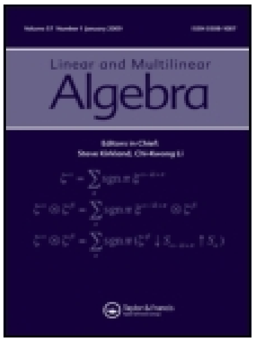

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Jordan isomorphisms of generalized structural matrix rings

S. Dăscălescu[a], S. Preduţ[a] and L. van Wyk[b]*

[a]*Faculty of Mathematics, University of Bucharest, Str. Academiei 14, Bucharest 1, RO – 010014, Romania;* [b]*Department of Mathematical Sciences (Mathematics Division), Stellenbosch University, Private Bag X1, Matieland 7602, Stellenbosch, South Africa*

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We describe the sub-bimodules of matrix bimodules over two structural matrix rings. Structural matrix bimodules arise as particular such sub-bimodules, and we discuss when such a bimodule is faithful or indecomposable. As an application, we obtain a large class of rings whose Jordan isomorphisms are either ring isomorphisms or ring anti-isomorphisms. Complete upper block triangular matrix rings over 2-torsion-free indecomposable rings are elements of this class.

Keywords: structural matrix ring; block triangular matrix ring; Jordan isomorphism; bimodule

AMS Subject Classifications: 16S50; 16D20; 15A04

1. Introduction and preliminaries

A Jordan isomorphism between two rings T and U is an isomorphism $f\colon T \to U$ of additive groups such that $f(xy+yx)=f(x)f(y)+f(y)f(x)$ for any $x, y \in T$. Ring isomorphisms and ring anti-isomorphisms are examples of Jordan isomorphisms. There has been extensive research to uncover conditions on T and U such that ring isomorphisms and anti-isomorphisms are the only Jordan isomorphisms. This was showed to be the case if $T=U$ is the ring of upper triangular matrices over a field $\neq \mathbf{F}_2$ [2], or more generally if T is the ring of upper triangular matrices over a 2-torsion-free indecomposable commutative ring C, and U is a C-algebra [1]. These results were extended in [4] to the case where $T=\left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ is a 2-torsion-free generalized triangular ring, where A and B are rings, and M is a left A, right B-bimodule which is faithful over A and over B, and U is any ring. Two particular cases where this result applies are when: (1) T is the upper triangular ring over a 2-torsion-free indecomposable ring; (2) T is a nest algebra and U is a complex algebra. We are interested to study the Jordan isomorphism problem for the case where T is a structural matrix ring, in particular when T is a complete upper block triangular matrix ring. Our approach works, in fact, for a larger class of rings.

*Corresponding author. Email: LvW@sun.ac.za

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A structural matrix ring A over a ring R is a subring of the matrix ring $\mathcal{M}_u(R)$ consisting of all matrices having zero on certain prescribed positions. Also, let B be a $v \times v$ structural matrix ring over a ring S. Let M be a left R, right S-bimodule. We consider the set $\mathcal{M}_{u,v}(M)$ of $u \times v$ matrices with entries in M, which is naturally a left A, right B-bimodule. We describe the sub-bimodules of $\mathcal{M}_{u,v}(M)$. In particular, structural matrix bimodules arise as such sub-bimodules, and all their sub-bimodules arise from here. As a special case, we recover the description of the two-sided ideals of a structural matrix ring [3]. We discuss when is a structural matrix bimodule faithful as a left A-module or as a right B-module, and when is it indecomposable as a bimodule, giving combinatorial characterizations which can be checked on the diagram of a certain partially ordered set. Combining these results and a result of Wong [4], we obtain our main result, Theorem 3.6, which describes a large class of rings T such that any Jordan isomorphism from T to another ring is either a ring isomorphism or a ring anti-isomorphism. This class includes complete upper block triangular matrix rings over 2-torsion-free indecomposable rings.

2. Sub-bimodules of matrix bimodules over structural matrix rings

Let R be a ring, and let u be a positive integer. Denote $I = \{1, \ldots, u\}$, and let $\mathcal{M}_u(R)$ be the ring of $u \times u$ matrices with entries in R. Denote by $e_{i,j}$ the matrix units. Let $\mathcal{B} \subseteq I \times I$ be such that $(i, i) \in \mathcal{B}$ for any $i \in I$, and $(i, k) \in \mathcal{B}$ whenever $(i, j) \in \mathcal{B}$ and $(j, k) \in \mathcal{B}$. Thus $\mathcal{B}$ is a preorder relation on I. We consider the structural matrix ring $A = \mathcal{M}(\mathcal{B}, R) = \sum_{(i,j) \in \mathcal{B}} Re_{i,j}$, which is the subring of $\mathcal{M}_u(R)$ consisting of all matrices whose (i,j) entries are zero for $(i,j) \notin \mathcal{B}$. We also consider a structural matrix ring $B = \mathcal{M}(\mathcal{B}', S)$ over a ring S, where $\mathcal{B}'$ is a preorder relation on the set $J = \{1, \ldots, v\}$ for some positive integer v. We denote by $f_{i,j}$ the matrix units in $\mathcal{M}_v(S)$.

Let M be a left R, right S-bimodule. We consider the set $\mathcal{M}_{u,v}(M)$ of all $u \times v$ matrices with entries in M, which is a left $\mathcal{M}_u(R)$, right $\mathcal{M}_v(S)$-bimodule with addition on positions and left and right action given by matrix-like multiplication. By restriction of scalars, $\mathcal{M}_{u,v}(M)$ becomes a left A, right B-bimodule. Our first aim is to describe the sub-bimodules of this A, B-bimodule. If $1 \le i \le u$, $1 \le j \le v$ and $m \in M$, we denote by $mM_{i,j}$ the matrix in $\mathcal{M}_{u,v}(M)$ having m on the (i,j)-th position, and zero elsewhere.

We consider the equivalence relation $\sim$ on I associated to the preorder relation $\mathcal{B}$, i.e. $i \sim j$ if and only if $(i,j) \in \mathcal{B}$ and $(j, i) \in \mathcal{B}$ (in other words both matrix units $e_{i,j}$ and $e_{j,i}$ lie in A). Let $I_1, \ldots, I_p$ be the associated equivalence classes, and fix some elements $i_1 \in I_1, \ldots, i_p \in I_p$. Then $\mathcal{B}$ induces a partial order $\preccurlyeq$ on $\{1, \ldots, p\}$ defined by $\alpha \preccurlyeq \alpha'$ if and only if $(i_\alpha, i_{\alpha'}) \in \mathcal{B}$. Clearly if $\alpha \preccurlyeq \alpha'$, then $(\delta, \delta') \in \mathcal{B}$ for any $\delta \in I_\alpha$ and $\delta' \in I_{\alpha'}$.

Similarly, we consider the equivalence relation $\approx$ on J associated to $\mathcal{B}'$, with equivalence classes $J_1, \ldots, J_q$. Fix some elements $j_1 \in J_1, \ldots, j_q \in J_q$, and let the partial order $\preccurlyeq$ on $\{1, \ldots, q\}$ be such that $\beta \preccurlyeq \beta'$ if and only if $(j_\beta, j_{\beta'}) \in \mathcal{B}'$.

Proposotion 2.1 *Let $\mathcal{F} = (N_{\alpha\beta})_{\substack{1 \le \alpha \le p \\ 1 \le \beta \le q}}$ be a family of sub-bimodules of ${}_R M_S$ such that $N_{\alpha\beta} \subseteq N_{\alpha'\beta'}$ for any $\alpha' \preccurlyeq \alpha$ and $\beta \preccurlyeq \beta'$. Then the set*

$$X_{\mathcal{F}} = \left\{ (m_{ij})_{\substack{1 \le i \le u \\ 1 \le j \le v}} \middle| m_{ij} \in N_{\alpha\beta} \text{ for any } 1 \le \alpha \le p, 1 \le \beta \le q \text{ and any } i \in I_\alpha, j \in J_\beta \right\}$$

is a sub-bimodule of ${}_A\mathcal{M}_{u,v}(M)_B$. Moreover, the correspondence $\mathcal{F} \mapsto X_{\mathcal{F}}$ is a bijection between the set of all such families $\mathcal{F}$ and the set of all sub-bimodules of ${}_A\mathcal{M}_{u,v}(M)_B$.

Proof It is straightforward to check that $X_{\mathcal{F}}$ is a sub-bimodule of ${}_A\mathcal{M}_{u,v}(M)_B$.

Now, let X be a sub-bimodule of ${}_A\mathcal{M}_{u,v}(M)_B$. For any $1 \le \alpha \le p$ and $1 \le \beta \le q$, we consider the set

$$N_{\alpha\beta} = \{m \in M |\ mM_{i_\alpha, j_\beta} \in X\}.$$

If $m \in N_{\alpha\beta}$, $r \in R$ and $s \in S$, we have that $(rms)M_{i_\alpha, j_\beta} = (re_{i_\alpha, i_\alpha})(mM_{i_\alpha, j_\beta})(sf_{j_\beta, j_\beta}) \in X$, so $rms \in N_{\alpha\beta}$, and thus $N_{\alpha\beta}$ is a sub-bimodule of ${}_RM_S$. If $\alpha' \preccurlyeq \alpha$ and $\beta \preccurlyeq \beta'$, let $m \in N_{\alpha\beta}$. Then

$$mM_{i_{\alpha'}, j_{\beta'}} = e_{i_{\alpha'}, i_\alpha}(mM_{i_\alpha, j_\beta})f_{j_\beta, j_{\beta'}} \in X,$$

and this shows that $m \in N_{\alpha'\beta'}$.

Denote by $\mathcal{F}$ the family $(N_{\alpha\beta})_{\substack{1\le\alpha\le p\\ 1\le\beta\le q}}$. We show that $X = X_{\mathcal{F}}$. Let first $U = (m_{ij})_{\substack{1\le i\le u\\ 1\le j\le v}} \in X_{\mathcal{F}}$. If $i \in I_\alpha$ and $j \in J_\beta$, then $m_{ij} \in N_{\alpha\beta}$, so $m_{ij}M_{i_\alpha, j_\beta} \in X$, and

$$m_{ij}M_{i,j} = e_{i,i_\alpha}(m_{ij}M_{i_\alpha, j_\beta})f_{j_\beta, j} \in X.$$

Then $U = \sum_{i,j} m_{ij}M_{i,j} \in X$.

Conversely, if $U = (m_{ij})_{\substack{1\le i\le u\\ 1\le j\le v}} \in X$, pick some $1 \le \alpha \le p$ and $1 \le \beta \le q$, and let $i \in I_\alpha$, $j \in J_\beta$. Then $m_{ij}M_{i_\alpha, j_\beta} = e_{i_\alpha, i}Uf_{j, j_\beta} \in X$, so $m_{ij} \in N_{\alpha\beta}$. Thus $U \in X_{\mathcal{F}}$.

We conclude that the correspondence $\mathcal{F} \mapsto X_{\mathcal{F}}$ is a bijection. ■

Example 2.2 The case where A and B are complete upper block triangular matrix rings (in particular when they are upper triangular matrix rings) is of special interest. Thus,

$$A = \begin{pmatrix} \mathcal{M}_{d_1}(R) & \mathcal{M}_{d_1,d_2}(R) & \cdots & \mathcal{M}_{d_1,d_p}(R) \\ 0 & \mathcal{M}_{d_2}(R) & \cdots & \mathcal{M}_{d_2,d_p}(R) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \mathcal{M}_{d_p}(R) \end{pmatrix}$$

for some positive integers $d_1, d_2, \ldots, d_p$ with $u = d_1 + \cdots + d_p$. In this case $I_1 = \{1, \ldots, d_1\}$, $I_2 = \{d_1 + 1, \ldots, d_1 + d_2\}, \ldots, I_p = \{d_1 + \cdots + d_{p-1} + 1, \ldots, u\}$ and $1 \preccurlyeq 2 \preccurlyeq \cdots \preccurlyeq p$. Similarly, let B be the complete upper-blocked triangular matrix ring of type $\delta_1, \delta_2, \ldots, \delta_q$ over S, where $v = \delta_1 + \cdots + \delta_q$. We have that $J_1 = \{1, \ldots, \delta_1\}$, $J_2 = \{\delta_1 + 1, \ldots, \delta_1 + \delta_2\}, \ldots, J_q = \{\delta_1 + \cdots + \delta_{q-1} + 1, \ldots, v\}$ and $1 \preccurlyeq 2 \preccurlyeq \cdots \preccurlyeq q$. Then the left A, right B-sub-bimodules of $\mathcal{M}_{u,v}(M)$ are the subsets of the form

$$\begin{pmatrix} \mathcal{M}_{d_1,\delta_1}(N_{11}) & \mathcal{M}_{d_1,\delta_2}(N_{12}) & \cdots & \mathcal{M}_{d_1,\delta_q}(N_{1q}) \\ \mathcal{M}_{d_2,\delta_1}(N_{21}) & \mathcal{M}_{d_2,\delta_2}(N_{22}) & \cdots & \mathcal{M}_{d_2,\delta_q}(N_{2q}) \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{M}_{d_p,\delta_1}(N_{p1}) & \mathcal{M}_{d_p,\delta_2}(N_{p2}) & \cdots & \mathcal{M}_{d_p,\delta_q}(N_{pq}) \end{pmatrix}$$

where $(N_{\alpha\beta})_{\substack{1\le\alpha\le p\\ 1\le\beta\le q}}$ is a family of sub-bimodules of M such that $N_{\alpha\beta} \subseteq N_{\alpha'\beta'}$ for any $\alpha' \le \alpha$ and $\beta \le \beta'$.

Now, we consider a special type of sub-bimodules of $\mathcal{M}_{u,v}(M)$, namely the ones corresponding to families $\mathcal{F}$ for which each $N_{\alpha\beta}$ is either 0 or M. Extending the terminology from rings, we call them structural matrix bimodules. By the description of sub-bimodules of $\mathcal{M}_{u,v}(M)$ given in Proposition 2.1, we see that if such a structural matrix bimodule has M on position (i,j), then it must have M on any position (i',j') with $i \sim i'$ and $j \approx j'$. Thus the structural matrix sub-bimodules of $\mathcal{M}_{u,v}(M)$ are the sets of the form

$$\mathcal{M}(\mathcal{P}, M) = \{(m_{ij})_{i,j} \mid m_{ij} = 0 \text{ if } (i,j) \notin \cup_{(\alpha,\beta)\in\mathcal{P}} \ I_\alpha \times J_\beta\}$$

where $\mathcal{P} \subseteq \{1,\ldots,p\} \times \{1,\ldots,q\}$ is a set such that for any $(\alpha,\beta) \in \mathcal{P}$ and any α' and β' such that $\alpha' \preccurlyeq \alpha$ and $\beta \precsim \beta'$, we also have that $(\alpha',\beta') \in \mathcal{P}$. With this notation, we get as a direct consequence of Proposition 2.1 the following.

Corollary 2.3 *The sub-bimodules of ${}_A\mathcal{M}(\mathcal{P}, M)_B$ are in bijection to the set of families $(N_{\alpha\beta})_{(\alpha,\beta)\in\mathcal{P}}$ of sub-bimodules of ${}_RM_S$ with the property that $N_{\alpha\beta} \subseteq N_{\alpha'\beta'}$ for any $(\alpha,\beta) \in \mathcal{P}$ and any $\alpha' \preccurlyeq \alpha$, $\beta \precsim \beta'$.*

Remark 2.4 In the particular case where $A = B$, $M = {}_RR_R$ and $\mathcal{M}(\mathcal{P}, M) = A$, our result describes the two-sided ideals of a structural matrix ring. This description was given in [3, Proposition 1.2].

3. Faithful structural matrix bimodules

We keep the notation of the previous section and discuss the following problem: when is the structural matrix bimodule $\mathcal{M}(\mathcal{P}, M)$ faithful as a left A-module, respectively, as a right B-module? The answer will be given as a consequence of the following description of the annihilator of a structural matrix bimodule.

Proposition 3.1

(1) *$ann_A(\mathcal{M}(\mathcal{P}, M))$ is the two-sided ideal of A corresponding to the family $(H_{\alpha\gamma})_{\substack{1\le\alpha,\gamma\le p\\ \alpha\preccurlyeq\gamma}}$, where $H_{\alpha\gamma} = ann_R(M)$ whenever $\alpha \preccurlyeq \gamma$ and there exists β with $(\gamma,\beta) \in \mathcal{P}$, and $H_{\alpha\gamma} = R$ whenever $\alpha \preccurlyeq \gamma$ and there does not exist β with $(\gamma,\beta) \in \mathcal{P}$.*

(2) *$ann_B(\mathcal{M}(\mathcal{P}, M))$ is the two-sided ideal of B corresponding to the family $(K_{\delta\beta})_{\substack{1\le\delta,\beta\le q\\ \delta\beta}}$, where $K_{\delta\beta} = ann_S(M)$ whenever $\delta \precsim \beta$ and there exists α with $(\alpha,\delta) \in \mathcal{P}$, and $K_{\delta\beta} = S$ whenever $\delta \precsim \beta$ and there does not exist α with $(\alpha,\delta) \in \mathcal{P}$.*

Proof

(1) By Corollary 2.3, we have that the ideal $H = ann_A(\mathcal{M}(\mathcal{P}, M))$ of A corresponds to a family $(H_{\alpha\gamma})_{\substack{1\le\alpha,\gamma\le p\\ \alpha\preccurlyeq\gamma}}$ of ideals of R such that $H_{\alpha\gamma} \subseteq H_{\alpha'\gamma'}$ whenever $\alpha' \preccurlyeq \alpha$ and $\gamma \precsim \gamma'$. Fix some i and t, with $i \in I_\alpha$, $t \in I_\gamma$ and $\alpha \preccurlyeq \gamma$.
If there is no β such that $(\gamma, \beta) \in \mathcal{P}$, then for any $r \in R$, we have that $(re_{i,t})(mM_{z,j}) = 0$ for any $mM_{z,j}$ lying in $\mathcal{M}(\mathcal{P}, M)$. Indeed, this is clearly 0 if $z \neq t$, while if $z = t$, then there is no j such that $mM_{z,j} \in \mathcal{M}(\mathcal{P}, M)$. We obtain that $H_{\alpha\gamma} = R$.
If there exists β such that $(\gamma,\beta) \in \mathcal{P}$, then for a fixed $r \in R$, we have that $r \in H_{\alpha\gamma}$ if and only if $(re_{i,t})(mM_{z,j}) = 0$ for any $mM_{z,j}$ lying in $\mathcal{M}(\mathcal{P}, M)$. This clearly holds if $z \neq t$, while if $z = t$, it is equivalent to $rM = 0$, i.e. $r \in ann_R(M)$;

(2) is similar. ∎

Corollary 3.2

(1) *If M is faithful as a left R-module, then $\mathcal{M}(\mathcal{P}, M)$ is faithful as a left A-module if and only if for any $1 \leq \alpha \leq p$ there exists β such that $(\alpha, \beta) \in \mathcal{P}$ (in other words any row of $\mathcal{M}(\mathcal{P}, M)$ is non-zero).*

(2) *If M is faithful as a right S-module, then $\mathcal{M}(\mathcal{P}, M)$ is faithful as a right B-module if and only if for any $1 \leq \beta \leq q$ there exists α such that $(\alpha, \beta) \in \mathcal{P}$ (in other words any column of $\mathcal{M}(\mathcal{P}, M)$ is non-zero).*

Example 3.3

(a) If A is complete upper block triangular, then $\mathcal{M}(\mathcal{P}, M)$ is faithful as a left A-module if and only if it has an entire column of M's. Indeed, faithfulness implies that there exists β such that $(p, \beta) \in \mathcal{P}$. Since $1 \preccurlyeq 2 \preccurlyeq \cdots \preccurlyeq p$, we obtain that $(i, \beta) \in \mathcal{P}$ for any i.

(b) If both A and B are complete upper block triangular, then $\mathcal{M}(\mathcal{P}, M)$ is faithful as a left A-module and as a right B-module if and only if the first row and the last column of $\mathcal{M}(\mathcal{P}, M)$ consist only of M's.

4. Indecomposable structural matrix bimodules

Let us consider a structural matrix bimodule $\mathcal{M}(\mathcal{P}, M)$, where A and B are structural matrix rings, and the notation is as in Section 2. We are interested to see when is $\mathcal{M}(\mathcal{P}, M)$ an indecomposable bimodule, i.e. when it cannot be written as a direct sum of two non-zero sub-bimodules. It is useful to regard $\mathcal{P}$ as a partially ordered set with the ordering relation $\leq$ defined by $(\alpha, \beta) \leq (\alpha', \beta')$ if and only if $\alpha' \preccurlyeq \alpha$ and $\beta \preccurlyeq \beta'$. Now, we have the following characterization of indecomposability.

Proposition 4.1 *Let $\mathcal{M}(\mathcal{P}, M)$ be a non-zero structural matrix bimodule over the structural matrix rings A and B. Then the following assertions are equivalent:*

(1) *$\mathcal{M}(\mathcal{P}, M)$ is an indecomposable bimodule.*

(2) *M is an indecomposable R, S-bimodule and $\mathcal{P}$ cannot be written as a disjoint union of two non-empty subsets $\mathcal{P}_1$ and $\mathcal{P}_2$ such that if $(\alpha, \beta) \in \mathcal{P}_i$, where $i \in \{1, 2\}$ and $(\alpha, \beta) \leq (\alpha', \beta')$, then $(\alpha', \beta') \in \mathcal{P}_i$.*

Proof (1) $\Rightarrow$ (2) If $M = X \oplus Y$, a direct sum of R, S-bimodules, we have that $\mathcal{M}(\mathcal{P}, M) = \mathcal{M}(\mathcal{P}, X) \oplus \mathcal{M}(\mathcal{P}, Y)$, a direct sum of A, B-bimodules. Since $\mathcal{M}(\mathcal{P}, M)$ is indecomposable, we must have either $\mathcal{M}(\mathcal{P}, X) = 0$ or $\mathcal{M}(\mathcal{P}, Y) = 0$, showing that $X = 0$ or $Y = 0$. Thus, M is indecomposable. On the other hand, if $\mathcal{P}$ would be the disjoint union of two non-empty sets $\mathcal{P}_1$ and $\mathcal{P}_2$ as described in (2), then $\mathcal{M}(\mathcal{P}, M) = \mathcal{M}(\mathcal{P}_1, M) \oplus \mathcal{M}(\mathcal{P}_2, M)$, contradicting the indecomposability of $\mathcal{M}(\mathcal{P}, M)$.

(2) $\Rightarrow$ (1) Let $\mathcal{M}(\mathcal{P}, M) = \mathcal{X} \oplus \mathcal{Y}$, a direct sum of sub-bimodules. By Corollary 2.3, $\mathcal{X}$ corresponds to a family $(X_{\alpha\beta})_{(\alpha,\beta) \in \mathcal{P}}$ of sub-bimodules of M, and $\mathcal{Y}$ corresponds to a family $(Y_{\alpha\beta})_{(\alpha,\beta) \in \mathcal{P}}$ of sub-bimodules. Then for any $(\alpha, \beta) \in \mathcal{P}$, we have that $X_{\alpha\beta} \oplus Y_{\alpha\beta} = M$, so either $X_{\alpha\beta} = 0$ (and then $Y_{\alpha\beta} = M$) or $Y_{\alpha\beta} = 0$ (and then $X_{\alpha\beta} = M$). Let $\mathcal{P}_1 = \{(\alpha, \beta) | \ X_{\alpha\beta} \neq 0\}$ and $\mathcal{P}_2 = \{(\alpha, \beta) | \ Y_{\alpha\beta} \neq 0\}$. It is clear that $\mathcal{P}$ is the disjoint union of $\mathcal{P}_1$ and $\mathcal{P}_2$. Since $X_{\alpha\beta} \subseteq X_{\alpha'\beta'}$ for any $(\alpha, \beta) \leq (\alpha', \beta')$, we see that if $(\alpha, \beta) \in \mathcal{P}_1$,

then also $(\alpha', \beta') \in \mathcal{P}_1$, and similarly for $\mathcal{P}_2$. Then, necessarily one of $\mathcal{P}_1$ and $\mathcal{P}_2$ is empty, which shows that either $\mathcal{X} = 0$ or $\mathcal{Y} = 0$. ■

Remark 4.2 The condition

(*) $\mathcal{P}$ cannot be written as a disjoint union of some non-empty subsets $\mathcal{P}_1$ and $\mathcal{P}_2$ such that if $(\alpha, \beta) \in \mathcal{P}_i$, where $i \in \{1, 2\}$ and $(\alpha, \beta) \leq (\alpha', \beta')$, then $(\alpha', \beta') \in \mathcal{P}_i$ means in fact that $\mathcal{P}$ is not the coproduct of two non-empty objects in the category of partially ordered sets, i.e. it is indecomposable in this category. We consider another condition

(**) For any minimal elements (α_1, β_1) and (α_2, β_2) in the partially ordered set $\mathcal{P}$ there exists $(\alpha, \beta) \in \mathcal{P}$ such that $(\alpha_1, \beta_1) \leq (\alpha, \beta)$ and $(\alpha_2, \beta_2) \leq (\alpha, \beta)$.

Then we have that (**) implies (*), thus (**) is a sufficient condition for the indecomposability of $\mathcal{M}(\mathcal{P}, M)$ in the case where M is indecomposable. Clearly, it is easy to check whether (**) holds or not by looking at the diagram associated to the partially ordered set $\mathcal{P}$. Nevertheless, (**) is not a necessary condition for (*). Indeed, we give the following example. Let

$$A = \begin{pmatrix} R & R & R & R \\ 0 & R & R & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & R \end{pmatrix}, \quad B = \begin{pmatrix} S & S & S & S \\ 0 & S & S & 0 \\ 0 & 0 & S & 0 \\ 0 & 0 & 0 & S \end{pmatrix},$$

thus $1 \preccurlyeq 2 \preccurlyeq 3$ and $1 \preccurlyeq 4$, and $1 \preccurlyeq 2 \preccurlyeq 3$ and $1 \preccurlyeq 4$. Let

$$\mathcal{M}(\mathcal{P}, M) = \begin{pmatrix} M & M & M & M \\ 0 & 0 & 0 & M \\ 0 & 0 & 0 & M \\ 0 & 0 & M & 0 \end{pmatrix},$$

thus $\mathcal{P} = \{(1,1), (1,2), (1,3), (1,4), (2,4), (3,4), (4,3)\}$.

$$\begin{tikzcd}
 & (1,3) & (1,4) \\
 & (1,2) \arrow[u] & (2,4) \arrow[u] \\
(4,3) \arrow[uur] & (1,1) \arrow[u] \arrow[uur] & (3,4) \arrow[u]
\end{tikzcd}$$

It is easy to see that $\mathcal{P}$ satisfies condition (*), but it does not satisfy (**), as we can see by considering the minimal elements (3, 4) and (4, 3).

It is possible to give a more approachable method to check whether $\mathcal{P}$ is indecomposable. In fact, we can consider a finite partially ordered set $(X, \leq)$, and let $Min(X)$ be the set of minimal elements of X. Consider the relation ρ on $Min(X)$, defined by $x \rho y$ if and only if there exists $z \in X$ such that $x \leq z$ and $y \leq z$. It is clear that ρ is reflexive and symmetric, but not necessarily transitive. Now, we can consider the induced equivalence relation $\asymp$ on $Min(X)$, defined by $x \asymp y$ if and only if there exist a positive integer n and $x_1, \dots, x_n \in Min(X)$ such that $x = x_1, x_1 \rho x_2, \dots, x_{n-1} \rho\ x_n$, $x_n = y$. Let $(M_i)_{i \in I}$ be the set of equivalence classes with respect to $\asymp$, and for any $i \in I$ let $X_i = \{z \in X |\ \exists\ x \in M_i \text{ with } x \leq z\}$. Then it is easy to see that each X_i is an indecomposable partially ordered set (with the order relation induced from X), and X is the disjoint union of all X_i's. In conclusion, $(X, \leq)$ is indecomposable if and only if $\asymp$ has only one equivalence class. Applying this to our context, we obtain the following, which checks the indecomposability of $\mathcal{P}$ by looking at its diagram.

Proposition 4.3 *Condition* (*) *is equivalent to the fact that for any minimal elements* (α, β) *and* (α', β') *of* $\mathcal{P}$, *there exist minimal elements* $(\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)$ *of* $\mathcal{P}$ *such that* $(\alpha_1, \beta_1) = (\alpha, \beta)$, $(\alpha_n, \beta_n) = (\alpha', \beta')$, *and for any* $1 \leq i \leq n-1$ *there exists* $(\gamma_i, \delta_i) \in \mathcal{P}$ *such that* $(\alpha_i, \beta_i) \leq (\gamma_i, \delta_i)$ *and* $(\alpha_{i+1}, \beta_{i+1}) \leq (\gamma_i, \delta_i)$.

We note that condition (**) in Remark 4.2 just says that any two minimal elements of $\mathcal{P}$ are in the relation ρ, which is clearly a stronger condition than being equivalent with respect to the relation $\asymp$.

Corollary 4.4 *If* $\mathcal{P}$ *has a smallest element, then* $\mathcal{M}(\mathcal{P}, M)$ *is an indecomposable bimodule if and only if* M *is an indecomposable bimodule.*

Remark 4.5 As a particular case of Corollary 4.4, we see that if A and B are complete upper block triangular matrix rings as in Example 2.2, then $\mathcal{M}_{u,v}(M)$ is an indecomposable A, B-bimodule if and only if M is an indecomposable R, S-bimodule. In particular, if $S = R$ and $M = R$, we have that $\mathcal{M}_{u,v}(R)$ is an indecomposable A, B-bimodule if and only if R is an indecomposable ring, i.e. it does not have non-trivial central idempotents. In the particular case of upper triangular matrix rings (i.e. A and B have only blocks of size 1), this recovers [4, Theorem 2.1].

Now using the result of Wong [4, Theorem 3.1], we obtain the following.

Theorem 4.6 *Let* M *be an indecomposable left* R, *right* S-*bimodule, such that* M *is faithful as a left* R-*module and as a right* S-*module, and* R, S *and* M *are* 2-*torsion-free. Let* A *and* B *be structural matrix rings over* R *and* S, *and let* $\mathcal{M}(\mathcal{P}, M)$ *be a structural matrix bimodule with non-zero rows and columns and such that* $\mathcal{P}$ *satisfies condition* (*). *Then any Jordan isomorphism from the triangular ring* $T = \left(\begin{smallmatrix} A & \mathcal{M}(\mathcal{P}, M) \\ 0 & B \end{smallmatrix}\right)$ *to another ring is either a ring isomorphism or a ring anti-isomorphism.*

Corollary 4.7 *Any Jordan isomorphism from an upper block triangular matrix ring* (*in particular an upper triangular matrix ring*) *over a* 2-*torsion-free indecomposable ring to another ring is either a ring isomorphism or a ring anti-isomorphism.*

Remark 4.8 Any structural matrix ring T, which is not the full matrix ring (over a ring R), can be regarded (in several ways) as a generalized triangular ring. Indeed, by a permutation of rows and columns, T is isomorphic to a (not necessarily complete) upper block triangular matrix ring. If there are h diagonal blocks, then we can split T as a generalized triangular ring $\left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$, where A is the ring obtained from T by taking the intersection of rows and columns of the first g diagonal blocks, where $g < h$, B is obtained similarly from the rest of $h - g$ diagonal blocks, and M is obtained by taking the intersection of the rows of the first g diagonal blocks and the columns of the other $h - g$ diagonal blocks. The Jordan isomorphism problem can be tested for any such representation of T, using Theorem 4.6.

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