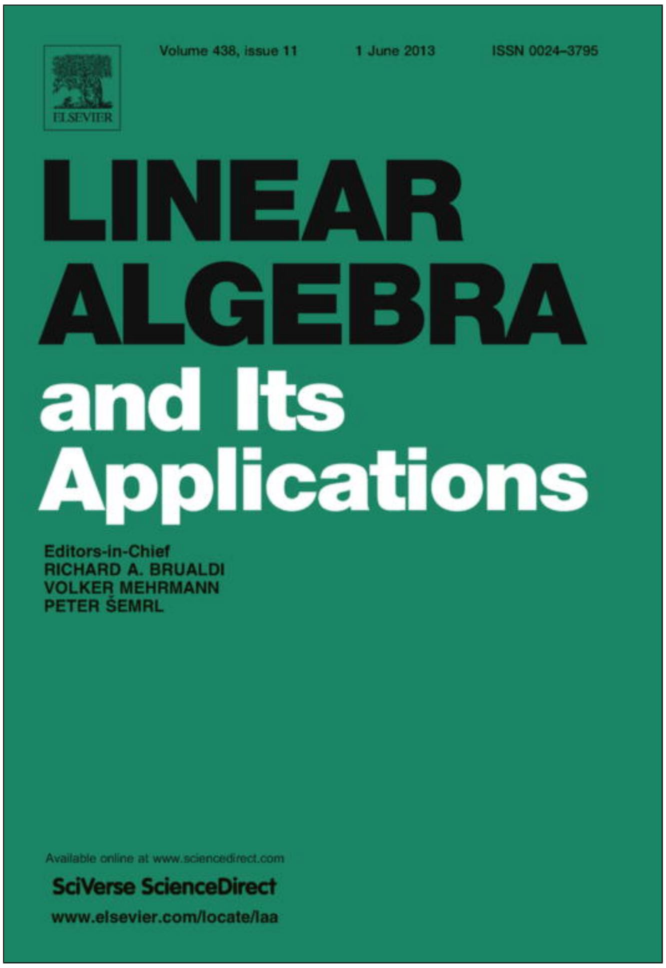

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Isomorphisms between strongly triangular matrix rings

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ABSTRACT

We describe isomorphisms between strongly triangular matrix rings that were defined earlier in Birkenmeier et al. (2000) [3] as ones having a complete set of triangulating idempotents, and we show that the so-called triangulating idempotents behave analogously to idempotents in semiperfect rings. This study yields also a way to compute theoretically the automorphism groups of such rings in terms of corresponding automorphism groups of certain subrings and bimodules involved in their structure, which completes the project started in Anh and van Wyk (2011) [1].

1. Introduction

Triangular matrix rings appear naturally in the theory of certain algebras, like nilpotent and solvable Lie algebras, Kac-Moody, Virasoro and Heisenberg algebras (see, for example, [6]), as well as in algebras of certain directed trees. In the latter case the triangular matrix rings may be seen to provide the abstract description of such quiver algebras without mentioning the associated directed tree and without appropriate numbering of the vertices.

Triangular matrix rings have become an important object of intense research, for example, it is a key tool in the description of semiprimary hereditary rings (see, for example, [4]), and certain triangular matrix rings are natural examples of representation-finite hereditary algebras (see, for example, [2,5]).

On the other hand, Birkenmeier et al. in [3] developed the general theory of generalized triangular matrix rings and used it to describe several particular classes of rings. Combining their terminology

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with ones (introduced later) in [1] we say that a ring A admits an *m-strongly (upper) triangular matrix decomposition with respect to an ordered sequence* $\{e_1, \ldots, e_m\}$ if the e_i's are pairwise orthogonal idempotents in A such that $1 = e_1 + \cdots + e_m,\ e_jAe_i = 0$ for all $j > i$ and e_iAe_i is semicentral reduced for every i, or equivalently, $\{e_1, \ldots, e_m\}$ is *a complete set of left triangulating idempotents* by terminology of [3] . Here, according to [3], an idempotent e in a ring A is called *semicentral* if $(1 - e)Ae = 0$, and A is called *semicentral reduced* if 0 and 1 are the only semicentral idempotents in A, i.e., A is semicentral reduced if and only if A is strongly indecomposable in the sense of [1]. Therefore, an idempotent $e \in A$ is semicentral reduced if it is semicentral and the subring eAe is a strongly indecomposable ring. If we set $R_i := e_iAe_i$ and $L_{ij} := e_iAe_j$ for $i < j$, then A can be written as a generalized upper triangular matrix ring

$$\begin{bmatrix} R_1 & L_{12} & L_{13} & \cdots & L_{1m} \\ 0 & R_2 & L_{23} & \cdots & L_{2m} \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & R_{m-1} & L_{m-1,m} \\ 0 & \cdots & \cdots & 0 & R_m \end{bmatrix}$$

with the obvious matrix addition and multiplication. It was pointed out in [3] that by reversing the order of the sequence $\{e_1, \ldots, e_m\}$ one obtains a new sequence providing the lower triangular matrix representation for A. Therefore it is not a restriction to study rings with a complete set of left triangulating idempotents.

The aim of this paper is to describe isomorphisms between strongly triangular matrix rings, thereby finishing the project initiated in [1]. As a by-product we show that triangulating idempotents behave similarly to idempotents in semiperfect rings. Namely, if one fixes a complete set $\{e_1, \ldots, e_m\}$ of triangulating idempotents, then a left ideal generated by any semicentral idempotent is isomorphic to one generated by an appropriate partial sum of some idempotents from the set $\{e_1, \ldots, e_m\}$.

For more information and detailed treatment of triangular matrix rings and their applications in other areas of mathematics we refer to [3], and for some interesting related questions on matrix rings we refer to [7].

2. Strongly triangular matrix rings

A strongly (upper) triangular matrix decomposition of a ring A depends essentially on the ordered sequence $\{e_1, \ldots, e_m\}$ of pairwise orthogonal idempotents with sum 1. However, in particular cases, another ordering of the set $\{e_1, \ldots, e_m\}$ may also give a strongly triangular matrix decomposition of A.

Furthermore, if there is no room for misunderstanding, then for short we sometimes say that a ring A is a *strongly triangular matrix ring*, without stating exactly the ordering on the set $\{e_1, \ldots, e_m\}$. Therefore one has to see clearly that all R_i are semicentral reduced, but all e_i, except e_1, need not be even semicentral idempotents of A, i.e., e_i for $i \geqq 1$ is certainly reduced semicentral only in the subring $A_i = (e_i + \cdots + e_m)A(e_i + \cdots + e_m)$ of $A = A_1$ but not necessarily in A_j with $j < i$. For example, if A is a strongly triangular matrix ring with respect to the ordered sequence $\{e_1, e_2, e_3\}$, then the generalized matrix decompositions of A with respect to the ordered sequence $\{e_2, e_1, e_3\}$ and $\{e_2, e_3, e_1\}$ are

$$\begin{bmatrix} R_2 & 0 & L_{23} \\ \\ L_{12} & R_1 & L_{13} \\ \\ 0 & 0 & R_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} R_2 & L_{23} & 0 \\ \\ 0 & R_3 & 0 \\ \\ L_{12} & L_{13} & R_1 \end{bmatrix},$$

respectively, which are definitely not triangular matrix decompositions of A.

Next, let B be an n-strongly triangulated matrix ring with respect to an ordered sequence $\{f_1, \ldots, f_n\}$, i.e., the f_i's are pairwise orthogonal idempotents in B with sum 1, $f_jBf_i = 0$ for all $j > i$, $S_i := f_iBf_i$ is semicentral reduced for every i, and f_i is a semicentral reduced idempotent of the ring $B_i = (f_i + \cdots + f_n)B(f_i + \cdots + f_n)$ for $i = 1, \ldots, n$. For each $i \neq j$, let $M_{ij} = f_iBf_j$. Therefore $M_{ij} = 0$ for all $j < i$. Moreover, for each i let M_i be the i-th truncated row of B, i.e.,

$$M_i = \oplus_{k>i} M_{ik} = \oplus_{k\neq i} M_{ik}.$$

If σ is any permutation on $\{1, \ldots, n\}$, then σ induces a new (generalized) matrix ring decomposition on B with respect to the ordered sequence $\{f_1^\sigma := f_{\sigma(1)}, \ldots, f_n^\sigma := f_{\sigma(n)}\}$. According to this notation, if we write $g_i = f_i^\sigma$, then one can identify the above convention as follows. Let $T_i = g_iBg_i,\ C_i = (g_i + \cdots + g_n)B(g_i + \cdots + g_n),\ N_{ij} = g_iBg_j$ for all $i \neq j,\ N_i = \oplus_{k\neq i} N_{ik}$. It is important to emphasize that B is not necessarily an n-strongly triangular matrix ring with respect to the ordered sequence $\{f_1^\sigma, \ldots, f_n^\sigma\}$. From the above one gets

$$T_i = S_i^\sigma := S_{\sigma(i)}, \quad C_i = B_i^\sigma := B_{\sigma(i)}, \quad N_i = M_i^\sigma := M_{\sigma(i)}.$$

Now we are in a position to state the main result precisely.

Theorem. *Let A and B be m- and n-strongly triangular matrix rings with respect to ordered sequences* $\{e_1, \ldots, e_m\}$ *and* $\{f_1, \ldots, f_n\}$, *respectively. Then A and B are isomorphic via an isomorphism* $\varphi : A \to B$ *iff* $m = n$ *and there is a permutation* σ *of* $\{1, \ldots, m\}$ *such that B is also an m-strongly triangular matrix ring with respect to the ordered sequence* $\{f_1^\sigma, \ldots, f_m^\sigma\}$, *there are ring isomorphisms* $\rho_i : R_i \to S_i^\sigma = S_{\sigma(i)},\ i = 1, \ldots, m = n$, *and for* $i = 1, \ldots, m-1$ *there are elements* $m_i \in M_i^\sigma$ *and ring isomorphisms* $\varphi_{i+1} : A_{i+1} \to B_{i+1}^\sigma$ *and* $R_i - A_{i+1}$*-bimodule isomorphisms* $\chi_i : e_iA_i(e_{i+1} + \cdots + e_{m(=n)}) = L_i \to M_i^\sigma$ *with respect to* ρ_i, φ_{i+1}, *such that for* $i = 1, \ldots, m-1$ *and*

$$a_i = \begin{bmatrix} r_i & \ell_i \\ \\ 0 & a_{i+1} \end{bmatrix} \in A_i = \begin{bmatrix} R_i & L_i \\ \\ 0 & A_{i+1} \end{bmatrix},$$

$$\varphi_i(a_i) = \begin{bmatrix} \rho_i(r_i) & \rho_i(r_i)m_i + \chi_i(\ell_i) - m_i\varphi_{i+1}(a_{i+1}) \\ \\ 0 & \varphi_{i+1}(a_{i+1}) \end{bmatrix}.$$

Moreover, all isomorphisms between isomorphic rings A and B can be described in this manner. (Keep in mind that $\varphi_1 = \varphi,\ \varphi_m = \rho_m;\ A_m = R_m$.*)*

Remark 1. The equality $m = n$ as well as some invariants associated to a complete set of triangulating idempotents up to a permutation σ were already obtained as Theorem 2.10 in [3], where an isomorphism is (surprisingly enough) an inner automorphism. However, these results are by-products of our description of general isomorphisms between such rings and our treatment is both elementary and direct. For further details for structural discussion we refer to Theorems 2.10, 3.3 and Corollary 3.4 in [3].

Proof of Theorem. We prove this theorem by induction on m, a number of pairwise orthogonal idempotents e_i in the ordered sequence $\{e_1, \ldots, e_m\}$ giving a strongly triangular matrix ring decomposition on A. The case $m = 1$ is obvious by the definition, because B must be also semicentral reduced, i.e., $m = n = 1$. Assume now that $m \geqslant 2$ and the theorem holds for $m - 1$.

The first induction step is the following obvious but interesting result (by direct computation, see also [1]). Because of its importance we state it separately as a self-contained assertion.

Proposition. *Let $e \in A$ and $f \in B$ be semicentral idempotents. Put $R = eAe$, $S = fBf$, $\bar{A} = (1-e)A(1-e)$, $\bar{B} = (1-f)B(1-f)$, $L = eA(1-e)$, $M = fB(1-f)$, i.e., $A = \begin{bmatrix} R & L \\ 0 & \bar{A} \end{bmatrix}$, $B = \begin{bmatrix} S & M \\ 0 & \bar{B} \end{bmatrix}$, and let $\varphi : A \to B$ be a ring isomorphism. Then $\varphi(e) \in f + M$ if and only if there are ring isomorphisms $\rho : R \to S$ and $\bar{\varphi} : \bar{A} \to \bar{B}$ and an $R - \bar{A}$-bimodule isomorphism $\chi : L \to M$ (M is an $R - \bar{A}$-bimodule via ρ and $\bar{\varphi}$) and an element m in M such that*

$$\varphi\left(\begin{bmatrix} r & \ell \\ 0 & a \end{bmatrix}\right) = \begin{bmatrix} \rho(r) & \rho(r)m + \chi(\ell) - m\bar{\varphi}(a) \\ 0 & \bar{\varphi}(a) \end{bmatrix}. \tag{1}$$

In particular, χ is just the restriction of φ to L. Moreover, all isomorphisms φ from A to B satisfying $\varphi(e) \in f + M$ can be obtained from a quadruple $(\rho, \bar{\varphi}, \chi, m)$ in this manner.

For the verification of the **Proposition** one observes $\varphi(e) \in f + M$ if and only if $\varphi(e) = f + m = f_m$ for some $m \in M$. Therefore $\varphi(1-e) = 1 - f - m = (1-f) - m = g_m$. Put $g = 1 - f$. Then by direct calculations (see also Lemma 2.2 in [1]) one has $M = fBg = f_mBg_m$ and canonical isomorphisms $S \cong f_mBf_m : v \in S \mapsto v + vm \in f_mBf_m$ and $\bar{B} \cong g_mBg_m : w \in B_1 \mapsto w - mw \in g_mBg_m$. Consequently, φ induces the isomorphisms $\rho : R \longrightarrow S : r \in R \mapsto \rho(r) = v$, $\bar{\varphi} : \bar{A} \longrightarrow \bar{B} : a \in \bar{A} \mapsto \bar{\varphi}(a) = w$ if $\varphi(r) = v + vm \in f_mBf_m = \varphi(e)\varphi(A)\varphi(e) = f_mBf_m$, $\varphi(a) = w - mw \in \varphi(1-e)\varphi(A)\varphi(1-e) = g_mBg_m$. Therefore for an arbitrary element of A, i.e., for an arbitrary triple $r \in R$, $\ell \in L$, $a \in A$, one obtains (1) immediately.

Finally, it is clear that every quadruple $(\rho, \bar{\varphi}, \chi, m)$ as described in the statement of the proposition leads to one of the desired isomorphisms, completing the justification of the proposition.

Now we continue with the proof of the **Theorem**. Consider the

Main Step. Let A and B be m-and n-strongly upper triangular matrix rings with respect to $\{e_1, \ldots, e_m\} \subseteq A$ and $\{f_1, \ldots, f_n\} \subseteq B$, respectively, and let $\varphi : A \to B$ be a ring isomorphism. Let $R_i = e_iAe_i$, $L_{ij} = e_iAe_j$ for $i < j$, and $S_i = f_iBf_i$, $M_{ij} = f_iBf_j$ for $i < j$, i.e.,

$$A = \begin{bmatrix} R_1 & L_{12} & L_{13} & \cdots & L_{1m} \\ 0 & R_2 & L_{23} & \cdots & L_{2m} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & R_m \end{bmatrix}, \quad B = \begin{bmatrix} S_1 & M_{12} & M_{13} & \cdots & M_{1n} \\ 0 & S_2 & M_{23} & \cdots & M_{2n} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & S_n \end{bmatrix}.$$

Then either $\varphi(e_1) \in f_1 + M_1$ or there is a $j \geqslant 2$ such that $\varphi(e_1) \in f_j + M_j$ and $M_{1j} = 0, \ldots, M_{j-1,j} = 0$.

Since $f = \varphi(e_1) \in B$ is a semicentral reduced idempotent, the statement of the **Main Step** can be reformulated in an equivalent, but little sharper, form, namely:

If f is a semicentral reduced idempotent in the n-strongly triangulated matrix ring B, then either $f \in f_1 + M_1$ or there is a $j \geqslant 2$ such that $f \in f_j + M_j$ and $M_{1j} = 0, \ldots, M_{j-1,j} = 0$.

Proof of the Main Step. Again we use induction for the verification. Let $F_1 = 1-f_1,\ B_2 = F_1BF_1,\ M = f_1BF_1$, i.e., $B = \begin{bmatrix} S_1 & M \\ 0 & B_2 \end{bmatrix}$. The statement is obvious for $n = 1$. Assume $n \geqslant 2$. Writing $f = \begin{bmatrix} \alpha & \mu \\ 0 & \beta \end{bmatrix} \in B = \begin{bmatrix} S_1 & M \\ 0 & B_2 \end{bmatrix}$ it follows from $f = f^2 = \begin{bmatrix} \alpha^2 & \alpha\mu + \mu\beta \\ & \\ 0 & \beta^2 \end{bmatrix}$ that $\alpha^2 = \alpha,\ \beta^2 = \beta$ and $\alpha\mu + \mu\beta = \mu$, and so $\alpha\mu\beta = 0$. Writing $s = \begin{bmatrix} \alpha & \alpha\mu \\ 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 0 & \mu\beta \\ 0 & \beta \end{bmatrix}$ we get $s^2 = s,\ b^2 = b$ and $sb = 0 = bs$. Hence, $f = s + b$ implies that $fs = s = sf$ and $fb = b = fb$, i.e., $s, b \in fBf$, which is semicentral reduced. Moreover,

$$\begin{bmatrix} 0 & \mu\beta \\ 0 & \beta \end{bmatrix} \begin{bmatrix} S_1 & M \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} \alpha & \alpha\mu \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mu\beta B_2 \\ 0 & \beta B_2 \end{bmatrix} \begin{bmatrix} \alpha & \alpha\mu \\ \alpha & 0 \end{bmatrix} = 0,$$

i.e., $s \in fBf$ is a semicentral. Consequently,

$$f = s = \begin{bmatrix} \alpha & \alpha\mu \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad f = \begin{bmatrix} 0 & \mu\beta \\ 0 & \beta \end{bmatrix}.$$

Assume the first case: $f = \begin{bmatrix} \alpha & \mu \\ 0 & 0 \end{bmatrix}$. Then $(1-f)Bf = 0$ implies that

$$\begin{bmatrix} 1-\alpha & -\mu \\ 0 & 1 \end{bmatrix} \begin{bmatrix} S_1 & M \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} \alpha & \mu \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (1-\alpha)S_1\alpha & * \\ 0 & 0 \end{bmatrix} = 0,$$

i.e., $(1-\alpha)S_1\alpha = 0$, hence α is a semicentral idempotent in S_1. Since $\alpha \neq 0$ and S_1 is semicentral reduced we obtain $\alpha = 1$, i.e., $f \in f_1B$.

Next, consider the case $f = \begin{bmatrix} 0 & \mu \\ 0 & \beta \end{bmatrix}$. Again $(1-f)Bf = 0$ implies that

$$\begin{bmatrix} 1 & -\mu \\ 0 & 1-\beta \end{bmatrix} \begin{bmatrix} S_1 & M \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} 0 & \mu \\ 0 & \beta \end{bmatrix} = \begin{bmatrix} 0 & * \\ 0 & (1-\beta)B_2\beta \end{bmatrix} = 0,$$

showing that $(1-\beta)B_2\beta = 0$, i.e., β is semicentral. Since $\beta B_2\beta = \beta B\beta = fBf$, we have that β is also reduced. Since B_2 is $(n-1)$-strongly triangular, the induction hypothesis shows that there is a $j \geqslant 2$ such that, in $B_2 = \begin{bmatrix} S_2 & M_{23} & \cdots & M_{2n} \\ 0 & S_3 & \cdots & M_{3n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & S_n \end{bmatrix}$, f is of the form

$$\begin{array}{c|ccccccc|} & 0 & \cdots & 0 & 0 & & & \\ & & \ddots & \vdots & \vdots & & \bigcirc & \\ & & & 0 & 0 & & & \\ j-1 & & & & 1 & * & \cdots & * \\ & & & & & 0 & \cdots & 0 \\ & & & & & & \ddots & \vdots \\ & & & & & & & 0 \end{array},$$

i.e., $M_{2j} = 0, \ldots, M_{j-1,j} = 0$. Therefore, in B, f is of the form

$$\begin{array}{c|ccccccccc|} & 0 & \cdots & \cdots & 0 & x_1 & & & & \\ & & \ddots & & \vdots & 0 & & & & \\ & & & \ddots & \vdots & \vdots & & \bigcirc & & \\ & & & & 0 & 0 & & & & \\ j & & & & & 1 & * & \cdots & \cdots & * \\ & & & & & & 0 & \cdots & \cdots & 0 \\ & & & & & & & \ddots & & \vdots \\ & & & & & & & & \ddots & \vdots \\ & & & & & & & & & 0 \end{array},$$

where $x_1 \in M_{1j}$. Now $0 = (1-f)Bf =$

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & -x_1 & & & & \\ & \ddots & & & 0 & & & & \\ & & \ddots & & \vdots & & \bigcirc & & \\ & & & 1 & 0 & & & & \\ & & & & 0 & * & \cdots & \cdots & * \\ & & & & & 1 & 0 & \cdots & 0 \\ & & & & & & \ddots & \ddots & \vdots \\ & & & & & & & \ddots & 0 \\ & & & & & & & & 1 \end{bmatrix} B \begin{bmatrix} 0 & \cdots & \cdots & 0 & x_1 & & & & \\ & \ddots & & \vdots & 0 & & & & \\ & & \ddots & \vdots & \vdots & & \bigcirc & & \\ & & & 0 & 0 & & & & \\ & & & & 1 & * & \cdots & \cdots & * \\ & & & & & 0 & \cdots & \cdots & 0 \\ & & & & & & \ddots & & \vdots \\ & & & & & & & \ddots & \vdots \\ & & & & & & & & 0 \end{bmatrix}$$

implies both $x_1 = 0$ and $M_{1j} = 0$, completing the proof of the **Main Step**. $\square$

The following observation is the last piece in the proof of the **Theorem**.

Lemma. *If*

$$B = \begin{bmatrix} S_1 & M_{12} & M_{13} & \cdots & M_{1n} \\ 0 & S_2 & M_{23} & \cdots & M_{2n} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & S_n \end{bmatrix}$$

is n-strongly triangular with respect to the ordered sequence $\{f_1, \ldots, f_n\}$ *of pairwise orthogonal idempotents such that* $M_{1j} = 0, \ldots, M_{j-1,j} = 0$ *for some index* $j > 1$*, then B is also n-strongly triangular with respect to the ordered sequence* $\{f_j, f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_n\}$*.*

Proof. Obvious by definition. $\square$

If we define now $\sigma(1) = j$, then the above **Lemma** together with the **Proposition** shows that φ induces the ring isomorphisms $\rho_1 : R_1 \cong S_j = S_1^\sigma = S_{\sigma(1)}$, $\varphi_2 : A_2 \cong \bar{B} = (1 - f_j)B(1 - f_j)$ and the bimodule isomorphism $\chi_1 : L_1 = e_1A(1 - e_1) \cong M_1^\sigma = M_{\sigma(1)} = f_jB(1 - f_j)$ together with an element $m_1 \in M_1^\sigma$ such that for an arbitrary $a = a_1 = \begin{bmatrix} r_1 & \ell_1 \\ 0 & a_2 \end{bmatrix} \in A = A_1 = \begin{bmatrix} R_1 & L_1 \\ 0 & A_2 \end{bmatrix}$, $\varphi = \varphi_1$ satisfies

$$\varphi(a) = \varphi_1(a_1) = \begin{bmatrix} \rho_1(r_1) & \rho_1(r_1)m_1 + \chi_1(\ell_1) - m_1\varphi_2(a_2) \\ 0 & \varphi_2(a_2) \end{bmatrix},$$

and every such φ can be described in this manner. Since A_1 is an $(m - 1)$-strongly triangular matrix ring and $\bar{B}$ is an $(n - 1)$-strongly triangular matrix ring, the theorem follows now immediately from the induction hypothesis which makes the proof of the **Theorem** complete. $\square$

We emphasize three important remarks.

Remark 2. If $M_{i,i+1} \neq 0$ for $i = 1, \ldots, n-1$, then $\{f_1, \ldots, f_n\}$ is the unique order (up to isomorphism) which induces the n-strongly triangular matrix decomposition on B.

Remark 3. If $i < j$ and $M_{ij} = 0, \ldots, M_{j-1,j} = 0$, then $\{f_1, \ldots, f_{i-1}, f_j, f_i, f_{i+1}, \ldots, f_{j-1}, f_{j+1}, \ldots, f_n\}$ also induces an n-strongly triangular matrix decomposition on B. In this case, the ring $(f_i + \cdots + f_{j-1} + f_j)B(f_i + \cdots + f_{j-1} + f_j)$ is the direct sum of the two rings $(f_i + \cdots + f_{j-1})B(f_i + \cdots + f_{j-1})$ and $S_j = f_jBf_j$. In particular, in the case $i = 1, j = n$ the ring B is the direct sum of $(f_1 + \cdots + f_{n-1})B(f_1 + \cdots + f_{n-1})$ and S_n if the truncated last column is 0.

Remark 4. Specializing the theorem for the case $A = B$ one obtains the description of the automorphism group of the strongly triangular matrix rings in terms of the corresponding automorphism groups of reduced rings R_i and of the corresponding bimodules L_i similar to one given in [1].

The proof of the **Main Step** shows also that for an arbitrary semicentral idempotent e in a strongly triangular matrix ring A each semicentral reduced idempotent $g \in A$ is either $g = \begin{bmatrix} \alpha & \mu \\ 0 & 0 \end{bmatrix}$ or $g = \begin{bmatrix} 0 & \nu \\ 0 & \beta \end{bmatrix}$ where $\alpha \in eAe$ and $\beta \in (1-e)A(1-e)$ are semicentral reduced idempotents in A, the associated subrings $gAg, \alpha A\alpha, \beta A\beta$ are isomorphic, and $\mu,\ \nu$ are appropriate elements in $eA(1-e)$. Observing that $C = (1-\alpha)A(1-\alpha)$ in the first case or $C = (1-\beta)A(1-\beta)$ in the second case is an $(m-1)$-strongly triangular matrix ring, by considering $\bar{e} = e - \alpha$ in the first case or in view of $e = (1-\beta)e(1-\beta)$ in the second case, respectively, the **Main Step** and the **Theorem** together with an obvious induction imply immediately the following

Corollary. *Any semicentral idempotent e in a m-strongly triangular matrix ring A with a complete set of triangulating idempotents can be written as a sum of l pairwise orthogonal idempotents $\{e_1, \ldots, e_l\}$ where $l \leqslant m$ is uniquely determined by e and this set of idempotents can be extended to the first l idempotents in a complete set of triangulating idempotents of A.*

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