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SPECIAL RADICALS IN STRUCTURAL MATRIX RINGS

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1. Introduction

It is well-known that the Jacobson radical $\mathcal{J}(T)$ of a ring T carries over to the complete matrix ring $\mathbb{M}_n(T)$ in the simplest possible way, i.e. $\mathcal{J}(\mathbb{M}_n(T)) = \mathbb{M}_n(\mathcal{J}(T))$. In fact, V.A. Andrunakievic [1] proved that for a special radical $\mathcal{R}$ determined by a special class $\mathcal{M}$ of rings such that $S \in \mathcal{M}$ iff $\mathbb{M}_n(S) \in \mathcal{M}$ whenever S has an identity, the equality $\mathcal{R}(\mathbb{M}_n(T)) = \mathbb{M}_n(\mathcal{R}(T))$ holds for every ring T.

However, for a field F the Jacobson radical of

$$\begin{bmatrix} F & 0 \\ F & F \end{bmatrix} \text{ is } \begin{bmatrix} 0 & 0 \\ F & 0 \end{bmatrix},$$

and so the Jacobson radical does not carry over in this way to the ring of all 2×2 lower triangular matrices over F. The main purpose of this paper is to study special radicals of structural matrix rings, i.e. subrings of $\mathbb{M}_n(T)$ which are rings solely by virtue of the shape of their matrices. Such a shape can be described by saying in which positions nonzero elements of the base ring T are allowed, i.e. by an $n \times n$ Boolean matrix B. The maximal left ideals of these rings (over a ring with 1) were characterized and studied in [2], where the relevant notation and definitions were introduced. For the convenience of the reader we provide the necessary background.

Throughout this paper the terms "ring" and "ideal" will mean "associative ring" and "two-sided ideal" respectively. T will be a generic symbol for a ring, and R for a ring with an identity. Every $n \times n$ Boolean matrix $B = [b_{ij}]$ determines and is determined by a binary relation $\leqslant_B$ on $\underline{n} := \{1,2,\ldots,n\}$ defined by $i \leqslant_B j :\Longleftrightarrow b_{ij} = 1$, and for a nonempty subset V of T, the *set associated with* B and V is the set

$$\mathcal{S}(B,V) := \{C = [c_{ij}] \in \mathbb{M}_n(V) : b_{ij} = 0 \rightarrow c_{ij} = 0\}.$$

Then $\mathcal{S}(B,T)$ is a ring if $\leqslant_B$ is transitive, and $\mathcal{S}(B,R)$ is a ring with an identity iff $\leqslant_B$ is a quasi-order relation, i.e. reflexive and transitive. Henceforth $\leqslant_B$ will be reflexive and transitive, and then we call $\mathcal{S}(B,T)$ a *structural matrix ring*.

In section 2 we give a characterization of the ideals of $\mathcal{S}(B,R)$ in terms of set-inclusion preserving functions. Although there is a 1 - 1 correspondence between the ideals of R and $\mathbb{M}_n(R)$ via $A \rightarrow \mathbb{M}_n(A)$, no other structural matrix ring enjoys this property and is therefore not Morita equivalent to R. We show that if R has finitely many ideals, then, starting with a quasi-order relation $\leq_B$ which is not antisymmetric, it is possible to construct a "more antisymmetric" quasi-order relation $\leq_{B'}$, differing slightly from $\leq_B$, such that $\mathcal{S}(B',R)$ has at least twice as many ideals as $\mathcal{S}(B,R)$; no surprise that $\mathbb{M}_n(R)$ has so "few" ideals.

In section 3 we characterize the prime ideals of $\mathcal{S}(B,R)$ and this leads to the main result of the paper which generalizes Andrunakievic's result to structural matrix rings. Our result states that for a special radical $\mathcal{R}$ determined by a special class $\mathcal{M}$ of rings such that $S \in \mathcal{M}$ iff $\mathbb{M}_n(S) \in \mathcal{M}$ whenever S has an identity, $\mathcal{R}(\mathcal{S}(B,T))$ is the sum of two ideals, namely $\mathcal{S}(B,\mathcal{R}(T))$ and the set of all matrices with entries from T in the "antisymmetric part" of B, i.e. the positions (r,s) such that $b_{rs} = 1$ and $b_{sr} = 0$, and zeroes elsewhere. Obviously the latter ideal, which we call the antisymmetric radical of $\mathcal{S}(B,T)$, is the zero ideal iff $\leq_B$ is symmetric, and so special radicals carry over to a structural matrix ring $\mathcal{S}(B,T)$ in the simplest possible way iff $\leq_B$ is an equivalence relation.

Notation

The quasi-order relation $\leq_B$ naturally gives rise to an equivalence relation $\sim_B$ on $\underline{n}$ defined by $i \sim_B j$ iff $i \leq_B j$ and $j \leq_B i$. If there can be no confusion, then we simply write $\sim$ instead of

$\sim_B$. The number of equivalence classes of $\underline{n}$ induced by $\sim_B$ will be denoted by β, and $z_1, z_2, \ldots, z_\beta$ will be representatives of the different equivalence classes, which we denote by $[z_\mu]_B$ (or $[z_\mu]$), $\mu \in \underline{\beta}$. Let $j_1 < j_2 < \ldots < j_{n_\mu}$ denote the different elements of $[z_\mu]$, i.e. $[z_\mu]$ has n_μ elements. We use E_{ij} for the Boolean matrix with 1 in position (i,j) and zeroes elsewhere.

2. Antisymmetry of B and ideals of $\mathcal{S}(B,R)$

For all $\mu, \xi \in \underline{\beta}$ such that $z_\mu \leqslant_B z_\xi$, we set

$$\Lambda_{\mu\xi}(B) := \{\nu \in \underline{\beta}: z_\mu \leqslant_B z_\nu \text{ and } z_\nu \leqslant_B z_\xi\}.$$

If there can be no confusion, then we simply write $\Lambda_{\mu\xi}$. Note that $\Lambda_{\mu\mu} = \{\mu\}$ for every $\mu \in \underline{\beta}$. Furthermore, all the $\Lambda_{\mu\xi}$'s are different.

Let us now consider any set-inclusion preserving function $\theta: \{\Lambda_{\mu\xi}: \mu, \xi \in \underline{\beta} \text{ and } z_\mu \leqslant_B z_\xi\} \to \{A: A \text{ is an ideal of } R\}$, i.e. θ is such that $\Lambda_{\nu\eta} \subseteq \Lambda_{\mu\xi}$ implies $\theta(\Lambda_{\nu\eta}) \subseteq \theta(\Lambda_{\mu\xi})$. We show that there is an ideal of $\mathcal{S}(B,R)$ associated with such a θ, viz

$\mathcal{A}_\theta := \{X = [x_{pq}] \in \mathcal{S}(B,R): x_{pq} \in \theta(\Lambda_{\mu\xi})$ if $p \sim z_\mu$, $q \sim z_\xi$ and $z_\mu \leqslant_B z_\xi\}$:

<u>Lemma 1.1</u> $\mathcal{A}_\theta$ is an ideal of $\mathcal{S}(B,R)$.

<u>Proof</u> We denote the ideal $\theta(\Lambda_{\mu\xi})$ of R by $A_{\mu\xi}$ for $\mu, \xi \in \underline{\beta}$ such that $z_\mu \leqslant_B z_\xi$. It is obvious that $\mathcal{A}_\theta$ is an additive subgroup

of $\mathcal{S}(B,R)$. Let $X = [x_{pq}] \in \mathcal{A}_\theta$ and let $Y = [y_{su}] \in \mathcal{S}(B,R)$. We show that $C = [c_{vw}] := XY$ is in $\mathcal{A}_\theta$. Let $p \leq_B q$, with $p \sim z_\mu$ and $q \sim z_\xi$. Then it is only necessary to prove that $c_{pq} \in A_{\mu\xi}$. First note that $c_{pq} = \sum_t x_{pt}y_{tq}$, and let $t \in [z_\nu]$ such that $p \leq_B t$ and $t \leq_B q$. Then $z_\mu \leq_B z_\nu$ and $z_\nu \leq_B z_\xi$, because $p \sim z_\mu$ and $q \sim z_\xi$. Hence, if $\eta \in \Lambda_{\mu\nu}$, then $z_\mu \leq_B z_\eta$ and $z_\eta \leq_B z_\nu$. But $\leq_B$ is transitive, which implies that $z_\eta \leq_B z_\xi$. This forces η to be in $\Lambda_{\mu\xi}$. Hence, $\Lambda_{\mu\nu} \subseteq \Lambda_{\mu\xi}$, and so $x_{pt} \in A_{\mu\nu} \subseteq A_{\mu\xi}$. Furthermore, $A_{\mu\xi}$ is an ideal, and so we conclude that $c_{pq} \in A_{\mu\xi}$. This proves that $\mathcal{A}_\theta$ is a right ideal of $\mathcal{S}(B,R)$. A similar argument shows that $\mathcal{A}_\theta$ is a left ideal of $\mathcal{S}(B,R)$. □

A standard argument using the matrix units shows that every ideal of $\mathcal{S}(B,R)$ has this form, and so we get

<u>Proposition 1.2</u> The set of $\mathcal{A}_\theta$, for $\theta\colon \{\Lambda_{\mu\xi}\colon\ \mu,\xi \in \underline{\beta}$ and $z_\mu \leq_B z_\xi\} \to \{A\colon A$ is an ideal of $R\}$ set-inclusion preserving, is the set of all the ideals of $\mathcal{S}(B,R)$. □

<u>Corollary 1.3</u> Let R have finitely many ideals. Then $\mathcal{S}(B,R)$ has $(\Sigma 1)^\beta$ ideals iff $\leq_B$ is symmetric, where the sum is taken over the ideals of R.

<u>Proof</u> If $\leq_B$ is symmetric, then the domain of θ consists of precisely β elements, viz $\Lambda_{\mu\mu} = \{\mu\}$ for $\mu = 1,2,\ldots,\beta$, and so the desired result follows. Conversely, if $\leq_B$ is not symmetric, then

$z_\mu \leqslant_B z_\xi$ for some $\mu,\xi \in \underline{\beta}$ with $z_\mu \not\sim z_\xi$, i.e. $z_\xi \not\leqslant_B z_\mu$. Let θ be defined as follows:

$$\theta(\Lambda_{\nu\eta}) = \begin{cases} R, & \text{if } \Lambda_{\mu\xi} \subseteq \Lambda_{\nu\eta} \\ 0, & \text{otherwise.} \end{cases}$$

A direct argument shows that θ is set-inclusion preserving, and so it gives rise to an ideal of $S(B,R)$. Now it is easy to see that $S(B,R)$ has at least $(\Sigma 1)^\beta + 1$ ideals. □

Example 1.4 Let F be a field, and let

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

$S(B,F)$ has five ideals, viz

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ R & 0 \end{bmatrix}$ (see θ constructed in corollary 1.3), $\begin{bmatrix} 0 & 0 \\ R & R \end{bmatrix}$, $\begin{bmatrix} R & 0 \\ R & 0 \end{bmatrix}$ and $\begin{bmatrix} R & 0 \\ R & R \end{bmatrix}$.

Corollary 1.3 and example 1.4 suggest that every quasi-order relation, which is not antisymmetric, can be changed into a quasi-order relation, which is "more antisymmetric", such that the latter relation gives rise to a structural matrix ring with more ideals. In fact,

Proposition 1.5 Let R have finitely many ideals, and suppose $\leqslant_B$ is not antisymmetric. There is a quasi-order C such that $S(C,R)$ has at least twice as many ideals as $S(B,R)$.

Proof As $\leqslant_B$ is not a partial order relation, there is an equivalence class which is not a singleton. We may assume, without loss of generality, that $[z_\beta]_B = \{j_1, j_2, \ldots, j_{n_\beta}\}$, with $n_\beta \geq 2$ and $j_{n_\beta} \neq z_\beta$. If we set $C := B - \Sigma E_{j_{n_\beta} k}$, where the sum is taken over the k's for which $j_{n_\beta} \leq_B k$ and $j_{n_\beta} \neq k$, then it follows almost immediately that C is still a quasi-order. Furthermore, $\sim_C$ induces $\beta + 1$ equivalence classes on $\underline{n}$. Let us leave the representatives $z_1, z_2, \ldots, z_\beta$ of the equivalence classes induced by $\sim_B$ unchanged for $\sim_C$, and let z_α be the new representative, with $\alpha := \beta + 1$. Then $z_\alpha = j_{n_\beta}$. The domain of the functions (in proposition 1.2) yielding the ideals of $S(C,R)$ is the union of the domain of the functions yielding the ideals of $S(B,R)$, and the set

$$\Lambda := \{\Lambda_{\mu\alpha}(C) : \; z_\mu \leq_B z_\beta \text{ and } \mu \in \underline{\beta}\} \cup \Lambda_{\alpha\alpha}(C).$$

Note that Λ has at least two elements, namely $\Lambda_{\beta\alpha}(C)$ and $\Lambda_{\alpha\alpha}(C)$. However, no element of Λ is contained in any $\Lambda_{\nu\eta}(B)$, since $\alpha \notin \Lambda_{\nu\eta}(B)$. For every ideal A_θ of $S(B,R)$, we now define θ' and θ'' to be the following extensions of θ:

$$\theta'(\Lambda_{\xi\alpha}(C)) = R \quad \text{for every } \Lambda_{\xi\alpha}(C) \in \Lambda;$$

$$\theta''(\Lambda_{\xi\alpha}(C)) = \begin{cases} R, & \text{if } \Lambda_{\xi\alpha}(C) \in \Lambda \text{ and } \xi \neq \alpha \\ 0, & \text{if } \xi = \alpha. \end{cases}$$

It is merely routine to check that θ' and θ'' are set-inclusion preserving, and so for every ideal A_θ of $S(B,R)$ we get the ideals $A_{\theta'}$ and $A_{\theta''}$ of $S(C,R)$. □

We conclude this section with some preliminary results for describing special radicals of structural matrix rings.

Let $\mu \in \underline{\beta}$, and let θ be the set-inclusion preserving function mapping $\Lambda_{\mu\mu}$ onto the zero ideal of R and everything else onto R. Henceforth we shall denote this resulting ideal A_θ of $S(B,R)$ by $K_\mu(R)$, or just K_μ. If we define $f_\mu : S(B,R) \to M_{n_\mu}(R)$ by $f_\mu([c_{qr}]) = [d_{tu}]$, where $d_{tu} = c_{j_t j_u}$, $1 \leq t,u \leq n_\mu$, then K_μ is the kernel of the ring epimorphism f_μ.

<u>Lemma 1.6</u> Let $0 \in V \subseteq R$. Then $S(B,V) + \bigcap_{\mu=1}^{\beta} K_\mu = \bigcap_{\mu=1}^{\beta} (S(B,V) + K_\mu)$.

<u>Proof</u> The inclusion $S(B,V) + \bigcap_{\mu=1}^{\beta} K_\mu \subseteq \bigcap_{\mu=1}^{\beta} (S(B,V) + K_\mu)$ is obvious. Therefore, let $X = [x_{pq}] \in \bigcap_{\mu=1}^{\beta} (S(B,V) + K_\mu)$, and let $\nu \in \underline{\beta}$. Then $X = Y + C$ for some $Y = [y_{tu}] \in S(B,V)$, $C = [c_{rs}] \in K_\nu$. Hence, if $r,s \in [z_\nu]$, then $c_{rs} = 0$, and so $x_{rs} = y_{rs} \in V$. This implies that

$$H := \sum_{\mu=1}^{\beta} \sum_{k,l \in [z_\mu]} x_{kl} E_{kl} \in S(B,V),$$

since $0 \in V$. Furthermore, $X - H \in \bigcap_{\mu=1}^{\beta} K_\mu$, and so

$$X = H + (X - H) \in S(B,V) + \bigcap_{\mu=1}^{\beta} K_\mu. \qquad \square$$

<u>Lemma 1.7</u> Let $0 \in V \subseteq R$, and let $\mu \in \underline{\beta}$. Then

$$f_\mu^{-1}(M_{n_\mu}(V)) = S(B,V) + K_\mu. \qquad \square$$

2. <u>Prime ideals of $S(B,R)$</u>

Let spec(R) denote the set of prime ideals of R.

<u>Lemma 2.1</u> Let $P \in \text{spec}(R)$, and let $\mu \in \underline{\beta}$. Then $S(B,P) + K_\mu \in \text{spec}(S(B,R))$.

Proof Let $\mathcal{A}_\theta$ be the ideal of $\mathcal{S}(B,R)$ with $\theta(\Lambda_{\mu\mu}) = P$ and $\theta(\Lambda_{\nu\nu}) = R$ for every $\nu \neq \mu$. Then $\mathcal{A}_\theta = \mathcal{S}(B,P) + K_\mu$. Let $\mathcal{A}_\psi$ and $\mathcal{A}_\chi$ be ideals of $\mathcal{S}(B,R)$ such that $\mathcal{A}_\psi \mathcal{A}_\chi \subseteq \mathcal{A}_\theta$, and set $A_1 := \psi(\Lambda_{\mu\mu})$ and $A_2 := \chi(\Lambda_{\mu\mu})$. It is easy to see that $A_1A_2 \subseteq P$, and so $A_1 \subseteq P$ or $A_2 \subseteq P$. This implies that $\mathcal{A}_\psi \subseteq \mathcal{A}_\theta$ or $\mathcal{A}_\chi \subseteq \mathcal{A}_\theta$, and so $\mathcal{A}_\theta$ is prime. □

It will turn out that these are all the prime ideals of $\mathcal{S}(B,R)$. We first need the following results:

Lemma 2.2 Let $\beta > 1$, and let $\mu,\nu \in \underline{\beta}$ with $\mu \neq \nu$. If A is a proper ideal of R, then ψ is set-inclusion preserving (i.e. $\mathcal{A}_\psi$ is an ideal of $\mathcal{S}(B,R)$) if ψ is defined as follows:

$$\psi(\Lambda_{\rho\sigma}) = \begin{cases} 0, & \text{if } \mu \notin \Lambda_{\rho\sigma} \text{ and } \nu \notin \Lambda_{\rho\sigma} \\ A, & \text{if } \mu \in \Lambda_{\rho\sigma} \text{ and } \nu \notin \Lambda_{\rho\sigma} \\ R, & \text{if } \nu \in \Lambda_{\rho\sigma}. \end{cases}$$ □

The following result is a trivial consequence, but we state it as we are going to use it directly in proposition 2.4:

Corollary 2.3 Let μ and ν be as in lemma 2.2, and let C be a proper ideal of R. Then $\mathcal{A}_\chi$ is an ideal of R if χ is defined as follows:

$$\chi(\Lambda_{\rho\sigma}) = \begin{cases} 0, & \text{if } \nu \notin \Lambda_{\rho\sigma} \text{ and } \mu \notin \Lambda_{\rho\sigma} \\ C, & \text{if } \nu \in \Lambda_{\rho\sigma} \text{ and } \mu \notin \Lambda_{\rho\sigma} \\ R, & \text{if } \mu \in \Lambda_{\rho\sigma}. \end{cases}$$ □

<u>Proposition 2.4</u> Let ψ and χ be the functions in lemma 2.2 and corollary 2.3 respectively, and let θ be a set-inclusion preserving function with $\theta(\Lambda_{\mu\mu}) = A$ and $\theta(\Lambda_{\nu\nu}) = C$, where $z_\nu \not\leqslant_B z_\mu$. Then $A_\psi A_\chi \subseteq A_\theta$.

<u>Proof</u> Let $\rho,\sigma \in \underline{\beta}$, and let $z_\rho \leqslant_B z_\sigma$. We consider the following four possibilities:

(i) $\mu \notin \Lambda_{\rho\sigma}$ and $\nu \notin \Lambda_{\rho\sigma}$:

We show that every matrix in $A_\psi A_\chi$ has 0 in position (z_ρ, z_σ). If $z_\rho \leqslant_B z_\omega$ and $z_\omega \leqslant_B z_\sigma$ for some $\omega \in \underline{\beta}$, then $\mu \notin \Lambda_{\rho\omega}$, otherwise $z_\rho \leqslant_B z_\mu$ and $z_\mu \leqslant_B z_\omega \leqslant_B z_\sigma$, which contradicts the assumption that $\mu \notin \Lambda_{\rho\sigma}$. Similarly, $\nu \notin \Lambda_{\rho\omega}$, and so $\psi(\Lambda_{\rho\omega}) = 0$, which shows that we kept our promise.

(ii) $\mu \notin \Lambda_{\rho\sigma}$ and $\nu \in \Lambda_{\rho\sigma}$:

Note that $\theta(\Lambda_{\rho\sigma}) \supseteq C$, because $\Lambda_{\nu\nu} \subseteq \Lambda_{\rho\sigma}$. We show that every matrix in $A_\psi A_\chi$ has an element of C in position (z_ρ, z_σ). If $z_\rho \leqslant_B z_\omega$ and $z_\omega \leqslant_B z_\sigma$ for some $\omega \in \underline{\beta}$, then an argument similar to one used in (i) shows that $\mu \notin \Lambda_{\omega\sigma}$, from which it follows that every matrix in A_χ has an element of C in position (z_ω, z_σ). But C is a left ideal, and so the desired result follows.

(iii) $\mu \in \Lambda_{\rho\sigma}$ and $\nu \notin \Lambda_{\rho\sigma}$:

In this case $\theta(\Lambda_{\rho\sigma}) \supseteq A$. An argument similar to the one in (ii) shows that every matrix in A_ψ has an element of A in position (z_ρ, z_ω) if $z_\rho \leqslant_B z_\omega$ and $z_\omega \leqslant_B z_\sigma (\omega \in \underline{\beta})$, from which it follows that every matrix in $A_\psi A_\chi$ has an element of A in position (z_ρ, z_σ), as A is a right ideal.

(iv) $\mu \in \Lambda_{\rho\sigma}$ and $\nu \in \Lambda_{\rho\sigma}$:

Now $\Lambda_{\mu\mu}, \Lambda_{\nu\nu} \subseteq \Lambda_{\rho\sigma}$, and so $\theta(\Lambda_{\rho\sigma}) \supseteq A + C$. We show that every matrix in $A_\psi A_\chi$ has an element of $A + C$ in position (z_ρ, z_σ). Let $z_\rho \leqslant_B z_\omega$ and $z_\omega \leqslant_B z_\sigma$, $\omega \in \underline{\beta}$. If $\mu \in \Lambda_{\omega\sigma}$ and $\nu \in \Lambda_{\rho\omega}$, then $z_\nu \leqslant_B z_\omega$ and $z_\omega \leqslant_B z_\mu$, and so $z_\nu \leqslant_B z_\mu$. But we assumed that $z_\nu \not\leqslant_B z_\mu$. Hence, $\nu \notin \Lambda_{\rho\omega}$ or $\mu \notin \Lambda_{\omega\sigma}$. This implies that every matrix in $A_\psi A_\chi$ has an element of $AR + RC \subseteq A + C$ in position (z_ρ, z_σ). □

<u>Theorem 2.5</u> The set of $\mathcal{S}(B,P) + K_\mu$, for $P \in \operatorname{spec}(R)$ and $\mu \in \underline{\beta}$, is the set of all the prime ideals of $\mathcal{S}(B,R)$.

<u>Proof</u> Let $\mathcal{P} = A_\theta \in \operatorname{spec}(\mathcal{S}(B,R))$ with $\theta(\Lambda_{\mu\mu}) =: A \neq R$ for some $\mu \in \underline{\beta}$. If $\beta = 1$, i.e. if B is the universal matrix, then there is nothing to prove. Therefore, suppose $\beta > 1$, and let $\nu \in \underline{\beta}$ such that $\mu \neq \nu$. Then $z_\nu \not\leqslant_B z_\mu$ or $z_\mu \not\leqslant_B z_\nu$. Set $\theta(\Lambda_{\nu\nu}) =: C$, and suppose $C \neq R$. Let ψ and χ be the functions in lemma 2.2 and corollary 2.3 respectively. Then by proposition 2.4 $A_\psi A_\chi \subseteq A_\theta$ or $A_\chi A_\psi \subseteq A_\theta$. However, $A_\psi \not\subseteq A_\theta$ and $A_\chi \not\subseteq A_\theta$, which contradicts the fact that A_θ is prime, and so $C = R$. Hence, $\mathcal{P} = A_\theta = \mathcal{S}(B,A) + K_\mu$. Furthermore, $f_\mu(\mathcal{P}) = M_{n_\mu}(A)$, which implies that $M_{n_\mu}(A) \in \operatorname{spec}(M_{n_\mu}(R))$, since $\mathcal{S}(B,R)/K_\mu \cong M_{n_\mu}(R)$. Therefore, $A \in \operatorname{spec}(R)$. This, together with lemma 2.1, proves the theorem. □

3. Specials radicals in structural matrix rings

Let M be a special class of rings satisfying the following condition:

I. If S is a ring with an identity, then $S \in M$ iff $\mathbb{M}_n(S) \in M$. Andrunakievic (see [1], theorem 12) proved that the equality $R(\mathbb{M}_n(T)) = \mathbb{M}_n(R(T))$ holds for every special radical R determined by a special class M of rings satisfying condition I. We shall use the characterization of the special radical $R(R)$ of R as the intersection of its M-special ideals (i.e. the ideals P of R such that $R/P \in M$) to generalize Andrunakievic's result to structural matrix rings.

Proposition 2.6 Let M be a special class of rings satisfying condition I. The set of $S(B,P) + K_\mu$, for P an M-special ideal of R and $\mu \in \underline{\beta}$, is the set of all the M-special ideals of $S(B,R)$.

Proof Let P be an M-special ideal of R, and let $\mu \in \underline{\beta}$. Then $R/P \in M$, and so $\mathbb{M}_{n_\mu}(R)/\mathbb{M}_{n_\mu}(P) \cong \mathbb{M}_{n_\mu}(R/P) \in M$. Hence, $\mathbb{M}_{n_\mu}(P)$ is an M-special ideal of $\mathbb{M}_{n_\mu}(R)$. Furthermore, $f_\mu^{-1}(\mathbb{M}_{n_\mu}(P)) = S(B,P) + K_\mu$, and so $(S(B,P) + K_\mu)/K_\mu$ is an M-special ideal of $S(B,R)/K_\mu$. But

$$(S(B,R)/K_\mu)/((S(B,P) + K_\mu)/K_\mu) \cong S(B,R)/(S(B,P) + K_\mu),$$

and so $S(B,R)/(S(B,P) + K_\mu) \in M$, i.e. $S(B,P) + K_\mu$ is an M-special ideal of $S(B,R)$. Suppose now that P is an M-special ideal of $S(B,R)$. Then $S(B,R)/P \in M$, and so $S(B,R)/P$ is a prime ring, i.e. $P \in \operatorname{spec}(S(B,R))$. Hence, by theorem 2.5 $P = S(B,P) + K_\mu$, for some

$P \in \operatorname{spec}(R)$ and $\mu \in \underline{\beta}$. The rest of the proof follows again from the two isomorphisms used above. □

<u>Theorem 2.7</u> Let $\mathcal{R}$ be a special radical determined by a special class $\mathcal{M}$ of rings satisfying condition I. Then

$$\mathcal{R}(\mathcal{S}(B,T)) = \mathcal{S}(B,\mathcal{R}(T)) + \bigcap_{\mu=1}^{\beta} \mathcal{K}_\mu(T)$$

for every ring T.

<u>Proof</u> We first suppose that the ring T has an identity. If we set $\mathcal{N}_\mu := \bigcap\{\mathcal{S}(B,P) + \mathcal{K}_\mu(T) : P$ an $\mathcal{M}$-special ideal of $T\}$, $\mu \in \underline{\beta}$, then by proposition 2.6 $\mathcal{R}(\mathcal{S}(B,T)) = \bigcap_{\mu=1}^{\beta} \mathcal{N}_\mu$. But the kernel of f_μ is $\mathcal{K}_\mu$ which is contained in every $\mathcal{S}(B,P) + \mathcal{K}_\mu(T)$, $\mu \in \underline{\beta}$, and so $\mathcal{R}(\mathcal{S}(B,T)) = \bigcap_{\mu=1}^{\beta} f_\mu^{-1}(f_\mu(\mathcal{N}_\mu)) = \bigcap_{\mu=1}^{\beta} f_\mu^{-1}(\bigcap\{f_\mu(\mathcal{S}(B,P) + \mathcal{K}_\mu(T)) : P$ an $\mathcal{M}$-special ideal of $T\})$

$$= \bigcap_{\mu=1}^{\beta} f_\mu^{-1}(\bigcap\{\mathbb{M}_{n_\mu}(P) : P \text{ an } \mathcal{M}\text{-special ideal of } T\}) = \bigcap_{\mu=1}^{\beta} f_\mu^{-1}(\mathbb{M}_{n_\mu}(\mathcal{R}(T))).$$

Hence, by lemmas 1.6 and 1.7

$$\begin{aligned}\mathcal{R}(\mathcal{S}(B,T)) &= \bigcap_{\mu=1}^{\beta} (\mathcal{S}(B,\mathcal{R}(T)) + \mathcal{K}_\mu(T)) \\ &= \mathcal{S}(B,\mathcal{R}(T)) + \bigcap_{\mu=1}^{\beta} \mathcal{K}_\mu(T).\end{aligned}$$

Now suppose that the ring T has no identity. It is well-known that T may be imbedded into a ring R with identity in such a way that T is an ideal of R. Since $\mathcal{R}$ is a hereditary radical, we have

$$\mathcal{R}(T) = T \cap \mathcal{R}(R),$$

and so $\mathcal{S}(B,\mathcal{R}(T)) + \bigcap_{\mu=1}^{\beta} \mathcal{K}_\mu(T) = (\mathcal{S}(B,T) \cap \mathcal{S}(B,\mathcal{R}(R))) + \bigcap_{\mu=1}^{\beta} \mathcal{K}_\mu(T)$. But $\bigcap_{\mu=1}^{\beta} \mathcal{K}_\mu(T) \subseteq \mathcal{S}(B,T)$, which implies that

$$\mathcal{S}(B,\mathcal{R}(T)) + \bigcap_{\mu=1}^{\beta} \mathcal{K}_\mu(T) = \mathcal{S}(B,T) \cap (\mathcal{S}(B,\mathcal{R}(R)) + \bigcap_{\mu=1}^{\beta} \mathcal{K}_\mu(T)).$$

Furthermore, $T = T \cap (\mathcal{R}(R) + T) = T \cap (\mathcal{R}(R) + R)$, and so

$$\mathcal{S}(B,\mathcal{R}(T)) + \bigcap_{\mu=1}^{\beta} \mathcal{K}_\mu(T) = \mathcal{S}(B,T) \cap (\mathcal{S}(B,\mathcal{R}(R)) + \bigcap_{\mu=1}^{\beta} \mathcal{K}_\mu(R))$$
$$= \mathcal{S}(B,T) \cap \mathcal{R}(\mathcal{S}(B,R)),$$

since R has an identity. Hence, as $\mathcal{S}(B,T)$ is an ideal of $\mathcal{S}(B,R)$, we get $\mathcal{S}(B,T) + \bigcap_{\mu=1}^{\beta} \mathcal{K}_\mu(T) = \mathcal{R}(\mathcal{S}(B,T))$. □

Example 2.8 The Jacobson radical of

$$\begin{bmatrix} T & T & T & 0 \\ T & T & T & 0 \\ 0 & 0 & T & 0 \\ T & T & T & T \end{bmatrix}$$

is

$$\begin{bmatrix} \mathcal{J}(T) & \mathcal{J}(T) & T & 0 \\ \mathcal{J}(T) & \mathcal{J}(T) & T & 0 \\ 0 & 0 & \mathcal{J}(T) & 0 \\ T & T & T & \mathcal{J}(T) \end{bmatrix}.$$

We call the ideal $\bigcap_{\mu=1}^{\beta} \mathcal{K}_\mu(T)$ the *antisymmetric radical* of $\mathcal{S}(B,T)$, since the factor ring $\mathcal{S}(B,T)/\bigcap_{\mu=1}^{\beta} \mathcal{K}_\mu(T)$ is isomorphic to the structural matrix ring $\mathcal{S}(C,T)$, C being the largest symmetric Boolean matrix Q satisfying $Q \leqslant B$.

Corollary 2.9 Let $\mathcal{R}$ and T be as in theorem 2.7. Then $\mathcal{R}(\mathcal{S}(B,T)) = \mathcal{S}(B,\mathcal{R}(T))$ iff $\leqslant_B$ is an equivalence relation.

Proof $\bigcap_{\mu=1}^{\beta} \mathcal{K}_\mu(T)$ is the zero ideal iff B is symmetric (unless, of course, if $T = 0$). □

Let I_n denote the $n \times n$ identity matrix. Then it is easy to see that $\bigcap_{\mu=1}^{\beta} \mathcal{K}_\mu(T) = \mathcal{S}(B - I_n,T)$ iff B is antisymmetric, and so the form of the Jacobson radical of the ring of all $n \times n$ lower triangular matrices over T should be no surprise. In fact,

Corollary 2.10 Let $\mathcal{R}$ and T be as in theorem 2.7. Then $\mathcal{R}(\mathcal{S}(B,T)) = \mathcal{S}(B,\mathcal{R}(T)) + \mathcal{S}(B - I_n,T)$ iff $\leqslant_B$ is a partial order relation. □

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