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Matrix Rings Satisfying Column Sum Conditions Versus Structural Matrix Rings

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ABSTRACT

The internal characterization of a structural matrix ring in terms of a set of matrix units associated with a quasiorder relation is used to obtain isomorphisms between seemingly different classes of subrings of a complete matrix ring. © *Elsevier Science Inc., 1996*

1. INTRODUCTION

Considerable interest has been directed toward providing conditions for a family of matrices to be reduced simultaneously to some prescribed form, for example, diagonal or triangular (see [10]). In particular, it is known from linear algebra (see, for example, [6, Chapter IV]) that if W is an n-dimensional vector space over a field F, $\mathscr{A}$ is a subalgebra of the full matrix algebra $\mathbb{M}_n(F)$, and $W = W_0 \supset W_1 \supset W_2 \supset \cdots \supset W_{l-1} \supset W_l = 0$ is a composition series for W as a right $\mathscr{A}$-module, with $\dim(W_{i-1}/W_i) = n_i$, $i = 1, \dots, l$,

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then there exists a transformation such that $\mathscr{A}$ has blocked triangular form with diagonal blocks of sizes $n_i \times n_i$, $i = 1, \ldots, l$.

In addition, several recent papers, for example, [1], [4], [8], and [11], either have explored new techniques enabling one to characterize complete matrix rings, or have applied these new techniques, as well as known techniques, to recognize disguised complete matrix rings. Although the class of structural matrix rings has been studied extensively in its own right (see, for example, [2], [3], [5], and [14]), a notable shortcoming had been the lack of an internal characterization of a structural matrix ring similar to the characterization of a complete matrix ring in terms of, for example, a set of matrix units. In [13] this problem is solved, viz., a characterization of a structural matrix ring in terms of a set of matrix units associated with a quasiorder relation is obtained. This characterization is used in [15] to generalize the linear algebraic result mentioned in the first paragraph; to be more precise, if $\mathscr{V}$ is a set of R-submodules of the free R-module R^n, R a commutative ring, then a sufficient condition on $\mathscr{V}$ in terms of a basis for R^n is provided so that the ring of all R-endomorphisms of R^n which leave every R-submodule in $\mathscr{V}$ invariant is isomorphic to a structural matrix ring.

Inspired by the foregoing results and the fact that the subring of the complete matrix ring $\mathbb{M}_2(R)$, R an arbitrary ring, comprising all the matrices with the property that the sum of the elements in the first column of such a matrix equals the sum of the elements in the second column, is isomorphic to the upper triangular matrix ring

$$\begin{bmatrix} R & R \\ 0 & R \end{bmatrix},$$

we provide in Section 2 a general application (see Theorem 2.4) of the mentioned internal characterization of a structural matrix ring. In Section 3 we apply Theorem 2.4 to obtain isomorphisms between seemingly different classes of subrings of a complete matrix ring, viz. structural matrix rings and matrix rings satisfying column sum conditions. Although the latter isomorphisms can also be obtained by conjugation by an invertible matrix, the purpose of this paper is to demonstrate the applicability of the internal characterization of a structural matrix ring. This should be viewed in the light of the success in obtaining new characterizations (of, for example, a complete matrix ring) using existing ones.

We work entirely in the category of associative rings. All rings contain an identity element, and this identity element is assumed to be inherited by all subrings. We use R, $\mathscr{R}$, and $\mathscr{S}$ as generic symbols for rings.

2. A GENERAL APPLICATION OF AN INTERNAL CHARACTERIZATION OF STRUCTURAL MATRIX RINGS

For the ease of the reader we provide the pertinent definitions. Let $B = [b_{ij}]$ be a reflexive (i.e., $b_{ii} = 1$ for every i) and transitive (i.e., if $b_{ij} = 1$ and $b_{jk} = 1$, then $b_{ik} = 1$) $n \times n$ Boolean matrix for some n, i.e., an $n \times n$ quasiorder. Then B determines and is determined by the quasiorder relation c_B (say) on the set $\{1, 2, \ldots, n\}$ defined by $i\, c_B\, j$ if and only if $b_{ij} = 1$. [We use the notation c_B to indicate that i and j are "connected" with respect to B, in the sense that B has a 1 in position (i, j).] The subset $\mathbb{M}_n(B, R)$ of $\mathbb{M}_n(R)$ of all matrices with (i, j)th entry equal to 0 if $b_{ij} = 0$ (i.e. if $i \not c_B\, j$), forms a ring, called a *structural matrix ring*.

The main result of this section is Theorem 2.4, which provides a general application of the internal characterization of structural matrix rings in [13].

Let $e := [e_{st}] \in \mathbb{M}_n(R)$, $n \geqslant 2$, with $e_{st} \in \{-1, 0, 1\}$ for all $s, t \in \{1, 2, \ldots, n\}$. Suppose further that $e_{l(e)m(e)} = 1$ for some $l(e)$ and $m(e)$, with $1 \leqslant l(e), m(e) \leqslant n$, and fix any such pair $(l(e), m(e))$. Assume also that

$$e_{st} = e_{sm(e)} e_{l(e)t} \tag{1}$$

for all $s, t \in \{1, 2, \ldots, n\}$. This implies that $e_{s*} = e_{sm(e)} e_{l(e)*}$ and $e_{*t} = e_{l(e)t} e_{*m(e)}$, where e_{s*} and e_{*t} denote the sth row and tth column of e respectively (see [7, Definition 1.2.2]). Let $f \in \mathbb{M}_n(R)$ be of the above form too, where $(l(f), m(f))$ again denotes the fixed pair for which $f_{l(f)m(f)} = 1$. We use E_{st} to denote the usual matrix unit, i.e. the matrix with 1 in position (s, t) and zeros elsewhere.

LEMMA 2.1. *$exf = (\sum_{u, v=1}^{n} e_{l(e)u} f_{vm(f)} x_{uv}) \sum_{s, t=1}^{n} e_{sm(e)} f_{l(f)t} E_{st}$ for every $x \in \mathbb{M}_n(R)$.*

Proof. Denoting e, x, and f by

$$\left[\begin{array}{c} e_{1*} \\ \hline \vdots \\ \hline e_{n*} \end{array}\right], \qquad \left[x_{*1} \mid \cdots \mid x_{*n}\right] \quad \text{and} \quad \left[f_{*1} \mid \cdots \mid f_{*n}\right]$$

respectively, it follows from (1) that the (s, t)th entry of $(ex)f$ is $e_{sm(e)}[e_{l(e)*} \cdot x_{*1} \; \cdots \; e_{l(e)*} \cdot x_{*n}] \cdot f_{l(f)t} f_{*m(f)}$, where $\cdot$ denotes the usual inner product. Since $(e_{l(e)*} \cdot x_{*v})f_{vm(f)} = (\sum_{u=1}^{n} e_{l(e)u} x_{uv}) f_{vm(f)}$ for every $v \in \{1, 2, \ldots, n\}$, and since $e_{sm(e)}, f_{l(f)t}, f_{vm(f)} \in \{-1, 0, 1\}$, the (s, t)th entry of exf is

$$\left[\sum_{v=1}^{n} \left(\sum_{u=1}^{n} e_{l(e)u} f_{vm(f)} x_{uv}\right)\right] e_{sm(e)} f_{l(f)t},$$

from which the result follows. ■

Denoting the trace of x by $\operatorname{tr} x$, it follows from Lemma 2.1 and (1) that

COROLLARY 2.2. *$exe = (\operatorname{tr} ex)e$ for every $x \in \mathbb{M}_n(R)$.*

Recall from [13, Definition 2.1] that a *set of matrix units in* a ring $\mathscr{S}$ *associated with* c_B is a subset $\{e^{(ij)} : i\, c_B\, j\}$ of $\mathscr{S}$, for some quasiorder relation c_B on $\{1, 2, \ldots, m\}$ (for some m), such that

$$\sum_{i=1}^{m} e^{(ii)} = 1, \qquad e^{(ij)} e^{(jk)} = e^{(ik)}, \quad \text{and} \quad e^{(ij)} e^{(j'k)} = 0$$

for all i, j, j', k with $i\, c_B\, j\, c_B\, k$, $j'\, c_B\, k$, and $j \neq j'$. (Since the $e^{(ij)}$'s will henceforth be matrices, we preserve, as usual, the subscripts to indicate the positions in such an $e^{(ij)}$.) For the sake of simplicity of notation we write $e^{(i)}$ instead of $e^{(ii)}$.

Let $\mathscr{R}$ be any subring of $\mathbb{M}_n(R)$ [which, by assumption, contains the identity element of $\mathbb{M}_n(R)$]. Assume that we can find p orthogonal idempotents $e^{(1)}, e^{(2)}, \ldots, e^{(p)}$, $2 \leqslant p \leqslant n$, in $\mathscr{R}$ of the form described in the paragraph preceding Lemma 2.1, with sum the identity element of $\mathbb{M}_n(R)$, such that $e^{(i)} \mathscr{R} e^{(i)} = R e^{(i)}$ for every $i \in \{1, 2, \ldots, p\}$. (Two remarks are in order at this stage. First, note that by Corollary 2.2 $e^{(i)} \mathscr{R} e^{(i)} \subseteq R e^{(i)}$. Second, if R happens to be a field, then, since every $e^{(i)}$ is a rank 1 matrix, and since the $e^{(i)}$'s sum to the identity matrix, we necessarily have $p = n$.) Lemma 2.1 leads us to defining the relation c_B on $\{1, 2, \ldots, p\}$ by

$$i\, c_B\, j \quad :\Leftrightarrow \quad S_{ij} \neq \{0\}, \tag{2}$$

where

$$S_{ij} := \left\{ \sum_{u, v=1}^{n} e^{(i)}_{l(e^{(i)})u} e^{(j)}_{vm(e^{(j)})} x_{uv} : x = [x_{uv}] \in \mathscr{R} \right\}. \tag{3}$$

If $i\, c_B\, j$, then, again encouraged by Lemma 2.1, we set

$$e^{(ij)} := \sum_{s,t=1}^{n} e^{(i)}_{sm(e^{(i)})} e^{(j)}_{l(e^{(j)})t} E_{st} \tag{4}$$

(Note that $e^{(ij)} \neq \{0\}$, since by assumption $e^{(i)}_{l(e^{(i)})m(e^{(i)})} e^{(j)}_{l(e^{(j)})m(e^{(j)})} = 1$.) Assume further that if $S_{ij} \neq \{0\}$, then $S_{ij} = R$. Thus, keeping [13, Theorem 2.9] in mind, we have

(i) by (4), $re^{(ij)} = e^{(ij)}r$ for every $r \in R$ and all i and j such that $i\, c_B\, j$;

(ii) from Lemma 2.1, (2)–(4), and the assumption that $S_{ij} = R$ if $S_{ij} \neq \{0\}$, it follows that $e^{(i)}\mathscr{R}e^{(j)} = Re^{(ij)}$ for all i and j such that $i\, c_B\, j$;

(iii) by Lemma 2.1 and (2)–(3), $e^{(j)}\mathscr{R}e^{(i)} = \{0\}$ if $j \not c_B\, i$;

(iv) if $r \in R$, $i\, c_B\, j$, and $re^{(ij)} = 0$, then by (4), $r = 0$.

Furthermore we claim that

Lemma 2.3. *c_B is a quasiorder relation on $\{1, 2, \ldots, p\}$, and $\{e^{(ij)} : i\, c_B\, j\}$ is a set of matrix units in $\mathscr{R}$ associated with c_B.*

Proof. Let $i, j, k, q \in \{1, 2, \ldots, p\}$. Since $e^{(i)} \in e^{(i)}\mathscr{R}e^{(i)}$, it follows that $S_{ii} \neq \{0\}$, and so $i\, c_B\, i$. Next, let $i\, c_B\, j$ and $k\, c_B\, q$. Then by (4),

$$\begin{aligned} e^{(ij)}e^{(kq)} &= \left(\sum_{s,t=1}^{n} e^{(i)}_{sm(e^{(i)})} e^{(j)}_{l(e^{(j)})t} E_{st} \right) \left(\sum_{y,z=1}^{n} e^{(k)}_{ym(e^{(k)})} e^{(q)}_{l(e^{(q)})z} E_{yz} \right) \\ &= \sum_{s,t=1}^{n} \left(e^{(i)}_{sm(e^{(i)})} e^{(j)}_{l(e^{(j)})t} \sum_{z=1}^{n} e^{(k)}_{tm(e^{(k)})} e^{(q)}_{l(e^{(q)})z} E_{sz} \right) \\ &= \left(\sum_{t=1}^{n} e^{(j)}_{l(e^{(j)})t} e^{(k)}_{tm(e^{(k)})} \right) \left(\sum_{s=1}^{n} \sum_{z=1}^{n} e^{(i)}_{sm(e^{(i)})} e^{(q)}_{l(e^{(q)})z} E_{sz} \right). \end{aligned}$$

If $j = k$, then $\sum_{t=1}^{n} e^{(j)}_{l(e^{(j)})t} e^{(k)}_{tm(e^{(k)})}$ is the $\big(l(e^{(j)}), m(e^{(j)})\big)$th entry of $e^{(j)}e^{(j)}$, which is 1, since $e^{(j)}$ is an idempotent. Hence

$$e^{(ij)}e^{(jq)} = \sum_{s,z=1}^{n} e^{(i)}_{sm(e^{(i)})} e^{(q)}_{l(e^{(q)})z} E_{sz}. \tag{5}$$

Since $i c_B j$ and $j c_B q$, we have by assumption $S_{ij} = R$ and $S_{jq} = R$, and so by Lemma 2.1,

$$R \sum_{s,z=1}^{n} e^{(i)}_{sm(e^{(i)})} e^{(q)}_{l(e^{(q)})z} E_{sz} \supseteq e^{(i)} \mathscr{R} e^{(q)}$$

$$\supseteq \left(e^{(i)}\mathscr{R}e^{(j)}\right)\left(e^{(j)}\mathscr{R}e^{(q)}\right)$$

$$= Re^{(ij)}Re^{(jq)}$$

$$= R \sum_{s,z=1}^{n} e^{(i)}_{sm(e^{(i)})} e^{(q)}_{l(e^{(q)})z} E_{sz},$$

where the last equality follows from (5). Hence $e^{(i)}\mathscr{R}e^{(q)} = R\sum_{s,z=1}^{n} e^{(i)}_{sm(e^{(i)})} e^{(q)}_{l(e^{(q)})z} E_{sz}$. Lemma 2.1 and (3) now imply that $S_{iq} = R$. Consequently by (2), $i c_B q$, and so by (4) and (5), $e^{(ij)}e^{(jq)} = e^{(iq)}$.

Finally, if $j \neq k$, then $\sum_{t=1}^{n} e^{(j)}_{l(e^{(j)})t} e^{(k)}_{tm(e^{(k)})} = 0$, since the $\left(l(e^{(j)}), (e^{(k)})\right)$th entry of $e^{(j)}e^{(k)}$ is 0, $e^{(j)}$ and $e^{(k)}$ being orthogonal. Hence $e^{(ij)}e^{(kq)} = 0$. ■

Taking (i)–(iv) above and Lemma 2.3 into account, we conclude that the conditions in [13, Theorem 2.9] are satisfied, with $\mathscr{R}$ containing the subring $Re^{(1)}$ $(= e^{(1)}\mathscr{R}e^{(1)})$, which is ring isomorphic to R via $re^{(1)} \mapsto r$. Now we invoke the proof of [13, Theorems 2.6]. First, $Re^{(1)}$ is ring isomorphic to $Re^{(i)}$ via $\psi_i : re^{(1)} \mapsto re^{(i)}$, and, for all i and j such that $i c_B j$, $Re^{(i)}$ is ismorphic to $Re^{(ij)}$ as right $Re^{(i)}$-modules via $\varphi_{ij} : re^{(i)} \mapsto re^{(ij)}$. [Here $Re^{(ij)}$ is considered as a right $Re^{(i)}$-module via $*_{ij}$ defined by $re^{(ij)} *_{ij} se^{(i)} = re^{(ij)}\psi_j\left(\psi_i^{-1}(se^{(i)})\right) = re^{(ij)}se^{(j)} = rse^{(ij)}$.] From Lemma 2.1 and (2)–(4) we have

$$x = \left(\sum_{i=1}^{p} e^{(i)}\right) x \left(\sum_{j=1}^{p} e^{(j)}\right) = \sum_{i=1}^{p}\left(\sum_{\substack{j \\ i c_B j}} \left[\left(\sum_{u,v=1}^{n} e^{(i)}_{l(e^{(i)})u} e^{(j)}_{vm(e^{(j)})} x_{uv}\right) e^{(ij)}\right]\right)$$

$$= \sum_{i=1}^{p}\left[\sum_{\substack{j \\ i c_B j}} \varphi_{ij}\left(\psi_i\left(\left(\sum_{u,v=1}^{n} e^{(i)}_{l(e^{(i)})u} e^{(j)}_{vm(e^{(j)})} x_{uv}\right) e^{(1)}\right)\right)\right]$$

for every $x \in \mathscr{R}$. Hence the proof of [13, Theorem 2.6], in particular [13, (3) and the subsequent definition of Θ], implies that $\mathscr{R}$ is (ring) isomorphic to the structural matrix ring $\mathbb{M}_p(B, R)$ via

$$[x_{st}] \mapsto \sum_{\substack{i,j \\ i c_B j}} \left(\sum_{u,v=1}^{n} e^{(i)}_{l(e^{(i)})u} e^{(j)}_{vm(e^{(j)})} x_{uv} \right) E_{ij}. \tag{6}$$

We summarize the foregoing results in

THEOREM 2.4. *Let $\mathscr{R}$ be a subring of $\mathbb{M}_n(R)$ containing p orthogonal idempotents $e^{(1)}, e^{(2)}, \ldots, e^{(p)}$, $2 \leqslant p \leqslant n$, with sum the identity element of $\mathbb{M}_n(R)$, such that the following conditions hold for every $i \in \{1, 2, \ldots, p\}$:*

(i) *$e^{(i)}_{st} \in \{-1, 0, 1\}$ for all $s, t \in \{1, 2, \ldots, n\}$;*
(ii) *$e^{(i)}_{l(e^{(i)})m(e^{(i)})} = 1$ for some fixed pair $(l(e^{(i)}), m(e^{(i)}))$;*
(iii) *$e^{(i)}_{st} = e^{(i)}_{sm(e^{(i)})} e^{(i)}_{l(e^{(i)})t}$ for all $s, t \in \{1, 2, \ldots, n\}$.*

Then $e^{(i)}\mathscr{R}e^{(j)} = S_{ij}\sum_{s,t=1}^{n} e^{(i)}_{sm(e^{(i)})} e^{(j)}_{l(e^{(j)})t} E_{st}$ for all $i, j \in \{1, 2, \ldots, p\}$, where

$$S_{ij} := \left\{ \sum_{u,v=1}^{n} e^{(i)}_{l(e^{(i)})u} e^{(j)}_{vm(e^{(j)})} x_{uv} : x = [x_{uv}] \in \mathscr{R} \right\}.$$

Define the relation c_B on the set $\{1, 2, \ldots, p\}$ by $i c_B j :\Leftrightarrow S_{ij} \neq \{0\}$, and assume that $S_{ij} = R$ if $S_{ij} \neq \{0\}$. Then c_B is a quasiorder relation on $\{1, 2, \ldots, p\}$, and if B is the Boolean matrix determined by c_B, then $\mathscr{R}$ is isomorphic to the structural matrix ring $\mathbb{M}_p(B, R)$ via

$$[x_{st}] \mapsto \sum_{\substack{i,j \\ i c_B j}} \left(\sum_{u,v=1}^{n} e^{(i)}_{l(e^{(i)})u} e^{(j)}_{vm(e^{(j)})} x_{uv} \right) E_{ij}.$$

3. MATRIX RINGS SATISFYING COLUMN SUM CONDITIONS

A subring $\mathscr{S}$ of $\mathbb{M}_n(R)$, $n \geqslant 2$, is said to *satisfy a column sum condition* if for some $j_1, j_2, \ldots, j_k$, with $1 \leqslant j_1, j_2, \ldots, j_k \leqslant n$, $2 \leqslant k \leqslant n$, we have

$$\sum_{i=1}^{n} x_{ij_1} = \sum_{i=1}^{n} x_{ij_2} = \cdots = \sum_{i=1}^{n} x_{ij_k}$$

for every matrix $x = [x_{st}] \in \mathscr{S}$.

We call a subring of the complete blocked triangular matrix ring

$$\begin{bmatrix} \mathbb{M}_{n_1}(R) & \mathbb{M}_{n_1\times n_2}(R) & \cdots & \mathbb{M}_{n_1\times n_l}(R) \\ 0 & \mathbb{M}_{n_2}(R) & \cdots & \mathbb{M}_{n_2\times n_l}(R) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \mathbb{M}_{n_l}(R) \end{bmatrix}$$

of the form

$$\begin{bmatrix} \mathbb{M}_{n_1}(R) & \mathbb{M}_{n_1\times n_2}(T_{12}) & \cdots & \mathbb{M}_{n_1\times n_l}(T_{1l}) \\ 0 & \mathbb{M}_{n_2}(R) & \cdots & \mathbb{M}_{n_2\times n_l}(T_{2l}) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \mathbb{M}_{n_l}(R) \end{bmatrix},$$

where, for all $i, j \in \{1, 2, \ldots, l\}$, $T_{ij} = \{0\}$ or R, a *blocked triangular matrix ring*. Here $n_1 + n_2 + \cdots + n_l = n$. (In [9] Li and Zelmanowitz characterized Artinian rings with certain restricted primeness conditions as complete blocked triangular matrix rings over division rings.) Every blocked triangular matrix ring is by defintiion a structural matrix ring, and every structural matrix ring $\mathbb{M}_n(B, R)$ is isomorphic to a blocked triangular matrix ring over R (see [13, pp. 5–6]). Therefore, without loss of generality, we restrict ourselves to blocked triangular matrix rings.

In Theorem 3.1 we invoke Theorem 2.4 and obtain an isomorphism between an arbitrary blocked triangular matrix ring $\mathbb{M}_n(B, R)$ (over an arbitrary ring R) and a subring $\mathscr{R}_B$ of $\mathbb{M}_n(R)$, satisfying the following conditions, such that $\mathbb{M}_n(B, R)$ and $\mathscr{R}_B$ are isomorphic:

(a) a column sum condition;

(b) if $1 \leqslant i \leqslant n$ and $b_{in} = 1$, then $\pi_{ij}(\mathscr{R}_B) = R$ for every $j \in \{1, 2, \ldots, n\}$.

Here π_{ij} denotes the (i, j)th projection map. Condition (b) above implies that if $\mathbb{M}_n(B, R)$ is a complete blocked triangular matrix ring, then π_{ij} is onto for all $i, j \in \{1, 2, \ldots, n\}$, since in this case $b_{ij} = 1$ for every i. In general condition (b) ensures that the matrices in $\mathscr{R}_B$ do not have an excess of zeros.

On the other hand, in Example 3.2 we shall exhibit a subring $\mathscr{R}$ of $\mathbb{M}_3(F)$, F a field, satisfying conditions (a) and (b) above for every reflexive and transitive 3×3 Boolean matrix B, which is not isomorphic to a structural matrix ring.

Let $\mathbb{M}_n(B, R)$ be a blocked triangular matrix ring, and let $\mathscr{R}_B$ be the set of all matrices $x = [x_{st}]$ in $\mathbb{M}_n(R)$ such that

(c) if $1 \leqslant i \leqslant n-1$ and $b_{in} = 0$, then $x_{ij} = 0$ for every $j \in \{1, 2, \ldots, n\}$ such that $b_{ij} = 0$;

(d) if $1 \leqslant i, j \leqslant n-1$, $b_{ij} = 0$, and $b_{in} = 1$, then $x_{ij} = x_{in}$;

(e) if $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n-1$, $b_{ij} = 0$, and $b_{in} = 1$, then $\sum_{s=1}^{n} x_{sj} = \sum_{s=1}^{n} x_{sn}$.

Tedious verification shows that $\mathscr{R}_B$ is a ring. The reason for emphasizing the last column is the assertion that if $b_{nj} = 1$ for some $j \in \{1, 2, \ldots, n-1\}$, then the jth column and nth column of B are equal, i.e., for every $k \in \{1, 2, \ldots, n\}$, $b_{kj} = 1$ if and only if $b_{kn} = 1$. Indeed, the (n, j)th position is below the main diagonal, and so if $b_{nj} = 1$, then, as we deal with blocks, the block with vertices at positions (j, j), (j, n), (n, j), and (n, n) has 1's everywhere. The assertion now follows easily, since B is transitive.

We are now in a position to use Theorem 2.4 in order to obtain an isomorphism between two seemingly different subrings of $\mathbb{M}_n(R)$.

THEOREM 3.1. *Let $\mathbb{M}_n(B, R)$ be a blocked triangular matrix ring. Then $\mathbb{M}_n(B, R)$ and $\mathscr{R}_B$ are isomorphic.*

Proof. For $i = 1, 2, \ldots, n$, set

$$e^{(i)} := \begin{cases} E_{ii} - E_{ni} & \text{if } 1 \leqslant i \leqslant n-1,\ b_{ki} = 0, \text{ and} \\ & \quad b_{kn} = 1 \text{ for some } k, \\ E_{nn} + \displaystyle\sum_{\substack{j \\ b_{kj}=0 \text{ and } b_{kn}=1 \text{ for some } k}} E_{nj} & \text{if } i = n, \\ E_{ii} & \text{otherwise.} \end{cases} \tag{7}$$

By the definition of $\mathscr{R}_B$ the $e^{(i)}$'s are in $\mathscr{R}_B$, and they are idempotents with sum the identity of $\mathbb{M}_n(R)$.

Now we show that the $e^{(i)}$'s are mutually orthogonal. Let $1 \leqslant i \leqslant n-1$, and suppose that $b_{ki} = 0$ and $b_{kn} = 1$ for some k. Since i is one of the j's and n is not one of the j's in the sum below, $-E_{ni}$ and E_{ni} are the only nonzero terms in the product

$$\left(E_{nn} + \sum_{\substack{j \\ b_{kj}=0 \text{ and } b_{kn}=1 \text{ for some } k}} E_{nj}\right)(E_{ii} - E_{ni}),$$

and so $e^{(n)}e^{(i)} = 0$. This argument serves to a large extent to showing that $e^{(1)}, \dots, e^{(n)}$ are mutually orthogonal idempotents.

Next, setting $(l(e^{(i)}), m(e^{(i)})) = (i, i)$ for every $i \in \{1, 2, \dots, n\}$, conditions (i)–(iii) in Theorem 2.4 are satisfied.

By (3) we have

$$S_{ij} = \left\{ \sum_{u,v=1}^{n} e_{iu}^{(i)} e_{vj}^{(j)} x_{uv} : x = [x_{uv}] \in \mathscr{R}_B \right\}. \tag{8}$$

We show that $S_{ij} = \{0\}$ if $b_{ij} = 0$, and $S_{ij} = R$ if $b_{ij} = 1$. This will imply that the $n \times n$ Boolean matrix $\overline{B}$ determined by the relation $c_{\overline{B}}$, defined on $\{1, 2, \dots, n\}$ by $i \, c_{\overline{B}} \, j :\Leftrightarrow S_{ij} = R$, is simply B. Suppose that

I. $b_{ij} = 0$, $1 \leqslant i, j \leqslant n - 1$: If $b_{in} = 1$, then by (7), $e^{(j)} = E_{jj} - E_{nj}$, and so by (8), $S_{ij} = \{x_{ij} - b_{in} : x \in \mathscr{R}_B\}$, since the only nonzero entry of $e^{(i)}$ in row i is 1 in position (i, i). Hence by (d), $S_{ij} = \{0\}$. If $b_{in} = 0$, then $S_{ij} = \{x_{ij} - x_{in} : x \in \mathscr{R}_B\} = \{0\}$ or $S_{ij} = \{x_{ij} : x \in \mathscr{R}_B\}$, depending on whether there is a k such that $b_{kj} = 0$ and $b_{kn} = 1$. Hence by (c), $S_{ij} = \{0\}$.

II. $b_{in} = 0$, $1 \leqslant i \leqslant n - 1$: The only nonzero entry of $e^{(n)}$ in column n is 1 in position (n, n), and so $S_{in} = \{x_{in} : x \in \mathscr{R}_B\}$. Consequently, by (c), $S_{ij} = \{0\}$.

III. $b_{nj} = 0$, $1 \leqslant j \leqslant n - 1$: It follows from (7) that $e^{(j)} = E_{jj} - E_{nj}$, and so for $x \in \mathscr{R}_B$,

$$\sum_{u,v=1}^{n} e_{nu}^{(n)} e_{vj}^{(j)} x_{uv} = x_{nj} - x_{nn} + \sum_{\substack{u \\ b_{ku}=0 \text{ and } b_{kn}=1 \text{ for some } k}} (x_{uj} - x_{un}).$$

If $\bar{u} \neq n$ and $\bar{u}$ is not one of the u's in the above summation on the right hand side, then $b_{n\bar{u}} = 1$, since $b_{nn} = 1$. Hence $b_{\bar{u}j} = 0$ (otherwise $b_{nj} = 1$, a contradiction to the hypothesis). By (c) and (d) we have $x_{\bar{u}j} = x_{\bar{u}n}$, irrespective of whether $b_{\bar{u}n} = 0$ or $b_{\bar{u}n} = 1$. Consequently, if we add $x_{\bar{u}j} - x_{\bar{u}n}$ to the right hand side of the above equality for all the mentioned $\bar{u}$'s, then we obtain from (e) that $\sum_{u,v=1}^{n} e_{nu}^{(n)} e_{vj}^{(j)} x_{uv} = \sum_{s=1}^{n} x_{sj} - \sum_{s=1}^{n} x_{sn} = 0$. We conclude from (8) that $S_{nj} = \{0\}$.

IV. $b_{ij} = 1,\ 1 \leqslant i \leqslant n-1,\ 1 \leqslant j \leqslant n$: Then

$$S_{ij} = \begin{cases} \{x_{ij} - x_{in} : x \in \mathscr{R}_B\} & \text{if } \ b_{kj} = 0 \text{ and } b_{kn} = 1 \text{ for some } k; \\ \{x_{ij} : x \in \mathscr{R}_B\} & \text{otherwise.} \end{cases} \tag{9}$$

Since $b_{ij} = 1$, it follows from the definition of $\mathscr{R}_B$ that $r(E_{ij} - E_{nj}) \in \mathscr{R}_B$ for every $r \in R$. Consequently, if $b_{kj} = 0$ and $b_{kn} = 1$ for some k, then $S_{ij} = R$. Next, if $1 \leqslant j \leqslant n-1$ and $b_{ln} = 0$ for every l such that $b_{lj} = 0$, then there is no restriction on the sum of the entries in column j of the matrices in $\mathscr{R}_B$. Therefore $rE_{ij} \in \mathscr{R}_B$ for every $r \in R$, and so $S_{ij} = R$ in this case too. Finally, if $j = n$, then by (9), $S_{in} = \{x_{in} : x \in \mathscr{R}_B\}$. Since by (d) and (e)

$$rE_{in} + \sum_{\substack{u \\ b_{ku}=0 \text{ and } b_{kn}=1 \text{ for some } k}} rE_{iu} \in \mathscr{R}_B,$$

it follows that $S_{in} = R$.

V. $b_{nj} = 1,\ 1 \leqslant j \leqslant n$: By transitivity $b_{ln} = 0$ for every l such that $b_{lj} = 0$. Hence by (7),

$$e^{(j)} := \begin{cases} E_{nn} + \displaystyle\sum_{\substack{u \\ b_{ku}=0 \text{ and } b_{kn}=1 \text{ for some } k}} E_{nj} & \text{if } \ j = n; \\ E_{jj} & \text{if } j \neq n. \end{cases} \tag{10}$$

Therefore, by (8) and (10),

$$S_{nj} = \left\{ x_{nj} + \sum_{\substack{u \\ b_{ku}=0 \text{ and } b_{kn}=1 \text{ for some } k}} x_{uj} : x \in \mathscr{R}_B \right\}.$$

Since $\sum_{s=1}^{n} rE_{ns} \in \mathscr{R}_B$ for every $r \in R$, it follows that $S_{nj} = R$.

Invoking (IV) and (V), we conclude from Theorem 2.4 that $\mathscr{R}_B \cong \mathbb{M}_n(B, R)$ via

$$[x_{st}] \mapsto \sum_{i=1}^{n-1} \left(\sum_{\substack{j \\ b_{ij}=1 \\ b_{kj}=0 \text{ and } b_{kn}=1 \text{ for some } k}} (x_{ij} - x_{in}) E_{ij} \right)$$

$$+ \sum_{i=1}^{n-1} \left(\sum_{\substack{j \\ b_{ij}=1 \\ b_{ln}=0 \text{ for every } l \text{ such that } b_{lj}=0}} x_{ij} E_{ij} \right)$$

$$+ \sum_{\substack{j \\ b_{nj}=1}} \left(x_{nj} + \sum_{\substack{u \\ b_{ku}=0 \text{ and } b_{kn}=1 \text{ for some } k}} x_{uj} \right) E_{nj}. \qquad \blacksquare$$

Theorem 3.1 can also be proved by conjugation by an invertible matrix, viz., the restriction of the inner automorphism $x \mapsto y^{-1}xy$ of $\mathbb{M}_n(R)$ to $\mathbb{M}_n(B, R)$ has $\mathscr{R}_B$ as image, where $y := I + E_{n1} + E_{n2} + \cdots + E_{n\,n-1}$, with I denoting the identity matrix.

In conclusion we mention that there are subrings of $\mathbb{M}_n(R)$ satisfying conditions (a) and (b) mentioned at the outset of this section, which are not isomorphic to structural matrix rings:

EXAMPLE 3.2. Let F be a field. It can be shown that every set, with cardinality at least 2, of mutually orthogonal idempotents in

$$\mathscr{R} := \left\{ \begin{bmatrix} a & b & b \\ c & d & c \\ e & e & f \end{bmatrix} \in \mathbb{M}_3(F) : a + c + e = b + d + e = b + c + f \right\},$$

such that the sum of its elements equals the identity element of $\mathbb{M}_3(F)$, has cardinality equal to 2. (Here idempotent means nonzero idempotent.) Furthermore, if $\{g^{(11)}, g^{(22)}\}$ is any such set, then $g^{(ii)}\mathscr{R}g^{(ii)}$ can be shown to be ring isomorphic to F for every i. Hence, if $\mathscr{R}$ is isomorphic to a structural matrix ring, then by [13, Theorem 2.6(i) and Proposition 2.8] the only candidates for such structural matrix rings (up to isomorphism) are

$$\begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix}, \quad \begin{bmatrix} F & F \\ F & F \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}, \tag{11}$$

depending on whether $\{g^{(11)}, g^{(22)}\}$, $\{g^{(11)}, g^{(12)}, g^{(21)}, g^{(22)}\}$, or $\{g^{(11)}, g^{(12)}, g^{(22)}\}$ is used as a set of matrix units in $\mathscr{R}$ (for some $g^{(12)}, g^{(21)} \in \mathscr{R}$) associated with the quasi-order relation $c_B = \{(1,1),(2,2)\}$, $\{(1,1),(1,2),(2,1),(2,2)\}$, or $\{(1,1),(1,2),(2,2)\}$, respectively, on the set $\{1,2\}$. If $i \neq j$, then it can be shown that $g^{(ii)}\mathscr{R}g^{(jj)}$ is isomorphic as F-module to $F \oplus F$, whereas $g^{(jj)}\mathscr{R}g^{(ii)} = \{0\}$, for an appropriate choice of the superscripts. Having chosen, without loss of generality,

$$\begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$$

as a candidate—and not

$$\begin{bmatrix} F & 0 \\ F & F \end{bmatrix}$$

—we may suppose that $g^{(11)}\mathscr{R}g^{(22)}$ is isomorphic as F-module to $F \oplus F$, whereas $g^{(22)}\mathscr{R}g^{(11)} = \{0\}$. Since $g^{(11)}\mathscr{R}g^{(22)} \neq \{0\}$, it follows from [13, Theorem 2.6(iv) and Proposition 2.8] that $1 \; c_B \; 2$, and so, firstly,

$$\begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix}$$

is immediately ruled out, and secondly, since F and $F \oplus F$ are not isomorphic as F-modules,

$$\begin{bmatrix} F & F \\ F & F \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$$

are ruled out by [13, Theorem 2.6(ii) and Proposition 2.8].

We conclude that there is no set, with cardinality at least 2, of mutually orthogonal idempotents in $\mathscr{R}(n)$ with sum equal to $1_{\mathscr{R}(n)}$, such that the conditions in [13, Theorem 2.6] are satisfied, and so [13, Proposition 2.8] implies that $\mathscr{R}$ is not isomorphic to a structural matrix ring.

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