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# ON IDEALS OF TRIANGULAR MATRIX RINGS

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**Abstract**

We provide a formula for the number of ideals of complete block-triangular matrix rings over any ring $R$ such that the lattice of ideals of $R$ is isomorphic to a finite product of finite chains, as well as for the number of ideals of (not necessarily complete) block-triangular matrix rings over any such ring $R$ with three blocks on the diagonal.

## 1. Introduction

It is well known that if $R$ is a ring with identity, then there is a one-to-one correspondence between the (two-sided) ideals of $R$ and those of $M_m(R)$, the full $m \times m$ matrix ring over $R$.

If we rather focus on the class of structural matrix rings, or incidence rings, which has been studied extensively (see, for example, [1], [2] and [6]), then the situation becomes more involved. There are in general a lot more ideals in a structural matrix ring over any ring $R$ than in the base ring $R$. It is known that every structural matrix ring is isomorphic to a block-triangular matrix ring (see [2]).

The purpose of this paper is to give a formula for the number of ideals of complete block-triangular matrix rings over any ring $R$ if the lattice of ideals of $R$ is isomorphic to a finite product of finite chains, for example, if $R = \mathbb{Z}_n$, the

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ring of integers modulo $n$, as well as for the number of ideals of (not necessarily complete) block-triangular matrix rings over any such ring $R$ with three blocks on the diagonal.

We recall the definition of a structural matrix ring and the description of its ideals in [6]. Let $m$ be a natural number, and let $\theta$ be a reflexive and transitive binary relation on the set $\underline{m} = \{1, 2, \ldots, m\}$. The subring

$$M_m(\theta, R) = \{[a_{i,j}] \in M_m(R) \,|\, a_{i,j} = 0 \text{ if } (i,j) \notin \theta\}$$

of $M_m(R)$ is called a structural matrix ring, and the ideals of $M_m(\theta, R)$ can be obtained as follows: For $i, j \in \underline{m}$ consider the (possibly empty) interval

$$[i,j]_\theta = \{k \in \underline{m} \,|\, (i,k), (k,j) \in \theta\}$$

and the set $I(\theta, m) = \{[i,j]_\theta \,|\, i,j \in \underline{m}\}$ of all such intervals. If $(\mathcal{I}(R), \subseteq)$ denotes the lattice of ideals of $R$, and

$$f\colon (I(\theta, m), \subseteq) \longrightarrow (\mathcal{I}(R), \subseteq)$$

is order preserving, then the set

$$M_m(\theta, R, f) = \{[a_{i,j}] \in M_m(\theta, R) \,|\, a_{i,j} \in f([i,j]_\theta)\}$$

is an ideal of $M_m(\theta, R)$. In fact, every ideal of $M_m(\theta, R)$ is of the form $M_m(\theta, R, f)$ for a unique $f$; in other words, there is a bijection between the considered order preserving maps $f$ and the ideals of $M_m(\theta, R)$.

Note that if $\theta = \underline{m} \times \underline{m}$, then $M_m(\theta, R) = M_m(R)$ and $[i,j]_\theta = \underline{m}$ for all $i, j \in \underline{m}$, from which it follows that $I(\theta, m)$ is a singleton, and so in this case we obtain the mentioned familiar one-to-one correspondence between the ideals of $R$ and $M_m(R)$.

Also, if $\theta = \{(i,j) \,|\, 1 \le i \le j \le m\}$, then

$$[i,j]_\theta = \{k \in \underline{m} \,|\, i \le k \le j\}$$

is either empty or the ordinary interval $[i,j]$. Hence $M_m(\theta, R)$ is the $m \times m$ upper triangular matrix ring $U_m(R)$ over $R$, and so a typical ideal of $U_m(R)$ is given by

$$I = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & \cdots & A_{1,m} \\ 0 & A_{2,2} & A_{2,3} & \cdots & A_{2,m} \\ 0 & 0 & A_{3,3} & \cdots & A_{3,m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{m,m} \end{bmatrix}, \tag{1}$$

where $A_{i,j} \lhd R$, $A_{i,j} \subseteq A_{i,j+1}$ for all $i = 1, 2, \ldots, m-1$ and $j = i, i+1, \ldots, m-1$, and $A_{i+1,j} \subseteq A_{i,j}$ for all $j = 2, 3, \ldots, m$ and $i = 1, 2, \ldots, j-1$. Note that the notation in (1) suggests that the ideal $I$ consists of all matrices $[x_{i,j}]$, where $x_{i,j} \in A_{i,j}$ if $i \leq j$, and $x_{i,j} = 0$ otherwise.

We recall the description of a complete block-triangular matrix ring. Let $b_1, b_2, \ldots, b_m$ be positive integers summing to $b$, say. Then every $b \times b$ matrix $X = (x_{i,j})$ can be viewed as a matrix of $m^2$ rectangular blocks $X_{k,l}, 1 \leq k, l \leq m$, with

$$X_{k,l} = (x_{b_1+\cdots+b_{k-1}+i, b_1+\cdots+b_{l-1}+j})_{1 \leq i \leq b_k, 1 \leq j \leq b_l}.$$

We call $X_{1,1}, \ldots, X_{m,m}$ the $m$ blocks of $X$ on the diagonal. The subring $T$ of $M_b(R)$ comprising all the matrices $X$ with $X_{k,l} = (0)_{b_k \times b_l}$ if $k > l$ is the $(b_1, b_2, \ldots, b_m)$ complete (upper) block-triangular matrix ring over $R$ (with $m$ blocks on the diagonal). According to the general description of the ideals of a structural matrix ring, in each "block" of an ideal of the complete block-triangular matrix ring we must have only one ideal of the base ring and so these "blocks" are collapsed to a single entry of the corresponding ideal of the "underlying" triangular matrix ring. It follows that the ideal structure of $T$ is precisely the same as that of $U_m(R)$, irrespective of the values of the $b_i$'s.

If, moreover, $X_{k,l} = (0)_{b_k \times b_l}$ for at least one pair $(k, l)$ for which $k < l$, then we merely have a $(b_1, b_2, \ldots, b_m)$ (upper) block-triangular matrix ring over $R$ (with $m$ blocks on the diagonal), which is not a complete block-triangular matrix ring.

A formula for the number of ideals of $U_m(F)$ for arbitrary $m$, and $F$ a field, has been determined by Shapiro in [4], and we state it here for the sake of reference:

Proposition 1.1. *$U_m(F)$ has $C_{m+1}$ ideals, where $C_l = \frac{1}{l+1}\binom{2l}{l}$ is the $l$-th Catalan number.*

For example, $C_3 = 5$, and the five ideals of $U_2(F)$ are as follows:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix} \text{ and } \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}.$$

## 2. The number of ideals

If $(P, \leq)$ is a poset and $(Q_i, \leq_i)$, $1 \leq i \leq k$, are chains, then an order preserving function

$$f \colon P \longrightarrow Q_1 \times Q_2 \times \cdots \times Q_k$$

is uniquely determined by the order preserving compositions $\pi_i \circ f \colon P \longrightarrow Q_i$, where

$$\pi_i \colon Q_1 \times Q_2 \times \cdots \times Q_k \longrightarrow Q_i,\ 1 \leq i \leq k,$$

are the natural projections. Hence the number of order preserving functions from $P$ to the product $Q_1 \times Q_2 \times \cdots \times Q_k$ is $q_1 q_2 \cdots q_k$, where $q_i$ denotes the number of order preserving functions from $P$ to $Q_i$. Thus, in order to determine the number of ideals of a structural matrix ring over a ring $R$ such that the lattice of ideals of $R$ is isomorphic to a finite product of finite chains, it suffices to restrict our consideration to the case of a *uniserial* (or a chain) ring $R$, i.e. $(\mathcal{I}(R), \subseteq)$ is a chain

$$\{0\} = I_0 \subset I_1 \subset \cdots \subset I_n = R.$$

Then, according to the description given in Section 1, all ideals of $U_m(R)$ are of the form

$$M_m(\theta, R, f) = \begin{bmatrix} f([1,1]) & f([1,2]) & f([1,3]) & \cdots & f([1,m]) \\ 0 & f([2,2]) & f([2,3]) & \cdots & f([2,m]) \\ 0 & 0 & f([3,3]) & \cdots & f([3,m]) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & f([m,m]) \end{bmatrix},$$

where $f([i,j]) \in \{I_0, I_1, \dots, I_n\}$ and $f$ is order preserving, i.e. $f([i,j]) \subseteq f([i',j'])$ if $[i,j] \subseteq [i',j']$.

Therefore, the number $\lambda(m,n)$ of ideals of $U_m(R)$ is precisely the number of plane partitions of the 'staircase shape' $\delta_{m+1} = (m, m-1, \dots, 1)$, allowing 0 as a part and with largest part at most $n$. [Following the notation used in [5], a *plane partition* is an array $\pi = (\pi_{ij})_{i,j \geq 1}$ of nonnegative integers such that $\pi$ has finite support (i.e., at most finitely many nonzero entries) and is weakly decreasing in both rows and columns. A *part* of a plane partition $\pi = (\pi_{ij})_{i,j \geq 1}$ is a positive entry $\pi_{ij} > 0$, but in some cases, such as ours, 0 is allowed as a part. The *shape* of $\pi$ is the ordinary partition $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_r)$ for which $\pi$ has $\gamma_i$ nonzero parts in the $i$-th row, $1 \leq i \leq r$. So, for the 'staircase shape' $\delta_{m+1} = (\gamma_1, \gamma_2, \dots, \gamma_m)$ we have $\gamma_i = m + 1 - i,\ 1 \leq i \leq m$.]

By [3] and [5],

$$\lambda(m,n) = \prod_{1 \leq i < j \leq m+1} \frac{2n + i + j - 1}{i + j - 1}.$$

Note that if $n = 1$ (which is the case when $R$ is a field), then it can be shown that

$$\lambda(m,1) = C_{m+1}.$$

This agrees with Proposition 1.1.

Combining the foregoing arguments, we obtain our main result:

THEOREM 2.1. *Let $m, b_1, \ldots, b_m \geq 1$, and let $R$ be a ring such that the lattice of ideals of $R$ is isomorphic to the product $Q_1 \times Q_2 \times \cdots \times Q_k$ of chains, where $Q_i$ has $n_i + 1$ elements for every $i$. The number of ideals of every $(b_1, b_2, \ldots, b_m)$ complete block-triangular matrix ring over $R$ is given by*

$$\lambda(m, n_1) \cdot \lambda(m, n_2) \cdots \lambda(m, n_k) = \prod_{t=1}^{k} \Big( \prod_{1 \leq i < j \leq m+1} \frac{2n_t + i + j - 1}{i + j - 1} \Big).$$

EXAMPLE 2.2. The number of ideals of every $(b_1, b_2, b_3)$ complete block-triangular matrix ring over $\mathbb{Z}_4$, for example $U_3(\mathbb{Z}_4)$, is

$$\lambda(3, 2) = \prod_{1 \leq i < j \leq 4} \frac{4 + i + j - 1}{i + j - 1} = 84,$$

while the number of ideals is

$$\lambda(3, 1) \cdot \lambda(3, 1) = 196$$

in case the base ring is $\mathbb{Z}_6$.

## 3. Block-triangular matrix rings with three blocks on the diagonal

In this section we are particularly interested in the number of ideals of the structural matrix rings

$$S_1 = \begin{bmatrix} R & 0 & R \\ 0 & R & R \\ 0 & 0 & R \end{bmatrix} \quad \text{and} \quad S_2 = \begin{bmatrix} R & R & R \\ 0 & R & 0 \\ 0 & 0 & R \end{bmatrix},$$

with $R$ as in Theorem 2.1; equivalently, as discussed in Section 2, the number of ideals of the block-triangular matrix rings

$$\begin{bmatrix} M_{b_1 \times b_1}(R) & (0)_{b_1 \times b_2} & M_{b_1 \times b_3}(R) \\ (0)_{b_2 \times b_1} & M_{b_2 \times b_2}(R) & M_{b_2 \times b_3}(R) \\ (0)_{b_3 \times b_1} & (0)_{b_3 \times b_2} & M_{b_3 \times b_3}(R) \end{bmatrix}$$

and

$$\begin{bmatrix} M_{b_1 \times b_1}(R) & M_{b_1 \times b_2}(R) & M_{b_1 \times b_3}(R) \\ (0)_{b_2 \times b_1} & M_{b_2 \times b_2}(R) & (0)_{b_2 \times b_3} \\ (0)_{b_3 \times b_1} & (0)_{b_3 \times b_2} & M_{b_3 \times b_3}(R) \end{bmatrix},$$

over $R$, with $b_1, b_2, b_3$ any positive integers.

REMARK 3.1. Some $3 \times 3$ structural matrix rings are direct sums of block-triangular matrix rings for which we have already determined the number of ideals. For example, the block-triangular matrix ring

$$\begin{bmatrix} R & 0 & R \\ 0 & R & 0 \\ 0 & 0 & R \end{bmatrix}$$

is the direct sum of $R$ and $U_2(R)$, and the lattices of ideals of the complete block-triangular matrix rings

$$S_3 = \begin{bmatrix} R & R & R \\ R & R & R \\ 0 & 0 & R \end{bmatrix} \quad \text{and} \quad S_4 = \begin{bmatrix} R & R & R \\ 0 & R & R \\ 0 & R & R \end{bmatrix}$$

are isomorphic to that of $U_2(R)$. (Note, however, that $S_3$ and $S_4$ are not isomorphic as rings. See [2].) We conclude from [2] that, up to isomorphism, the rings $S_1$ and $S_2$ are the only $3 \times 3$ structural matrix rings for which we have not yet determined the number of ideals.

By Section 1, every ideal of $S_1$ (respectively $S_2$) is of the form $M_3(\theta_1, R, f_1)$ (respectively $M_3(\theta_2, R, f_2)$), where $\theta_1$ and $\theta_2$ are the appropriate relations and $f_1$ and $f_2$ are the appropriate order preserving functions. If $R$ is a uniserial ring with a chain $\{0\} = I_0 \subset I_1 \subset \cdots \subset I_n = R$ of ideals as in Section 2, then a typical ideal of $S_1$ is given by

$$\begin{bmatrix} A_{1,1} & 0 & A_{1,3} \\ 0 & A_{2,2} & A_{2,3} \\ 0 & 0 & A_{3,3} \end{bmatrix},$$

where each $A_{i,j} = I_r$ can be identified with $r$. Thus an ideal of $S_1$ can be visualized as a matrix

$$\begin{bmatrix} a & 0 & b \\ 0 & c & d \\ 0 & 0 & e \end{bmatrix},$$

where $a, b, c, d, e \in \{0, 1, \ldots, n\}$, and $a \le b,\ c \le d,\ e \le d$ and $e \le b$. A formula for the number of such matrices can be obtained as the following nested sequence of summations:

$$\sum_{a=0}^{n} \sum_{b=a}^{n} \sum_{c=0}^{n} \sum_{d=c}^{n} \sum_{e=0}^{\min(b,d)} 1,$$

which simplifies to

$$\xi(n) = \tfrac{1}{30}(n+1)(n+2)(2n+3)(2n^2+6n+5).$$

As in Theorem 2.1, the number of ideals of $S_1$ is the product $\xi(n_1) \cdot \xi(n_2) \cdots \xi(n_k)$, where the lattice of ideals of $R$ is isomorphic to $Q_1 \times Q_2 \times \cdots \times Q_k$, with $Q_i$ a chain of $n_i + 1$ elements.

Although $S_1$ and $S_2$ are not isomorphic as rings, it is straightforward to see that there is an anti-isomorphism between the rings $S_2^*$ (obtained from $S_2$ by replacing each $R$ by $R^{op}$) and $S_1$. Since $R$ and $R^{op}$ have the same lattice of ideals, the same is true for $S_1$ and $S_2$, and so the number of ideals of $S_2$ is the same as the number of ideals of $S_1$ (using the same base ring $R$).

EXAMPLE 3.2. If $R = \mathbb{Z}_4$ (respectively $\mathbb{Z}_6$), then we get that the number of ideals of $S_1$ is $\xi(2) = 70$ (respectively $\xi(1) \cdot \xi(1) = 169$). Naturally, one expects these numbers to be somewhat less than the values obtained in Example 2.2 for the full triangular case.

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