1, Lemma 3.1(7)] $f^1_{ii} = f^1_{ii}U + f^1_{ii}V \in \mathscr{M}$.

THEOREM 3.16. *The set of* $(R^n(a, M):\alpha)$, *for* M *a strictly maximal left ideal of* R *and* $\alpha \in R^n(a, R)\backslash R^n(a, M)$, $a \in \underline{b}$, *is the set of all the strictly maximal left ideals of* $\mathbb{M}(B, R)$.

Proof. Let $\mathscr{M}$ be a strictly maximal left ideal of $\mathbb{M}(B, R)$ with $f^1_{ii} \notin \mathscr{M}$ for some $i \in [z_a]$, $a \in \underline{b}$. Then by Proposition 3.15 $\mathscr{K}_a \subseteq \mathscr{M}$, and so $\mathscr{M}/\mathscr{K}_a$ is a strictly maximal left ideal of $\mathscr{M}(B, R)/\mathscr{K}_a$. Hence $\phi(M)$ is strictly maximal left ideal of $\mathbb{M}_{n_a}(R)$. Therefore by Theorem 3.3 $\Phi(\mathscr{M}) = (M^{n_a}:\beta)$ for some strictly maximal left ideal M of R and $\beta = \langle \alpha_1, \alpha_2, ..., \alpha_{n_a} \rangle \in R^{n_a}\backslash M^{n_a}$. We assert that $\mathscr{M} = (R^n(a, M):\alpha)$, where α is the element $\iota_{j_1}(\alpha_1) + \iota_{j_2}(\alpha_2) + \cdots + \iota_{j_{n_a}}(\alpha_{n_a})$ in $R^n(a, R)\backslash R^n(a, M)$. Let $U \in \mathbb{M}(B, R)$. Then $U \in (R^n(a, M):\alpha)$ iff $\pi_{j_m}(U\alpha) \in M$ for every $m = 1, 2, ..., n_a$. Hence the argument used in the second part of the proof of Theorem 3.10 shows that $U \in (R^n(a, M):\alpha)$ iff $\Phi(U) \in \Phi(\mathscr{M})$, and so the result follows from Proposition 3.15.

The main result of the paper can now be stated:

THEOREM 3.17. $J_2(\mathbb{M}(B, R)) = \bigcap_{a \in \underline{b}} (R^n(a, J_2(R)):R^n(a, R))$.

Proof. By Theorem 3.16 $J_2(\mathbb{M}(B, R)) = \bigcap \{(R^n(a, M):\alpha) \mid \alpha \in R^n(a, R)\backslash R^n(a, M)$, M is a strictly maximal left ideal of R, and $a \in \underline{b}\} = \bigcap \{(R^n(a, M):\beta) \mid \beta \in R^n(a, R)$, M is a strictly maximal left ideal of R, and $a \in \underline{b}\} = \bigcap \{(R^n(a, M):R^n(a, R)) \mid M$ is a strictly maximal left ideal of R, and $a \in \underline{b}\} = \bigcap_{a \in \underline{b}} (R^n(a, J_2(R)):R^n(a, R))$.

We do not know at present whether in general $J_2(\mathbb{M}(B, R))$ can be expressed as the sum of two two-sided ideals, one of which is nilpotent as in the ring case, where it is called the antisymmetric radical (see [8]).

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