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Lie solvability in matrix algebras

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ABSTRACT

If an algebra \mathcal{A} satisfies the polynomial identity

$$[x_1, y_1][x_2, y_2] \cdots [x_{2^m}, y_{2^m}] = 0$$

(for short, \mathcal{A} is D_{2^m}), then \mathcal{A} is trivially Lie solvable of index $m + 1$ (for short, \mathcal{A} is Ls_{m+1}). We prove that the converse holds for subalgebras of the upper triangular matrix algebra $U_n(R)$, R any commutative ring, and $n \geq 1$. We also prove that if a ring S is D_2 (respectively, Ls_2), then the subring $U_m^*(S)$ of $U_m(S)$ comprising the upper triangular $m \times m$ matrices with constant main diagonal, is $D_{2^{\lceil \log_2 m \rceil}}$ (respectively, $Ls_{\lceil \log_2 m \rceil + 1}$) for all $m \geq 2$. We also study two related questions, namely whether, for a field F , an Ls_2 subalgebra of $M_n(F)$, for some n , with (F) -dimension larger than the maximum dimension $2 + \lfloor \frac{3n^2}{8} \rfloor$ of a D_2 subalgebra of $M_n(F)$, exists, and whether a D_2 subalgebra of $U_n(F)$ with (the mentioned) maximum dimension, other than the typical D_2 subalgebras of $U_n(F)$ with maximum dimension, which were described by Domokos and refined by van Wyk and Ziembowski, exists. Partial results with regard to these two questions are obtained.

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1. Notation

Throughout this paper, R denotes a (not necessarily commutative) ring with identity 1, F denotes a field, and $[x, y] = xy - yx$ denotes the additive commutator of elements $x, y \in R$.

For $n \geq 1$ we use $M_n(R)$ for the full matrix ring of all $n \times n$ matrices over R , $U_n(R)$ for the subring of $M_n(R)$ comprising all the upper triangular matrices, and $U_n^*(R)$ for the subring of $U_n(R)$ consisting of all the matrices in $U_n(R)$ with constant main diagonal, i.e. $A_{1,1} = \cdots = A_{n,n}$ for every $A \in U_n^*(R)$, where $X_{i,j}$ (or $(X)_{i,j}$ in case of possible confusion) denotes the entry of a matrix X in $M_n(R)$ in position (i, j) . The customary notation I_n (or simply I in case of no ambiguity) for the $n \times n$ identity matrix, and $e_{i,j}$ for the (i, j) -th matrix unit, i.e. the matrix with 1 in position (i, j) and zeroes elsewhere, will be used.

If R is commutative, then $M_n(R)$ is an R -algebra, and the mentioned subrings are R -subalgebras.

2. Introduction

Define inductively the Lie central and Lie derived series of a ring R as follows:

$$\mathfrak{C}^0(R) := R, \mathfrak{C}^{q+1}(R) := [\mathfrak{C}^q(R), R] \text{ (central series),} \tag{1}$$

and

$$\mathfrak{D}^0(R) := R, \mathfrak{D}^{q+1}(R) := [\mathfrak{D}^q(R), \mathfrak{D}^q(R)] \text{ (derived series).} \tag{2}$$

We say that R is Lie nilpotent (respectively, Lie solvable) of index q (for short, R is Ln_q ; respectively, R is Ls_q) if $\mathfrak{C}^q(R) = 0$ (respectively, $\mathfrak{D}^q(R) = 0$). It is evident from (1) and (2) that R is Ln_q or Ls_q if and only if R satisfies the corresponding (polynomial) identity (PI), and that if R is Ln_q , then R is Ls_q , or, for short, $\text{Ln}_q \Rightarrow \text{Ls}_q$; in particular,

$$\text{Ln}_2 \Rightarrow \text{Ls}_2. \tag{3}$$

The PI

$$[x_1, y_1][x_2, y_2] \cdots [x_q, y_q] = 0 \tag{4}$$

features prominently in numerous papers, e.g. [1–4]. Mal’sev proved in [1] that all the polynomial identities of $U_q(F)$ are consequences of only one identity, namely the identity in (4). For an explicit form of a finite set of generators of an ideal of identities of the algebra $U_q^*(R)$ over a commutative integral domain R , see [5].

If a ring R satisfies the identity in (4) for some $q \geq 1$, then we say that R is D_q . (We opted for the letter ‘D’ (in ‘ D_q ’), since this identity was studied extensively by Domokos in [3].)

Note that if a ring R is Ln_q (respectively, Ls_q or D_q) for some q , then R is $\text{Ln}_{q'}$ (respectively, $\text{Ls}_{q'}$ or $\text{D}_{q'}$) for all q' such that $q' > q$.

It follows trivially that

$$\text{D}_{2^m} \Rightarrow \text{Ls}_{m+1}; \tag{5}$$

in particular,

$$\text{D}_2 \Rightarrow \text{Ls}_2. \tag{6}$$

Although, in general, $\text{Ls}_2 \not\Rightarrow \text{D}_2$, it was shown in [6, Theorem 4] that the implication $\text{Ls}_2 \Rightarrow \text{D}_2$ does hold for any block triangular structural matrix subalgebra of $M_n(R)$, R any commutative ring. Moreover (see [6, Corollary 6]), this implication also holds for any R -subalgebra of $U_n(R)$, R any commutative ring, with structural Jacobson radical. In Section 3 we will prove in Theorem 4 that, for all $m, n \geq 1$, every Ls_{m+1} structural matrix subalgebra of $U_n(R)$ is D_{2^m} . (See (5) in this regard.) In order to prove Theorem 4, we invoke the notion of the (s, t) -reduction of a (triangular) matrix.

In the light of (3) and (6), the R -subalgebra $U_3^*(U_3^*(R))$ of $U_9^*(R)$, R any commutative ring, was exhibited in [4, Corollary 2] as an example of a matrix algebra which is Ls_2 , but neither D_2 nor Ln_2 . This example has given rise to the directions of study pursued in Section 4.

To wit, apart from the mentioned example illustrating that, in general, $\text{Ls}_2 \not\Rightarrow \text{Ln}_2$, it was noted in [6] that the D_2 (and hence Ls_2) algebra $\mathcal{A} = U_4^*(F)$ (with dimension 7) is an example of a subalgebra of $M_n(F)$ with dimension larger than the maximum dimension $1 + \left\lfloor \frac{n^2}{3} \right\rfloor$ (which, for $n = 4$, is equal to 6) of an Ln_2 subalgebra of $M_n(F)$. In this vein,

also considering the fact that, in general, $Ls_2 \not\cong D_2$, one notes that, if R above is a field F , then the dimension of the mentioned algebra in [4], namely the subalgebra $U_3^*(U_3^*(F))$ of $U_9^*(F)$, is 16, while the maximum dimension of a D_2 subalgebra of $M_9(F)$ is $2 + \lfloor \frac{3 \cdot 9^2}{8} \rfloor = 32$ (see [2,3,6]).

These observations have led to the following analogous question (see [6, Problem 17]) in connection with the Ls_2 identity and the maximum dimension of a D_2 subalgebra of $M_n(F)$:

Question 1: *For a field F , does an Ls_2 F -subalgebra of $M_n(F)$ (for some n) with dimension larger than the maximum dimension $2 + \lfloor \frac{3n^2}{8} \rfloor$ of a D_2 F -subalgebra of $M_n(F)$ exist?*

A (possible) negative answer to the above question, in conjunction with a description of ‘typical’ such Ls_2 subalgebras $M_n(F)$, could perhaps pave the way to finding a sharp upper bound for the dimension of an Ls_2 subalgebra of $M_n(F)$.

It seems that a solution to Question 1 above will be facilitated by an answer to the question of whether a D_2 F -subalgebra of $U_n(F)$, for some n , with the mentioned maximum dimension, which is not a typical D_2 subalgebra (the type mentioned in [3] and refined in [6]; see also Section 4 in this regard) of $U_n(F)$ with maximum dimension, exists.

Partial results (see Theorems 15 and 16) with regard to these two questions are obtained in Section 4. In this section, we also prove the converse of [4, Theorem 2.1], which was an important tool in [4] in obtaining a matrix algebra which is Ls_2 , but neither D_2 nor Ln_2 , and we prove that if a ring S is D_2 (respectively, Ls_2), then $U_m^*(S)$ is $D_{2^{\lceil \log_2 m \rceil}}$ (respectively, $Ls_{\lceil \log_2 m \rceil + 1}$) for all $m \geq 2$.

3. Ls_{m+1} structural matrix subalgebras of $U_n(R)$ are D_{2^m}

In this section, we will consider structural R -subalgebras of $M_n(R)$, R a commutative ring, and R -subalgebras of $M_n(R)$ such that their Jacobson radicals are ‘structural’. The class of structural matrix rings or incidence rings has been studied extensively. See for example, [7–12].

A structural matrix ring over a (not necessarily commutative) ring R is a subring of full the matrix ring $M_n(R)$ consisting of all matrices having zero in certain prescribed positions and any elements of R in all the other positions. To be more precise, for a reflexive and transitive binary relation θ on the set $\{1, 2, \dots, n\}$, the structural matrix subring $M_n(\theta, R)$ of $M_n(R)$ is defined as follows:

$$M_n(\theta, R) = \{A \in M_n(R) \mid A_{ij} = 0 \text{ if } (i, j) \notin \theta\}.$$

Note that if, for any ordered pair (i, j) , there is a matrix A in a structural matrix ring $M_n(\theta, R)$ such that $A_{ij} \neq 0$, then $Re_{ij} \subseteq M_n(\theta, R)$, i.e.

$$\pi_{i,j}(M_n(\theta, R)) \neq \{0\} \implies Re_{ij} \subseteq M_n(\theta, R). \tag{7}$$

(Here $\pi_{i,j}$ is the natural projection onto the (i, j) -entry, and Re_{ij} is the set comprising all the matrices with any element of R in position (i, j) , and zeroes elsewhere.)

It can be shown (see [7, p.1386] or [10, p.5604]) that, for some k , there are positive integers n_1, \dots, n_k such that $n_1 + \dots + n_k = n$ and $M_n(\theta, R)$ is (isomorphic to) a block(ed)

triangular matrix ring

$$\begin{pmatrix} M_{n_1}(R) & M_{n_1 \times n_2}(X_{(1,2)}) & \cdots & M_{n_1 \times n_k}(X_{(1,k)}) \\ 0 & M_{n_2}(R) & \ddots & \vdots \\ \vdots & \ddots & \ddots & M_{n_{k-1} \times n_k}(X_{(k-1,k)}) \\ 0 & \cdots & 0 & M_{n_k}(R) \end{pmatrix},$$

where $X_{(i,j)} = \{0\}$ or $X_{(i,j)} = R$ for all i, j with $1 \leq i < j \leq k$. (See also [8].) By, e.g. [12, Theorem 2.7],

$$J(M_n(\theta, R)) \cong \begin{pmatrix} M_{n_1}(J(R)) & M_{n_1 \times n_2}(X_{(1,2)}) & \cdots & M_{n_1 \times n_k}(X_{(1,k)}) \\ 0 & M_{n_2}(J(R)) & \ddots & \vdots \\ \vdots & \ddots & \ddots & M_{n_{k-1} \times n_k}(X_{(k-1,k)}) \\ 0 & \cdots & 0 & M_{n_k}(J(R)) \end{pmatrix}, \tag{8}$$

implying that the quotient ring $M_n(\theta, R)/J(M_n(\theta, R))$ is (isomorphic to) a direct sum of full matrix rings:

$$M_n(\theta, R)/J(M_n(\theta, R)) \cong M_{n_1}(R/J(R)) \oplus \cdots \oplus M_{n_k}(R/J(R)). \tag{9}$$

At this point we note that complete block triangular matrix rings $M_n(\theta, F)$ over a field F (that is the case when $X_{(i,j)} = F$ for all i, j with $1 \leq i < j \leq k$ in the block triangular matrix ring $M_n(\theta, R)$ above) feature prominently in [13], where it is proved that $Id(M_n(\theta, F)) = Id(M_{n_1}(F)) \cdots Id(M_{n_k}(F))$. Here, $Id(\mathcal{A})$ denotes the set of all polynomial identities of \mathcal{A} (for an algebra \mathcal{A}), which is a two-sided ideal of the free (associative) algebra $F\langle X \rangle$ of polynomials in the non-commuting indeterminates $x \in X$ (for a set X).

In order to prove in Theorem 4 that, for all $m, n \geq 1$, every Ls_{m+1} structural matrix subring of $U_n(R)$, R a commutative ring, is D_{2^m} (see (5) in this regard), we invoke the notion of the (s, t) -reduction of a (triangular) matrix:

Let

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ 0 & A_{2,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{n-1,n} \\ 0 & \cdots & 0 & A_{n,n} \end{pmatrix} \in U_n(R),$$

and let $s, t \in \{1, \dots, n\}$, with $s < t$. We define the (s, t) -reduction of A as the following $n \times n$ matrix:

$$\text{red}_{s,t}A = \begin{pmatrix} & \begin{matrix} s \\ \downarrow \end{matrix} & & & \begin{matrix} t \\ \downarrow \end{matrix} \\ s \rightarrow & A_{s,s} & \cdots & & A_{s,t} \\ & & & \ddots & \vdots \\ & & & & A_{t,t} \\ t \rightarrow & & & & \end{pmatrix}.$$

Lemma 2: If $A, B \in U_n(R)$ are such that $[A, B]_{s,t} \neq 0$ for some s and t such that $s < t$, then

- (1) $[\text{red}_{s,t}A, \text{red}_{s,t}B]_{s,t} = [A, B]_{s,t} \neq 0$,
- (2) $[\text{red}_{s,t}A, \text{red}_{s,t}B]_{w,t} = [A, B]_{w,t}$ for any $w > s$,
- (3) $[\text{red}_{s,t}A, \text{red}_{s,t}B]_{p,t} = 0$ for any $p < s$,
- (4) $[\text{red}_{s,t}A, \text{red}_{s,t}B]_{s,q} = 0$ for any $q > t$.

Also, if $P = [\text{red}_{s,t}A, \text{red}_{s,t}B]$, then, keeping in mind that $P_{s,t} \neq 0$,

$$P = \begin{pmatrix} & \begin{matrix} s \\ \downarrow \end{matrix} & & & \begin{matrix} t \\ \downarrow \end{matrix} \\ s \rightarrow & 0 & P_{s,s+1} & \cdots & P_{s,t} \\ & & & \ddots & \vdots \\ & & & & P_{t-1,t} \\ t \rightarrow & & & & 0 \end{pmatrix}. \quad (10)$$

Proof: By assumption we have

$$[A, B]_{s,t} = (AB)_{s,t} - (BA)_{s,t} = \sum_{i=1}^n A_{s,i}B_{i,t} - \sum_{i=1}^n B_{s,i}A_{i,t} \neq 0,$$

and so the fact that $A, B \in U_n(F)$ implies that

$$\sum_{i=s}^t A_{s,i}B_{i,t} - \sum_{i=s}^t B_{s,i}A_{i,t} \neq 0.$$

Now, by the construction it is clear that

$$[\text{red}_{s,t}A, \text{red}_{s,t}B]_{s,t} = \sum_{i=s}^t A_{s,i}B_{i,t} - \sum_{i=s}^t B_{s,i}A_{i,t} \neq 0.$$

Also, by the construction it is easy to see that (2), (3) and (4) hold.

The form of $[\text{red}_{s,t}A, \text{red}_{s,t}B]$ should be also clear. □

Lemma 3: *If $A, B, C, D \in U_n(R)$ and s, t, z are integers such that $1 \leq s < t < z \leq n$, then*

$$[\text{red}_{t,z}A, \text{red}_{t,z}B][\text{red}_{s,t}C, \text{red}_{s,t}D] = 0.$$

Proof: For $P = [\text{red}_{t,z}A, \text{red}_{t,z}B]$ and $Q = [\text{red}_{s,t}C, \text{red}_{s,t}D]$, using (10) we get

$$PQ = \begin{pmatrix} 0 & P_{t,t+1} & \cdots & P_{t,z} \\ & \ddots & \ddots & \vdots \\ & & \ddots & P_{z-1,z} \\ & & & 0 \end{pmatrix} \begin{pmatrix} 0 & Q_{s,s+1} & \cdots & Q_{s,t} \\ & \ddots & \ddots & \vdots \\ & & \ddots & Q_{t-1,t} \\ & & & 0 \end{pmatrix} = 0.$$

□

Theorem 4: *If \mathcal{A} is an Ls_{m+1} (for some $m \geq 1$) structural matrix subring of $U_n(R)$, R a commutative ring and $n \geq 1$, then \mathcal{A} is D_{2^m} .*

Proof: First notice that if $[A, B]_{s,t} \neq 0$, then $s < t$.

We use induction on n . For $n = 1$ there is nothing to prove. Hence, consider a fixed n such that $n \geq 2$, and assume that the assertion is true for all k such that $k < n$. Let \mathcal{A} be an Ls_{m+1} (for some $m \geq 1$) structural matrix subring of $U_n(R)$, R a commutative ring. Let $\ell = 2^m$, and suppose that \mathcal{A} is not D_ℓ . Then there are matrices $A_1, B_1, \dots, A_\ell, B_\ell$ in \mathcal{A} such that

$$[A_1, B_1][A_2, B_2] \cdots [A_\ell, B_\ell] \neq 0.$$

Since every matrix A in \mathcal{A} is upper triangular, there are an $(n - 1) \times (n - 1)$ upper triangular matrix \bar{A} and a $1 \times (n - 1)$ matrix N_A such that

$$A = \left(\begin{array}{c|c} A_{1,1} & N_A \\ \hline 0 & \bar{A} \end{array} \right).$$

It is not hard to see that

$$\bar{\mathcal{A}} := \{\bar{A} : A \in \mathcal{A}\}$$

is an Ls_{m+1} structural subring of $U_{n-1}(R)$, and so by the induction hypothesis $\bar{\mathcal{A}}$ is D_ℓ . Hence,

$$0 \neq [A_1, B_1][A_2, B_2] \cdots [A_\ell, B_\ell] = \left(\begin{array}{c|c} 0 & N \\ \hline 0 & 0 \end{array} \right),$$

where $N := N_{[A_1, B_1][A_2, B_2] \cdots [A_\ell, B_\ell]} \in M_{1, n-1}(R)$.

Let $X_i = [A_i, B_i]$, $i = 1, \dots, \ell$, and let t_1 be the largest integer such that $N_{1, t_1-1} \neq 0$. Then $(X_1 X_2 \cdots X_\ell)_{1, t_1} \neq 0$, which gives

$$\sum_{j=1}^n (X_1 X_2 \cdots X_{\ell-1})_{1, j} (X_\ell)_{j, t_1} = \sum_{j=1}^{t_1-1} (X_1 X_2 \cdots X_{\ell-1})_{1, j} (X_\ell)_{j, t_1} \neq 0.$$

Let t_2 now be the largest integer such that

$$(X_1 X_2 \cdots X_{\ell-1})_{1, t_2} (X_\ell)_{t_2, t_1} \neq 0. \quad (11)$$

Obviously we have $t_2 < t_1$.

Let $X'_\ell = [\text{red}_{t_2, t_1} A_\ell, \text{red}_{t_2, t_1} B_\ell]$. Then by Lemma 2,

$$\begin{aligned} (X'_\ell)_{t_2, t_1} &= (X_\ell)_{t_2, t_1} \neq 0, \\ (X'_\ell)_{w, t_1} &= (X_\ell)_{w, t_1} \text{ for any } w > t_2, \\ (X'_\ell)_{s, t_1} &= 0 \text{ for any } s < t_2, \\ \text{and } (X'_\ell)_{t_2, q} &= 0 \text{ for any } q > t_1. \end{aligned}$$

This consideration leads us to the following:

$$\begin{aligned} (X_1 X_2 \cdots X_{\ell-1} X'_\ell)_{1, t_1} &= \sum_{j=1}^{t_1-1} (X_1 X_2 \cdots X_{\ell-1})_{1, j} (X'_\ell)_{j, t_1} \\ &= \sum_{j=t_2}^{t_1-1} (X_1 X_2 \cdots X_{\ell-1})_{1, j} (X'_\ell)_{j, t_1} \\ &= (X_1 X_2 \cdots X_{\ell-1})_{1, t_2} (X'_\ell)_{t_2, t_1} \\ &= (X_1 X_2 \cdots X_{\ell-1})_{1, t_2} (X_\ell)_{t_2, t_1} \\ &\neq 0 \text{ (by (11))}, \end{aligned}$$

and so

$$(X_1 X_2 \cdots X_{\ell-1})_{1, t_2} = \sum_{j=1}^{t_2-1} (X_1 X_2 \cdots X_{\ell-2})_{1, j} (X_{\ell-1})_{j, t_2} \neq 0.$$

Next, let t_3 be the largest integer such that

$$(X_1 X_2 \cdots X_{\ell-2})_{1, t_3} (X_{\ell-1})_{t_3, t_2} (X'_\ell)_{t_2, t_1} \neq 0. \quad (12)$$

Notice that $t_3 < t_2 < t_1$. Similarly as above, let

$$X'_{\ell-1} = [\text{red}_{t_3,t_2} A_{\ell-1}, \text{red}_{t_3,t_2} B_{\ell-1}].$$

Then by Lemma 2,

$$\begin{aligned} (X'_{\ell-1})_{t_3,t_2} &= (X_{\ell-1})_{t_3,t_2} \neq 0, \\ (X'_{\ell-1})_{w,t_2} &= (X_{\ell-1})_{w,t_2} \text{ for any } w > t_3, \\ (X'_{\ell-1})_{s,t_2} &= 0 \text{ for any } s < t_3, \\ (X'_{\ell-1})_{t_3,q} &= 0 \text{ for any } q > t_2. \end{aligned}$$

Next,

$$\begin{aligned} &(X_1 X_2 \cdots X_{\ell-2} X'_{\ell-1} X'_{\ell})_{1,t_1} \\ &= \sum_{j=1}^{t_1-1} (X_1 X_2 \cdots X'_{\ell-1})_{1,j} (X'_{\ell})_{j,t_1} \\ &= (\star) \sum_{j=t_2}^{t_1-1} \left[\sum_{i=1}^{j-1} (X_1 X_2 \cdots X_{\ell-2})_{1,i} (X'_{\ell-1})_{i,j} \right] (X'_{\ell})_{j,t_1} \\ &= (X_1 X_2 \cdots X_{\ell-2})_{1,t_3} (X'_{\ell-1})_{t_3,t_2} (X'_{\ell})_{t_2,t_1} \\ &= (X_1 X_2 \cdots X_{\ell-2})_{1,t_3} (X_{\ell-1})_{t_3,t_2} (X_{\ell})_{t_2,t_1} \neq 0. \end{aligned}$$

Notice that if $j < t_2$ in the sum (\star) , then $(X'_{\ell})_{j,t_1} = 0$, and for $j > t_2$ we have $(X'_{\ell-1})_{i,j} = 0$. Thus $j = t_2$. Now if $i < t_3$, then $(X'_{\ell-1})_{i,t_2} = 0$, while if $i > t_3$, then $(X'_{\ell-1})_{i,t_2} = (X_{\ell-1})_{i,t_2}$, and by the maximality of t_3 (see (12)) we must have

$$(X_1 X_2 \cdots X_{\ell-2})_{1,i} (X_{\ell-1})_{i,t_2} (X'_{\ell})_{t_2,t_1} = 0.$$

Thus, indeed $(X_1 X_2 \cdots X_{\ell-2} X'_{\ell-1} X'_{\ell})_{1,t_1} \neq 0$.

We now show the next step to describe the procedure we are dealing with:

Let t_4 be the largest integer such that

$$(X_1 X_2 \cdots X_{\ell-3})_{1,t_4} (X_{\ell-2})_{t_4,t_3} (X'_{\ell-1})_{t_3,t_2} (X'_{\ell})_{t_2,t_1} \neq 0.$$

Obviously $t_4 < t_3 < t_2 < t_1$. As one can expect, we set

$$X'_{\ell-2} = [\text{red}_{t_4,t_3} A_{\ell-2}, \text{red}_{t_4,t_3} B_{\ell-2}].$$

Then by Lemma 2,

$$\begin{aligned} (X'_{\ell-2})_{t_4,t_3} &= (X_{\ell-2})_{t_4,t_3} \neq 0, \\ (X'_{\ell-2})_{w,t_3} &= (X_{\ell-2})_{w,t_3} \text{ for any } w > t_4, \\ (X'_{\ell-2})_{s,t_3} &= 0 \text{ for any } s < t_4, \\ (X'_{\ell-2})_{t_4,q} &= 0 \text{ for any } q > t_3. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &(X_1 X_2 \cdots X_{\ell-3} X'_{\ell-2} X'_{\ell-1} X'_\ell)_{1,t_1} \\ &= \sum_{j=1}^{t_1-1} (X_1 X_2 \cdots X'_{\ell-1})_{1,j} (X'_\ell)_{j,t_1} \\ &= \sum_{j=t_2}^{t_1-1} (X_1 X_2 \cdots X'_{\ell-1})_{1,j} (X'_\ell)_{j,t_1} \\ &= \sum_{j=t_2}^{t_1-1} \left[\sum_{i=1}^{j-1} (X_1 X_2 \cdots X'_{\ell-2})_{1,i} (X'_{\ell-1})_{i,j} \right] (X'_\ell)_{j,t_1} \\ &= \left[\sum_{i=1}^{j-1} (X_1 X_2 \cdots X'_{\ell-2})_{1,i} (X'_{\ell-1})_{i,t_2} \right] (X'_\ell)_{t_2,t_1} \\ &= \left[\sum_{i=t_3}^{j-1} (X_1 X_2 \cdots X'_{\ell-2})_{1,i} (X'_{\ell-1})_{i,t_2} \right] (X'_\ell)_{t_2,t_1} \\ &= \sum_{i=t_3}^{t_2-1} \left[\sum_{k=1}^{i-1} (X_1 X_2 \cdots X_{\ell-3})_{1,k} (X'_{\ell-2})_{k,i} \right] (X'_{\ell-1})_{i,t_2} (X'_\ell)_{t_2,t_1} \\ &= (X_1 X_2 \cdots X_{\ell-3})_{1,t_4} (X'_{\ell-2})_{t_4,t_3} (X'_{\ell-1})_{t_3,t_2} (X'_\ell)_{t_2,t_1} \\ &= (X_1 X_2 \cdots X_{\ell-3})_{1,t_4} (X_{\ell-2})_{t_4,t_3} (X'_{\ell-1})_{t_3,t_2} (X'_\ell)_{t_2,t_1} \neq 0. \end{aligned}$$

Continuing in the above way, after finitely many steps we get positive integers $t_1, t_2, \dots, t_{\ell-1}, t_\ell$, such that $1 < t_\ell < t_{\ell-1} < \dots < t_2 < t_1$, and reduced matrices X'_1, \dots, X'_ℓ such that

$$X'_q = [\text{red}_{t_\ell-q+2, t_\ell-q+1} A_q, \text{red}_{t_\ell-q+2, t_\ell-q+1} B_q]$$

for any $q = 1, \dots, \ell$. Moreover,

$$(X'_1 X'_2 \cdots X'_\ell)_{1,t_1} = (X'_1)_{1,t_\ell} (X'_2)_{t_\ell, t_{\ell-1}} \cdots (X'_{\ell-1})_{t_3, t_2} (X'_\ell)_{t_2, t_1} \neq 0.$$

Using Lemma 3 we can easily see that for any permutation $\sigma \in S_\ell$, with $\sigma \neq id$, we have

$$X'_{\sigma(1)} X'_{\sigma(2)} \cdots X'_{\sigma(\ell)} = 0.$$

For a sequence Y_1, Y_2, \dots of elements of $M_n(F)$ and positive integers p_1, p_2, \dots , let

$$S(p_1 p_2) := [Y_{p_1}, Y_{p_2}], \quad S(p_1 p_2 p_3 p_4) := [S(p_1 p_2), S(p_3 p_4)],$$

and if $S(p_1 p_2 \cdots p_{2^q})$ has been defined for some q , then

$$S(p_1 p_2 \cdots p_{2^q} p_{2^{q+1}} \cdots p_{2^{q+1}}) := [S(p_1 p_2 \cdots p_{2^q}), S(p_{2^{q+1}} \cdots p_{2^{q+1}})].$$

Using the above notation and results, we have, for the sequence X'_1, \dots, X'_ℓ (recall that $\ell = 2^m$), $S(123 \dots \ell) = X'_1 \cdots X'_\ell$ and

$$(S(123 \dots \ell))_{1,t_1} = (X'_1 \cdots X'_\ell)_{1,t_1} \neq 0.$$

Thus \mathcal{A} is not $L_{S_{m+1}}$, a contradiction. We conclude that \mathcal{A} is D_{2^m} . □

4. D_2 subalgebras and L_{S_2} subalgebras of $M_n(F)$

Let F be any field, and let $n_1 \geq 2$. Define the rectangular array \mathfrak{B} by

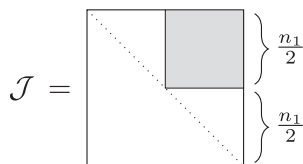
$$\mathfrak{B} := \{(i, j) : 1 \leq i \leq \lfloor \frac{n_1}{2} \rfloor < j \leq n_1\},$$

and the subset \mathcal{J} of $M_{n_1}(F)$ by

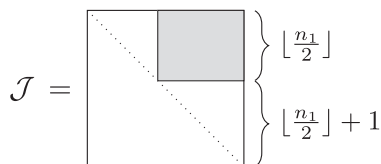
$$\mathcal{J} := \left\{ \sum_{(i,j) \in \mathfrak{B}} a_{i,j} e_{i,j} : a_{i,j} \in F \text{ for all } (i, j) \in \mathfrak{B} \right\}. \tag{13}$$

This means that \mathcal{J} comprises the set of all block upper triangular matrices that correspond with \mathfrak{B} , which has the following illuminating pictorial representations (the unshaded regions in the pictures below correspond with zero entries):

(1) n_1 is even:



(2) n_1 is odd:



Denote by

$$FI_{n_1} := \{aI_{n_1} : a \in F\}$$

the set of all $n_1 \times n_1$ scalar matrices over F . By [14,15], the upper triangular matrix F -subalgebra $FI_{n_1} + \mathcal{J}$ of $M_{n_1}(F)$ is an example of a commutative F -subalgebra of $M_{n_1}(F)$ with maximum dimension $1 + \lfloor \frac{n_1^2}{4} \rfloor$.

Next, let n_1 and n_2 be positive integers, let $n = n_1 + n_2$ and let \mathcal{A} be an R -subalgebra of $U_n(R)$, R any commutative ring. Every matrix A in \mathcal{A} is of the form

$$A = \left(\begin{array}{c|c} A_1 & A_3 \\ \hline 0 & A_2 \end{array} \right), \tag{14}$$

with $A_1 \in U_{n_1}(R)$, $A_2 \in U_{n_2}(R)$ and $A_3 \in M_{n_1 \times n_2}(R)$. For $i = 1, 2, 3$, we set

$$\mathcal{A}_i := \left\{ A_i : \left(\begin{array}{c|c} A_1 & A_3 \\ \hline 0 & A_2 \end{array} \right) \in \mathcal{A} \right\}. \tag{15}$$

Then \mathcal{A}_i is an R -subalgebra of $U_{n_i}(R)$, $i = 1, 2$. (Note that the commutative algebra \mathcal{C} above may also be presented in this way.)

For a particular such R -subalgebra \mathcal{A} of $U_n(R)$, it is in principle possible that, for every matrix A in \mathcal{A} , there may be ‘ties’ (or ‘links’) between the matrices A_1 , A_2 and A_3 (respectively, between *some* of the matrices A_1 , A_2 and A_3), as is indeed the case in the algebras $U_3^*(U_3^*(F))$ (respectively $U_9^*(F)$) and $FI_{n_1} + \mathcal{J}$ mentioned above, and so the containment $\mathcal{A} \subseteq \mathcal{A}'$ of the algebras \mathcal{A} and \mathcal{A}' , where

$$\mathcal{A}' := \left\{ \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & 0 \end{array} \right) + \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & A_2 \end{array} \right) + \left(\begin{array}{c|c} 0 & A_3 \\ \hline 0 & 0 \end{array} \right) : A_i \in \mathcal{A}_i, i = 1, 2, 3 \right\} \tag{16}$$

may be proper. If there are none of the possible mentioned ‘ties’, then clearly $\mathcal{A} = \mathcal{A}'$. This observation leads to:

Definition 5: If, for every matrix $A = \left(\begin{array}{c|c} A_1 & A_3 \\ \hline 0 & A_2 \end{array} \right) \in \mathcal{A}$ above, there are no ‘ties’ between the matrices A_1 , A_2 and A_3 , then we say that \mathcal{A}_1 , \mathcal{A}_2 and \mathcal{A}_3 are independent.

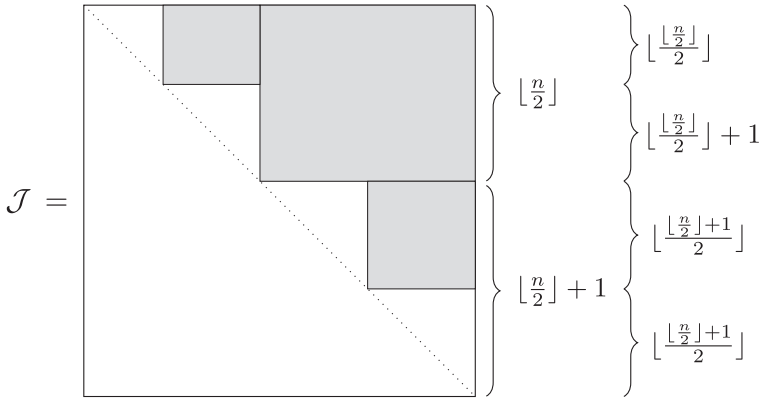
It is not hard to see that the above ‘independence’ is equivalent to the fact that the idempotent $e = e_{1,1} + e_{2,2} + \dots + e_{n_1,n_1}$ is an element of \mathcal{A} , which obviously implies that $e_{n_1+1,n_1+1} + \dots + e_{n,n} = 1 - e$ is also in \mathcal{A} . We then have $\left(\begin{array}{c|c} I_{n_1} & 0 \\ \hline 0 & 0 \end{array} \right), \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & I_{n_2} \end{array} \right) \in \mathcal{A}$.

For $q = 2$, the D_q subalgebras of $M_n(F)$ (with $n = n_1 + n_2$) of maximum dimension $2 + \lfloor \frac{3n^2}{8} \rfloor$ in [3, p.157] are of the form in (16) if the commutative subalgebras of $M_{n_i}(F)$, $i = 1, 2$, of maximum dimension $1 + \lfloor \frac{n_i^2}{4} \rfloor$ in the main diagonal blocks in [3, p.157] are taken to be upper triangular matrix algebras, for example, as in (17).

Such a selection implies that there are no ties between any of the matrices A_1 , A_2 and A_3 in such (i.e. the form in (16)) a chosen D_2 subalgebra $\mathcal{A} = \mathcal{A}'$ of $U_n(F)$ of maximum dimension. By [6, Theorem 14 and Remark 15], for a given n , and for such a chosen D_2 subalgebra of $U_n(F)$ of maximum dimension, we may take $n_1 = n_2 = \frac{n}{2}$, if n is even, and we may take $n_1 = \lfloor \frac{n}{2} \rfloor$ and $n_2 = n_1 + 1$, if n is odd. This gives rise to:

Definition 6: We call such (as in the preceding paragraph) a D_2 subalgebra of $U_n(F)$ with maximum dimension a typical D_2 subalgebra of $U_n(F)$ with maximum dimension.

The analogue of the block upper triangular pictorial representations of \mathcal{J} above, for the case where both n and $\lfloor \frac{n}{2} \rfloor$ are odd (and hence $\lfloor \frac{n}{2} \rfloor + 1$ is even), is the following:



By [3, p.157] and [6, Theorem 14 and Remark 15], the F -subalgebra

$$\left\{ a \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} e_{i,i} : a \in F \right\} + \left\{ b \sum_{j=\lfloor \frac{n}{2} \rfloor+1}^n e_{j,j} : b \in F \right\} + \mathcal{J} \tag{17}$$

of $M_n(F)$ is an example of a typical D_2 F -subalgebra of $M_n(F)$ with maximum dimension $2 + \lfloor \frac{3n^2}{8} \rfloor$. Of course, the other possibilities, i.e. when n and/or $\lfloor \frac{n}{2} \rfloor$ are even/odd, can be treated similarly. Instead of taking only two positive integers n_1 and n_2 such that $n_1 + n_2 = n$ (for a fixed n), we could have generalized the above setting in the framework of D_q subalgebras of $M_n(F)$ as in [3, p.157] by taking q positive integers with sum equal to n . However, since the focus in this section is on D_2 subalgebras, we have only treated the $n_1 + n_2 = n$ case.

It is immediately clear that if \mathcal{A} is an Ln_2 (respectively Ls_2, D_2) R -subalgebra of $U_n(R)$, then the algebras \mathcal{A}_i above (even considering the case $n_1 + n_2 + \dots + n_q = n$) are also Ln_2 (respectively Ls_2, D_2). In this vein, [4, Theorem 2.1] states that if S is a ring which is both D_2 and Ln_2 , then $U_3^*(S)$ is Ls_2 . This result was an important tool in [4] in obtaining a matrix algebra which is Ls_2 , but neither D_2 nor Ln_2 . In Theorem 8 we explore $U_m^*(S)$ (for $m \geq 2$) if S is not required to be Ln_2 , but merely D_2 (respectively Ls_2). However, we first prove that the converse of [4, Theorem 2.1] is also true:

Proposition 7: *If S is a ring such that $U_3^*(S)$ is Ls_2 , then S is D_2 and Ln_2 .*

Proof: For all $a, d, c, d \in S$ we have

$$0 = [[aI_3, be_{1,2}], [cI_3, de_{2,3}]] = [[a, b]e_{1,2}, [c, d]e_{2,3}] = [a, b][c, d]e_{1,3}$$

and

$$0 = [[aI_3, bI_3], [ce_{1,2}, e_{2,3}]] = [[a, b]I_3, ce_{1,3}] = [a, b, c]e_{1,3},$$

implying that $[a, b][c, d] = 0$ and $[a, b, c] = 0$. □

Theorem 8: *If a ring S is D_2 (respectively Ls_2), then $U_m^*(S)$ is $D_2^{\lfloor \log_2 m \rfloor}$ (respectively, $Ls_{\lfloor \log_2 m \rfloor+1}$) for all $m \geq 2$.*

Proof: We first prove by induction that if a ring S is D_2 (respectively Ls_2), then

$$U_{2^n}^*(S) \text{ is } D_{2^n} \text{ (respectively, } Ls_{n+1}) \text{ for all } n \geq 1. \tag{18}$$

The case $n = 1$ is direct computation. So, suppose that S is D_2 (respectively Ls_2), and for some fixed positive integer k , $U_{2^k}^*(S)$ is D_{2^k} (respectively Ls_{k+1} , i.e. by (2), $\mathfrak{D}^{k+1}(U_{2^k}^*(S)) = 0$.) Since (an isomorphic copy of) $U_{2^{k+1}}^*(S)$ is contained in the $2^{k+1} \times 2^{k+1}$ upper triangular matrix ring

$$\left(\begin{array}{c|c} U_{2^k}^*(S) & M_{2^k}(S) \\ \hline 0 & U_{2^k}^*(S) \end{array} \right) = \left\{ \left(\begin{array}{c|c} A_1 & A_3 \\ \hline 0 & A_2 \end{array} \right) : A_i \in U_{2^k}^*(S), i = 1, 2, \text{ and } A_3 \in M_{2^k}(S) \right\}, \tag{19}$$

it follows from the induction hypothesis, for the D_2 case, that for all matrices X_i, Y_i, Z_i and $W_i, i = 1, 2, \dots, 2^k$, in $U_{2^{k+1}}^*(S)$,

$$[X_1, Y_1][X_2, Y_2] \cdots [X_{2^k}, Y_{2^k}] \in \left(\begin{array}{c|c} 0 & M_{2^k}(S) \\ \hline 0 & 0 \end{array} \right),$$

and

$$[Z_1, W_1][Z_2, W_2] \cdots [Z_{2^k}, W_{2^k}] \in \left(\begin{array}{c|c} 0 & M_{2^k}(S) \\ \hline 0 & 0 \end{array} \right),$$

and so

$$[X_1, Y_1][X_2, Y_2] \cdots [X_{2^k}, Y_{2^k}][Z_1, W_1][Z_2, W_2] \cdots [Z_{2^k}, W_{2^k}] = 0,$$

which implies that $U_{2^{k+1}}^*(S)$ is $D_{2^{k+1}}$, and thus proves the D_2 version of (18).

Since, by (19),

$$\mathfrak{D}^{k+1}(U_{2^{k+1}}^*(S)) \subseteq \left(\begin{array}{c|c} \mathfrak{D}^{k+1}(U_{2^k}^*(S)) & M_{2^k}(S) \\ \hline 0 & \mathfrak{D}^{k+1}(U_{2^k}^*(S)) \end{array} \right),$$

the induction hypothesis for the Ls_2 case implies that

$$\mathfrak{D}^{k+1}(U_{2^{k+1}}^*(S)) \subseteq \left(\begin{array}{c|c} 0 & M_{2^k}(S) \\ \hline 0 & 0 \end{array} \right),$$

and so

$$\mathfrak{D}^{k+2}(U_{2^{k+1}}^*(S)) = \left[\mathfrak{D}^{k+1}(U_{2^{k+1}}^*(S)), \mathfrak{D}^{k+1}(U_{2^{k+1}}^*(S)) \right] = 0,$$

i.e. $U_{2^{k+1}}^*(S)$ is Ls_{k+2} , establishing the Ls_2 version of (18).

Next, since (an isomorphic copy of) $U_{2^{n-1+i}}^*(S), i = 1, \dots, 2^{n-1}$, is contained in $U_{2^n}^*(S)$, it follows from (18) that if S is D_2 (respectively, Ls_2), then

$$U_{2^{n-1+i}}^*(S) \text{ is } D_{2^n} \text{ (respectively, } Ls_{n+1}), i = 1, \dots, 2^{n-1}, n \geq 1.$$

Since $\lceil \log_2(2^{n-1} + i) \rceil = n$ for all such i and n , it follows that if S is D_2 (respectively, Ls_2), then

$$U_{2^{n-1}+i}^*(S) \text{ is } D_{2^{\lceil \log_2(2^{n-1}+i) \rceil}} \text{ (respectively, } Ls_{\lceil \log_2(2^{n-1}+i) \rceil+1}).$$

Setting $m := 2^{n-1} + i$ for a fixed pair (n, i) , and noting that

$$\{2^{n-1} + i : i = 1, \dots, 2^{n-1}, n \geq 1\} = \{m : m \geq 2\},$$

we get that if S is D_2 (respectively, Ls_2), then $U_m^*(S)$ is $D_{2^{\lceil \log_2 m \rceil}}$ (respectively, $Ls_{\lceil \log_2 m \rceil+1}$) for all $m \geq 2$. □

It seems that a solution to Question 1 (see Section 2) will be facilitated by an answer to the following question:

Question 9: For a field F , does a D_2 F -subalgebra of $U_n(F)$, for some n , with maximum dimension $2 + \lfloor \frac{3n^2}{8} \rfloor$ and which is not a typical D_2 subalgebra of $U_n(F)$ with maximum dimension (see Definition 6 and the paragraph preceding it), exist?

In particular, we do not know if there is a D_2 subalgebra \mathcal{A} of $U_n(F)$ (for some n) with maximum dimension such that

$$\left(\begin{array}{c|c} 0 & M_{\frac{n}{2}}(F) \\ \hline 0 & 0 \end{array} \right) \not\subseteq \mathcal{A} \quad \text{or} \quad \left(\begin{array}{c|c} 0 & M_{\lfloor \frac{n}{2} \rfloor \times (\lfloor \frac{n}{2} \rfloor + 1)}(F) \\ \hline 0 & 0 \end{array} \right) \not\subseteq \mathcal{A}, \tag{20}$$

depending on whether n is even or odd.

Neither do we know whether a D_2 subalgebra \mathcal{A} of $U_n(F)$ (for some n , with n even (say)) with maximum dimension exists such that

$$\left(\begin{array}{c|c} 0 & M_{\frac{n}{2}}(F) \\ \hline 0 & 0 \end{array} \right) \subseteq \mathcal{A} \quad \text{and} \quad \dim_F \mathcal{A}_i > 1 + \left\lfloor \frac{(\frac{n}{2})^2}{4} \right\rfloor, \tag{21}$$

which would imply that the algebra \mathcal{A}_i is not commutative, and that

$$\dim_F \mathcal{A}_j < 1 + \left\lfloor \frac{(\frac{n}{2})^2}{4} \right\rfloor,$$

with $i \neq j$ and $\{i, j\} = \{1, 2\}$. (For the sake of brevity we have mentioned only the case when n is even, but the corresponding question, for when is odd, is clear.)

However, if we do not require a D_2 subalgebra of $U_n(F)$ to have maximum dimension, then a D_2 algebra \mathcal{A} as in (21) does exist, as shown in Example 10 below.

Note that in this case, for every matrix $A = \left(\begin{array}{c|c} A_1 & A_3 \\ \hline 0 & A_2 \end{array} \right) \in \mathcal{A}$, there are ‘ties’ between A_1 and A_2 .

Example 10: The R -subalgebra

$$\mathcal{A} := \left\{ \left(\begin{array}{ccc|ccc} a & b & c & e & f & g \\ 0 & a & d & h & k & \ell \\ 0 & 0 & a & p & q & r \\ \hline 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & a \end{array} \right) : a, b, \dots, r \in R \right\}$$

of $U_6^*(R)$, R any commutative ring, is easily seen to be D_2 . To wit, every matrix in \mathcal{A} is of the form

$$X_{a,U,B} := \left(\begin{array}{ccc|ccc} aI_3 + U & & & B & & \\ & & & 0 & & \\ & & & & & aI_3 \end{array} \right),$$

with U a strictly upper triangular 3×3 matrix, and $B \in M_3(R)$, and so the commutator

$$[X_{a,U,B}, X_{c,V,D}] = \left[\left(\begin{array}{ccc|ccc} aI_3 + U & & & B & & \\ & & & 0 & & \\ & & & & & aI_3 \end{array} \right), \left(\begin{array}{ccc|ccc} cI_3 + V & & & D & & \\ & & & 0 & & \\ & & & & & cI_3 \end{array} \right) \right]$$

of two such matrices is equal to

$$\left(\begin{array}{ccc|ccc} [U, V] & aD + UD + Bc - cB - VB - Da & & & & \\ & & & 0 & & \\ & & & & & 0 \end{array} \right) = \left(\begin{array}{ccc|ccc} [U, V] & UD - VB & & & & \\ & & & 0 & & \\ & & & & & 0 \end{array} \right),$$

since aI_3 and cI_3 are in the centre of $M_3(R)$. Furthermore, $[U, V] = xe_{1,3}$ for some $x \in R$, and the 3rd rows of UD and VB are zero rows. Hence,

$$\begin{aligned} & [X_{a,U,B}, X_{c,V,D}] \cdot [X_{e,W,G}, X_{f,Y,H}] \\ &= \left(\begin{array}{ccc|ccc} [U, V] & UD - VB & & & & \\ & & & 0 & & \\ & & & & & 0 \end{array} \right) \left(\begin{array}{ccc|ccc} [W, Y] & WH - YG & & & & \\ & & & 0 & & \\ & & & & & 0 \end{array} \right) \\ &= \left(\begin{array}{ccc|ccc} [U, V][W, Y] & [U, V](WH - YG) & & & & \\ & & & 0 & & \\ & & & & & 0 \end{array} \right) \\ &= 0, \end{aligned}$$

since, as above, the 3rd rows of WH and YG are zero rows.

However, $\mathcal{A}_1 = U_3^*(R)$, which is not commutative.

Note that $U_6^*(F)$, with F a field, is not an example illustrating the phenomenon in Example 10. To wit, $U_6^*(F)$ is not D_2 , because the maximum dimension of a D_2 (F -)subalgebra of $M_6(F)$ is $2 + \lfloor \frac{3 \cdot 6^2}{8} \rfloor = 15$, but $\dim_F U_6^*(F) = 16$.

The last two theorems in the paper, namely Theorems 15 and 16, provide partial answers to Questions 9 and 1, respectively. In particular, Theorem 15 should also be viewed in the context of (21), and Theorem 16 should be seen against the background of (20).

However, we first need some preliminary results, including the following result from [16]:

Theorem 11: *If \mathcal{A} is a finite-dimensional commutative algebra over a field F , and V is a faithful \mathcal{A} -module of finite F -dimension, then*

$$\dim_F \mathcal{A} \leq \left\lfloor \frac{(\dim_F V)^2}{4} \right\rfloor + 1.$$

Lemma 12: *If \mathcal{A} is a D_2 subalgebra of $U_n(F)$, $V = F^n$ and \mathcal{C} is the ideal of \mathcal{A} generated by all commutators $[x, y]$, $x, y \in \mathcal{A}$, then*

$$\dim_F \mathcal{A} \leq \left\lfloor \frac{(\dim_F(V\mathcal{C}))^2}{4} \right\rfloor + \left\lfloor \frac{(\dim_F(V/V\mathcal{C}))^2}{4} \right\rfloor + 1 + \dim_F(V\mathcal{C}) \cdot \dim_F(V/V\mathcal{C}).$$

Proof: If $\mathcal{C} = 0$ (which means that \mathcal{A} is commutative), then the desired inequality has the form

$$\dim_F \mathcal{A} \leq \left\lfloor \frac{(\dim_F V)^2}{4} \right\rfloor + 1,$$

which is valid by Theorem 11.

Suppose now that $V\mathcal{C} \neq 0$, and consider the F -algebra homomorphism $\phi : \mathcal{A} \rightarrow \text{End}(V\mathcal{C})$ induced by the \mathcal{A} -module structure of $V\mathcal{C}$. Obviously,

$$\dim_F \mathcal{A} = \dim_F(\text{im}(\phi)) + \dim_F(\ker(\phi)). \tag{22}$$

Since $V\mathcal{C}$ is a faithful $\text{im}(\phi)$ -module, and $\text{im}(\phi)$ is commutative (because $\mathcal{C}^2 = 0$), we get

$$\dim_F(\text{im}(\phi)) \leq \left\lfloor \frac{(\dim_F(V\mathcal{C}))^2}{4} \right\rfloor + 1. \tag{23}$$

Consider now the ideal $\ker(\phi)$ of \mathcal{A} . Obviously, $I_n \notin \ker(\phi)$. Thus, considering the subalgebra \mathcal{E} of \mathcal{A} generated by $\ker(\phi)$, we have

$$\dim_F(\ker(\phi)) < \dim_F \mathcal{E}. \tag{24}$$

As $(V\mathcal{C})\ker(\phi) = 0$, we can investigate the F -algebra homomorphism $\varphi : \mathcal{E} \rightarrow \text{End}(V/V\mathcal{C})$. Then

$$\dim_F \mathcal{E} = \dim_F(\text{im}(\varphi)) + \dim_F(\ker(\varphi)). \tag{25}$$

Again, since $V/V\mathcal{C}$ is a faithful $\text{im}(\varphi)$ -module, and $\text{im}(\varphi)$ is commutative, it follows from Theorem 11 that

$$\dim_F(\text{im}(\varphi)) \leq \left\lfloor \frac{(\dim_F(V/V\mathcal{C}))^2}{4} \right\rfloor + 1. \tag{26}$$

Notice that every element of $\ker(\varphi)$ can be viewed as a linear map from V/VC to VC , which implies that

$$\dim_F(\ker(\varphi)) \leq \dim_F(V/VC) \cdot \dim_F(VC). \quad (27)$$

Putting together (22)–(27) we finally obtain the desired inequality in the case $VC \neq 0$. \square

Lemma 13: *If \mathcal{A} is a D_2 subalgebra of $U_n(F)$, and \mathcal{A}_1 , \mathcal{A}_2 and \mathcal{A}_3 are independent (see Definition 5), then $\mathcal{C}_1\mathcal{A}_3 = \mathcal{A}_3\mathcal{C}_2 = 0$, where \mathcal{C}_i is the ideal of \mathcal{A}_i generated by all commutators $[x, y]$, $x, y \in \mathcal{A}_i$, $i = 1, 2$.*

Proof: We will only show that $\mathcal{C}_1\mathcal{A}_3 = 0$, since the second case can be treated similarly.

Suppose, for a contradiction, that $\mathcal{C}_1\mathcal{A}_3 \neq 0$. Let $W \in \mathcal{C}_1$ and $G \in \mathcal{A}_3$ be such that $WG \neq 0$. Firstly, for the identity $I \in \mathcal{A}_2$ (see the paragraph immediately following Definition 5) we consider the commutator

$$\left[\left(\begin{array}{c|c} 0 & G \\ \hline 0 & 0 \end{array} \right), \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & I \end{array} \right) \right] = \left(\begin{array}{c|c} 0 & G \\ \hline 0 & 0 \end{array} \right).$$

As \mathcal{A} is D_2 and $W \in \mathcal{C}_1$ we get

$$\left(\begin{array}{c|c} 0 & WG \\ \hline 0 & 0 \end{array} \right) = \left(\begin{array}{c|c} W & 0 \\ \hline 0 & 0 \end{array} \right) \cdot \left(\begin{array}{c|c} 0 & G \\ \hline 0 & 0 \end{array} \right) = 0;$$

a contradiction. \square

Henceforth, for every integer $n \geq 2$, we write $D_2(n)$ for the maximum dimension $\left\lfloor \frac{3n^2}{8} \right\rfloor + 2$ of a D_2 subalgebra of $U_n(F)$.

Remark 14:

- (1) Let $V_1 = F^{n_1}$, $V_2 = F^{n_2}$, and let \mathcal{A} be a linear subspace of $M_{n_1 \times n_2}(F)$. Suppose that W_1 is a subspace of V_1 such that $W_1\mathcal{A} = 0$. Taking a complement U_1 of W_1 in V_1 , we have $V_1 = W_1 \oplus U_1$. It can be seen that \mathcal{A} can be embedded into the linear space $\mathcal{L}(U_1, V_2)$ of all linear transformations from U_1 into V_2 , and

$$\dim_F \mathcal{A} \leq n_2 \cdot \dim_F U_1 = n_2 \cdot (n_1 - \dim_F W_1).$$

- (2) Keeping the notation as above, and considering a subspace W_2 of V_2 such that $\mathcal{A}W_2 = 0$, we have, for a complement U_2 of W_2 ,

$$\dim_F \mathcal{A} \leq n_1 \cdot \dim_F U_2 = n_1 \cdot (n_2 - \dim_F W_2).$$

In the next result, which provides a partial answer to Question 9, we follow, without loss of generality, the ‘convention’ described in the paragraph immediately preceding Definition 6, namely the sizes of \mathcal{A}_1 and \mathcal{A}_2 are equal when n is even, and the sizes differ by 1 when n is odd.

Theorem 15: *If \mathcal{A} is a D_2 subalgebra of $U_n(F)$ with maximum dimension $D_2(n)$, such that \mathcal{A}_1 , \mathcal{A}_2 and \mathcal{A}_3 are independent, then \mathcal{A}_1 and \mathcal{A}_2 are commutative.*

Proof: We will consider two cases.

Case 1. n is even: Then, for $m := \frac{n}{2}$: $\mathcal{A}_1, \mathcal{A}_2 \subseteq U_m(F)$.

By Lemma 12 and its symmetric version we have

$$\dim_F \mathcal{A}_1 \leq \left\lfloor \frac{p^2}{4} \right\rfloor + \left\lfloor \frac{(m-p)^2}{4} \right\rfloor + 1 + p(m-p) =: P,$$

and

$$\dim_F \mathcal{A}_2 \leq \left\lfloor \frac{q^2}{4} \right\rfloor + \left\lfloor \frac{(m-q)^2}{4} \right\rfloor + 1 + q(m-q) =: Q,$$

where for $V = F^m$, $p := \dim_F(V\mathcal{C}_1)$, $q := \dim_F(\mathcal{C}_2V)$, and \mathcal{C}_i is the ideal of \mathcal{A}_i generated by all commutators $[x, y]$, $x, y \in \mathcal{A}_i$, $i = 1, 2$.

Using Lemma 13 and Remark 14 we have

$$\dim_F \mathcal{A}_3 \leq m(m-p), \text{ and } \dim_F \mathcal{A}_3 \leq m(m-q).$$

Without loss of generality we may assume that $Q \leq P$. Then

$$\begin{aligned} \dim_F \mathcal{A} &= \dim_F \mathcal{A}_1 + \dim_F \mathcal{A}_2 + \dim_F \mathcal{A}_3 \leq P + Q + m(m-p) \\ &\leq 2 \left(\left\lfloor \frac{p^2}{4} \right\rfloor + \left\lfloor \frac{(m-p)^2}{4} \right\rfloor + 1 + p(m-p) \right) + m(m-p) \\ &\leq 2 \left(\frac{p^2}{4} + \frac{(m-p)^2}{4} + 1 + p(m-p) \right) + m(m-p) \\ &= \frac{3}{2}m^2 - p^2 + 2. \end{aligned}$$

Thus, if $m = 2s$ for some s , then $\dim_F \mathcal{A} \leq 6s^2 - p^2 + 2$, and if $m = 2s + 1$, then $\dim_F \mathcal{A} \leq 6s^2 + 6s - p^2 + 3\frac{1}{2}$. On the other hand, since $n = 2m$ and $D_2(n) = \left\lfloor \frac{3n^2}{8} \right\rfloor + 2$, we have

$$D_2(n) = \begin{cases} 6s^2 + 2, & \text{if } m = 2s, \\ 6s^2 + 6s + 3, & \text{if } m = 2s + 1. \end{cases}$$

Therefore, we conclude that $p = 0$, which means that \mathcal{A}_1 is commutative. Using $p = 0$ and the assumption that $Q \leq P$, it is not hard to show that $q = 0$, whence \mathcal{A}_2 is also commutative. The proof is thus complete for the case when n is even.

Case 2. n is odd: Then, for $m := \lfloor \frac{n}{2} \rfloor$: $\mathcal{A}_1 \in U_m(F)$ and $\mathcal{A}_2 \subseteq U_{m+1}(F)$.

Notice that in this case, $\mathcal{A}_3 \subseteq M_{m,m+1}(F)$. (The second pictorial representation following (13) may be helpful.) By the same argument as above,

$$\dim_F \mathcal{A}_1 \leq \left\lfloor \frac{p^2}{4} \right\rfloor + \left\lfloor \frac{(m-p)^2}{4} \right\rfloor + 1 + p(m-p) =: P,$$

and

$$\dim_F \mathcal{A}_2 \leq \left\lfloor \frac{q^2}{4} \right\rfloor + \left\lfloor \frac{(m+1-q)^2}{4} \right\rfloor + 1 + q(m+1-q) =: Q,$$

where for $V_1 = F^m$ and $V_2 = F^{m+1}$, $p := \dim_F(V_1\mathcal{C}_1)$, $q := \dim_F(\mathcal{C}_2V_2)$ and \mathcal{C}_i is the ideal of \mathcal{A}_i generated by all commutators $[x, y]$, $x, y \in \mathcal{A}_i$, $i = 1, 2$.

Also by similar arguments as above we get

$$\dim_F \mathcal{A}_3 \leq \min\{(m + 1)(m - p), m(m + 1 - q)\},$$

and so

$$\dim_F \mathcal{A} \leq P + Q + \min\{(m + 1)(m - p), m(m + 1 - q)\}.$$

Now we consider all the possible cases related to the parity of m and the value of $\min\{(m + 1)(m - p), m(m + 1 - q)\}$.

Subcase A. $m = 2s$ (for some s) and $(m + 1)(m - p) \leq m(m + 1 - q)$:

The assumption $(m + 1)(m - p) \leq m(m + 1 - q)$ implies that $-\frac{1}{2}p \leq s(p - q)$. Thus if $p = 0$, then $q = 0$, and we are done. So suppose that $p \neq 0$. Then, keeping in mind that $\dim_F \mathcal{A} = D_2(n) = D_2(4s + 1)$, we have

$$\begin{aligned} D_2(n) - \dim_F \mathcal{A} &\geq D_2(4s + 1) - P - Q - (2s - p)(2s + 1) \\ &\geq \frac{1}{2}p^2 + \frac{1}{2}q^2 + p - \frac{1}{2}q - \frac{1}{4} + s(p - q) \\ &\geq \frac{1}{2}p^2 + \frac{1}{2}q^2 + p - \frac{1}{2}q - \frac{1}{4} - \frac{1}{2}p \\ &= \frac{1}{2}p^2 + \frac{1}{2}q^2 + \frac{1}{2}p - \frac{1}{2}q - \frac{1}{4} > 0. \end{aligned}$$

Thus $D_2(n) - \dim_F \mathcal{A} > 0$, and we have a contradiction.

Subcase B. $m = 2s$ and $(m + 1)(m - p) > m(m + 1 - q)$:

The assumption $(m + 1)(m - p) > m(m + 1 - q)$ yields $\frac{1}{2}p < s(q - p)$. In this case,

$$\begin{aligned} D_2(4s + 1) - P - Q - 2s(2s + 1 - q) \\ &\geq \frac{1}{2}p^2 + \frac{1}{2}q^2 - \frac{1}{2}q - \frac{1}{4} + s(q - p) \\ &\geq \frac{1}{2}p^2 + \frac{1}{2}p + \frac{1}{2}q^2 - \frac{1}{2}q - \frac{1}{4}. \end{aligned}$$

If $p \neq 0$, then by the above, $D_2(4s + 1) - \dim_F \mathcal{A} > 0$, and we again have a contradiction. If $p = 0$ and $q > 1$, then we also get a contradiction by direct computation. If $p = 0$ and $q = 1$ then we obtain that $D_2(4s + 1) - P - Q - 2s(2s + 1 - q) = s > 0$; again a contradiction. Hence we conclude that $p = q = 0$, which implies that \mathcal{A}_1 and \mathcal{A}_2 are commutative.

The situation where $m = 2s + 1$ can be considered in a similar way. Thus the proof is complete. □

The following result gives a partial answer to Question 1.

Theorem 16: *Suppose there is an Ls_2 F -subalgebra \mathcal{A} of $U_n^*(F)$, for some n , with $\dim_F \mathcal{A} > 2 + \lfloor \frac{3n^2}{8} \rfloor$, i.e. the maximum dimension of a D_2 subalgebra of $M_n(F)$, and consider the smallest such n . Then $n = 2k$ or $n = 2k + 1$ for some k . Let $m \in \{k, k + 1\}$. Then $\left(\begin{array}{c|c} 0 & M_{k \times m}(F) \\ \hline & 0 \end{array} \right) \notin \mathcal{A}$.*

Proof: Suppose that $\left(\begin{array}{c|c} 0 & M_{k \times m}(F) \\ \hline & 0 \end{array}\right) \subseteq \mathcal{A}$. Since \mathcal{A} is not D_2 , we have that \mathcal{A}_1 or \mathcal{A}_2 is not commutative.

We only consider the case where \mathcal{A}_1 is not commutative, since the case where \mathcal{A}_2 is not commutative can be handled in a similar way. Then there are matrices A and A' in \mathcal{A} such that $[A, A']_1 \neq 0$, and $([A, A']_1)_{ij} \neq 0$ for some pair (i, j) (see (14)), with $1 \leq i < j \leq k$. Since A and A' have constant main diagonals, we may assume that both are strictly upper triangular as far as $[A, A']$ is concerned. Moreover, since $\left(\begin{array}{c|c} 0 & M_{k \times m}(F) \\ \hline & 0 \end{array}\right) \subseteq \mathcal{A}$, we may assume that $A_3 = 0 = A'_3$, and so

$$[A, A'] = \left(\begin{array}{c|c} [A_1, A'_1] & 0 \\ \hline & [A_2, A'_2] \end{array}\right).$$

Next, we will show that there is a matrix in \mathcal{A}_2 which is not a scalar matrix. Suppose, for a contradiction, that this is not the case. Then $\dim_F \mathcal{A}_2 = 1$. We will consider the following two cases.

Case 1. $m = k$: Then we have

$$\dim_F \mathcal{A} > 2 + \left\lfloor \frac{12k^2}{8} \right\rfloor > 1 + \frac{12k^2}{8}, \quad (28)$$

and by the minimality of n ,

$$\dim_F \mathcal{A}_1 \leq 2 + \left\lfloor \frac{3k^2}{8} \right\rfloor \leq 2 + \frac{3k^2}{8}.$$

Thus, noticing that we may omit below the dimension of \mathcal{A}_2 (since, by the assumption above, all matrices in \mathcal{A}_2 are scalar), we have

$$\dim_F \mathcal{A} \leq \dim_F \mathcal{A}_1 + \dim_F(M_{k \times k}(F)) \leq 2 + \frac{11}{8}k^2,$$

which together with (28) gives us $k^2 < 8$. Thus $k \in \{1, 2\}$; i.e. in both cases \mathcal{A}_1 is commutative, a contradiction.

Case 2. $m = k + 1$: In this case, using similar arguments as above, we get $k^2 + 4k - 5 < 0$; a contradiction.

The above arguments show that there is indeed a matrix in \mathcal{A}_2 which is not scalar. Hence there is a p , with $k + 1 \leq p < k + m$, such that the p -th row of some matrix in \mathcal{A} has a nonzero non-diagonal entry. Fix p , and let

$$q = \min\{\ell : \ell > p, \text{ and } A_{p,\ell} \neq 0 \text{ for some } A \in \mathcal{A}\}. \quad (29)$$

Let $C = \left(\begin{array}{c|c} C_1 & C_3 \\ \hline 0 & C_2 \end{array} \right) \in \mathcal{A}$ be a matrix such that $C_{p,q} \neq 0$. A similar argument as above shows that we may assume that C is of the form

$$C = \left(\begin{array}{c|c} C_1 & 0 \\ \hline & C_2 \end{array} \right).$$

With $E := e_{j,p}$ (note that $E \in \mathcal{A}$, because $\left(\begin{array}{c|c} 0 & M_{k \times m}(F) \\ \hline & 0 \end{array} \right) \subseteq \mathcal{A}$), we have

$$[[A, A'], [C, E]]_{i,q} = ([A, A'] [C, E])_{i,q} - ([C, E] [A, A'])_{i,q}.$$

Direct calculation shows that

$$([A, A'] [C, E])_{i,q} = \sum_t [A, A']_{i,t} [C, E]_{t,q} = [A, A']_{i,j} [C, E]_{j,q} \neq 0,$$

and

$$([C, E] [A, A'])_{i,q} = C_{i,j} [A, A']_{p,q}.$$

Notice that, by (29), and the fact that A and A' are (without loss of generalization) strictly upper triangular,

$$[A, A']_{p,q} = \sum_t A_{p,t} A'_{t,q} - \sum_s A'_{p,t} A_{t,q} = 0,$$

Therefore

$$[[A, A'], [C, E]]_{i,q} = [A, A']_{i,j} [C, E]_{j,q} \neq 0,$$

which contradicts the assumption that \mathcal{A} is Ls_2 . □

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