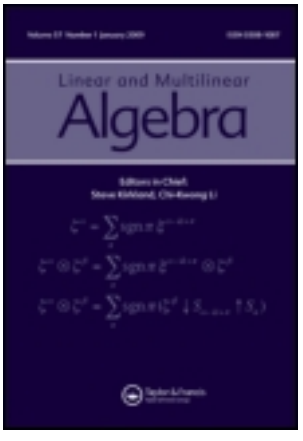

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Isomorphisms between Morita context rings

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Isomorphisms between Morita context rings

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Let $(R, S, {}_RM_S, {}_SN_R, f, g)$ be a general Morita context, and let $T = \left[\begin{smallmatrix} R & {}_RM_S \\ {}_SN_R & S \end{smallmatrix}\right]$ be the ring associated with this context. Similarly, let $T' = \left[\begin{smallmatrix} R' & M' \\ N' & S' \end{smallmatrix}\right]$ be another Morita context ring. We study the set $\mathrm{Iso}(T, T')$ of ring isomorphisms from T to T'. Our interest in this problem is motivated by: (i) the problem to determine the automorphism group of the ring T, and (ii) the recovery of the non-diagonal tiles problem for this type of generalized matrix rings. We introduce two classes of isomorphisms from T to T', the disjoint union of which is denoted by $\mathrm{Iso}_0(T, T')$. We describe $\mathrm{Iso}_0(T, T')$ by using the $\mathbb{Z}$-graded ring structure of T and T'. Our main result characterizes $\mathrm{Iso}_0(T, T')$ as the set consisting of all semigraded isomorphisms and all anti-semigraded isomorphisms from T to T', provided that the rings R' and S' are indecomposable and at least one of M' and N' is nonzero; in particular, $\mathrm{Iso}_0(T, T')$ contains all graded isomorphisms and all anti-graded isomorphisms from T to T'. We also present a situation where $\mathrm{Iso}_0(T, T') = \mathrm{Iso}(T, T')$. This is in the case where R, S, R' and S' are rings having only trivial idempotents and all the Morita maps are zero. In particular, this shows that the group of automorphisms of T is completely determined.

Keywords: Morita context; bimodule; graded ring; semigraded isomorphism; automorphism

AMS Subject Classifications: 16W20; 16W50; 16S50; 15A33; 16D20

1. Introduction

Morita contexts appeared as a key ingredient in the work of Morita that described equivalences between full categories of modules over rings with identities. One of the fundamental results in this direction says that the categories of left modules over the rings R and S are equivalent if and only if there exists a strict Morita context connecting R and S.

Throughout the sequel $(R, S, {}_RM_S, {}_SN_R, f, g)$ will be a general Morita context, i.e. R and S are rings with identity, M is a left R, right S-bimodule, N is a left S, right R-bimodule, $f: M \otimes_S N \to R$ is a morphism of R, R-bimodules and $g: N \otimes_R M \to S$ is

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a morphism of S, S-bimodules, such that if we denote $[m, n] := f(m \otimes n)$ and $(n, m) := g(n \otimes m)$, we have that

$$[m, n]m' = m(n, m') \quad \text{and} \quad n[m, n'] = (n, m)n' \tag{1}$$

for all $m, m' \in M$ and all $n, n' \in N$.

With such a Morita context we associate the ring $T = \left[\begin{smallmatrix} R & M \\ N & S \end{smallmatrix}\right]$ with operations the formal operations of 2×2-matrices, using $[\,,]$ and $(\,,)$ in defining the multiplication (see, e.g. [15, p. 12]), more precisely

$$\begin{bmatrix} r & m \\ n & s \end{bmatrix}\begin{bmatrix} r' & m' \\ n' & s' \end{bmatrix} = \begin{bmatrix} rr' + [m, n'] & rm' + ms' \\ nr' + sn' & (n, m') + ss' \end{bmatrix}.$$

T is called the Morita context ring associated with the given Morita context. Such rings (especially in the case where both Morita maps f and g are zero, or even more particularly when $N = 0$) have been intensively used to provide examples and counterexamples in ring theory [15].

The aim of this article is to investigate isomorphisms between two Morita context rings T as above and $T' = \left[\begin{smallmatrix} R' & M' \\ N' & S' \end{smallmatrix}\right]$. This investigation is motivated by at least the following two types of problems:

- *What are the automorphisms of a Morita context ring T?*
- *Can we recover the tiles in Morita context rings? For example, if* $\left[\begin{smallmatrix} R & M \\ N & S \end{smallmatrix}\right]$ *and* $\left[\begin{smallmatrix} R & M' \\ N' & S \end{smallmatrix}\right]$ *are two isomorphic Morita context rings, when are M and M′ (respectively N and N′) isomorphic in some sense?*

Automorphisms of various kinds of algebraic structures have been extensively studied in the literature. Knowing the group of automorphisms of a certain object can be key information about it. It is a very difficult problem to find all the automorphisms of T. In fact, as we explain in Remark 4.8, Morita context rings are up to isomorphism just the rings with non-trivial idempotents (indeed a very large class of rings), and so there is virtually no hope of finding the automorphisms of T in the general case. Even in a very particular case, where the rings R and S are the same and both bimodules M and N are also R, while the Morita maps are just the multiplication of the ring, the Morita ring is the full 2×2 matrix ring $\mathbb{M}_2(R)$, whose automorphisms are known only for special rings R.

The Skolem–Noether theorem (see, e.g. [8] or [18]) states that if A is a simple artinian algebra which is finite-dimensional over its centre F, then all F-automorphisms of A are inner. Jøndrup [10] showed that if A is a simple artinian algebra which is finite-dimensional over its centre F, then all F-automorphisms of the ring $\mathbb{U}_n(A)$ of $n \times n$ upper triangular matrices over such a ring A are also inner. (Note that when $n = 1$, this is the Skolem–Noether theorem.) Jøndrup also computed the automorphism groups of certain non-semiprime rings, for example,

$$\mathbb{U}_2(k[X]), \quad \begin{bmatrix} k[X] & k[X, Y] \\ 0 & k[Y] \end{bmatrix},$$

where k is a field and X and Y are indeterminates. The automorphism groups of $k[X, Y]$ and $k\langle X, Y\rangle$ (the free algebra) are known, and in fact equal [7].

Rosenberg and Zelinsky [17] obtained information about the extent to which it can be true that not all automorphisms of central separable algebras are inner.

These algebras constitute a class which is more general than the class of matrix algebras $\mathbb{M}_n(R)$ over commutative rings R, considered by Isaacs [9], who showed that, although not all R-algebra automorphisms of $\mathbb{M}_n(R)$ are inner, the extent of this failure is somehow under control. For example, the commutator of any two automorphisms and the nth power of each of them are inner.

For any algebra A over a commutative ring R such that every nontrivial R-algebra endomorphism of A is an R-algebra automorphism, Barker and Kezlan [2] showed that every R-algebra automorphism ϕ of $\mathbb{U}_n(A)$ factors as $\phi = \varphi\psi$, where φ is inner and ψ is an R-algebra automorphism of $\mathbb{U}_n(A)$ which is induced (componentwise) from some R-algebra automorphism of A, and they provided an example showing that ϕ itself need not be inner. Jøndrup [11] showed that the above hypothesis on R-algebra endomorphisms of A, used in obtaining the factorization $\phi = \varphi\psi$, can be removed in the case where A is a prime R-algebra.

Other papers on automorphisms of algebras of upper triangular matrices, automorphisms of structural matrix algebras and automorphisms of other types of subalgebras of full matrix algebras include, for example, [4,5,12,14].

As far as the possible recovery of the tiles is concerned, it was shown in [6] that it can happen that

$$\begin{bmatrix} R & R \\ R & R \end{bmatrix} \simeq \begin{bmatrix} R & 0 \\ R & R \end{bmatrix} \simeq \begin{bmatrix} R & R \\ 0 & R \end{bmatrix} \simeq \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$$

for certain rings R, so the tiles in the non-diagonal positions of a Morita context ring cannot be recovered in general. However, in [13] a positive recovery result in this vein was obtained for a generalized triangular matrix ring $\left[\begin{smallmatrix} R & {}_RM_S \\ 0 & S \end{smallmatrix}\right]$ over rings R and S having only the idempotents 0 and 1, in particular, over indecomposable commutative rings or over local rings (not necessarily commutative). In addition, the automorphism group of such a generalized triangular matrix ring was obtained. In [1] this result was extended to the case where the diagonal rings R and S are strongly indecomposable (not necessarily commutative) rings, which include rings with only the trivial idempotents, as well as endomorphism rings of vector spaces, or more generally, semiprime indecomposable rings. We note that strongly indecomposable rings are called semicentral reduced rings in [3].

In this sequel, we consider ring isomorphisms between two Morita context rings T and T'. We denote the set of all such isomorphisms by $\mathrm{Iso}(T, T')$. We introduce two classes $\mathrm{Iso}_0^0(T, T')$ and $\mathrm{Iso}_0^1(T, T')$ of such isomorphisms. The disjoint union of these two classes is denoted by $\mathrm{Iso}_0(T, T')$. As Remark 2.5 shows, $\mathrm{Iso}_0(T, T')$ is much smaller than $\mathrm{Iso}(T, T')$ if $[,] \neq 0$ or $(,) \neq 0$. In the case where $T' = T$, we denote $\mathrm{Iso}_0(T, T)$ by $\mathrm{Aut}_0(T)$, and we show that it is a subgroup of $\mathrm{Aut}(T)$, and $\mathrm{Iso}_0^0(T, T)$ is a normal subgroup of $\mathrm{Aut}_0(T)$.

For understanding $\mathrm{Iso}_0(T, T')$ we emphasize the structure of T and T' as $\mathbb{Z}$-graded rings. There are some other isomorphisms associated with this graded structure: the set of graded isomorphisms, denoted by $\mathrm{Iso}_g^+(T, T')$, and the set of anti-graded isomorphisms, denoted by $\mathrm{Iso}_g^-(T, T')$ (which we define in Section 3). We consider $\mathrm{Iso}_g(T, T') = \mathrm{Iso}_g^+(T, T') \cup \mathrm{Iso}_g^-(T, T')$. In the case where $T' = T$, $\mathrm{Aut}_g(T) = \mathrm{Iso}_g(T, T)$ is a subgroup of $\mathrm{Aut}(T)$. We show that in the case where one of the Morita contexts with which T and T' are associated is strict, we have that $\mathrm{Iso}_0(T, T') \subseteq \mathrm{Iso}_g(T, T')$.

The new concepts of a semigraded isomorphism and an anti-semigraded isomorphism are introduced in Definition 3.5, and our main result describes $\mathrm{Iso}_0(T, T')$ in terms of these isomorphisms from T to T' in the case where the rings R' and S' in the Morita context ring T' are indecomposable and at least one of the bimodules M' and N' is nonzero. To be precise, Theorem 3.6 says that, under these conditions, $\mathrm{Iso}_0^0(T, T')$ is the set of all semigraded isomorphisms from T to T', and $\mathrm{Iso}_0^1(T, T')$ is the set of all anti-semigraded isomorphisms from T to T', and so $\mathrm{Iso}_g(T, T') \subseteq \mathrm{Iso}_0(T, T')$. In particular, this result can be seen as a way to recover tiles from certain types of isomorphisms.

In the last section, we present a situation where $\mathrm{Iso}_0(T, T') = \mathrm{Iso}(T, T')$. This is in the case where R, S, R' and S' are rings having only trivial idempotents, and all the Morita maps are zero. In particular, this shows that the tiles can be recovered from any isomorphism (not only from semigraded or anti-semigraded ones), and also that the group of automorphisms of T is completely determined.

Throughout this article by ring we understand a ring with identity $1 \neq 0$.

2. Two classes of isomorphisms between Morita context rings

Let $(R, S, {}_RM_S, {}_SN_R, f, g)$ be a Morita context, with $[\,,]$ and $(\,,)$ the maps defined as in Section 1, and let T be the associated Morita context ring. Consider another Morita context $(R', S', {}_{R'}M'_{S'}, {}_{S'}N'_{R'}, f', g')$, and for simplicity we denote also by $[\,,]$ and $(\,,)$ the maps defined by this second Morita context, and let T' be the associated Morita context ring.

Recall that if $\gamma : R \to R'$ and $\delta \in S \to S'$ are ring isomorphisms, we say that a morphism $u : (M, +) \to (M', +)$ is a γ–δ-bimodule isomorphism if

$$u(rms) = \gamma(r)u(m)\delta(s)$$

for all $m \in M$, $r \in R$, $s \in S$. This is in fact equivalent to the fact that u is an isomorphism of R–S-bimodules when M' is regarded as such a bimodule via γ and δ.

We consider two classes of ring isomorphisms between the Morita context rings T and T', constructed in the following two propositions.

PROPOSITION 2.1 *Let $(\gamma, \delta, u, v, m'_0, n'_0)$ be a 6-tuple such that $\gamma : R \to R'$ and $\delta : S \to S'$ are ring isomorphisms, $u : M \to M'$ is a γ–δ-bimodule isomorphism, $v : N \to N'$ is a δ–γ-bimodule isomorphism, and $m'_0 \in M'$ and $n'_0 \in N'$ are fixed elements, such that the following conditions are satisfied*:

(i) $[m'_0, N'] = 0$ *and* $(N', m'_0) = 0$,
(ii) $[M', n'_0] = 0$ *and* $(n'_0, M') = 0$,
(iii) $[u(m), v(n)] = \gamma([m, n])$ *and* $(v(n), u(m)) = \delta((n, m))$ *for all* $m \in M$, $n \in N$.

Then the map $\phi : T \to T'$ defined by

$$\phi\left(\begin{bmatrix} r & m \\ n & s \end{bmatrix}\right) = \begin{bmatrix} \gamma(r) & \gamma(r)m'_0 - m'_0\delta(s) + u(m) \\ n'_0\gamma(r) - \delta(s)n'_0 + v(n) & \delta(s) \end{bmatrix}$$

is a ring isomorphism.

Proof The additivity (respectively the injectivity) of γ, δ, u and v ensure that ϕ is additive (respectively injective).

Next, let $r_1, r_2 \in R$, $s_1, s_2 \in S$, $m_1, m_2 \in M$, $n_1, n_2 \in N$. For the multiplicativity of ϕ we only show that the entries of

$$\phi\left(\begin{bmatrix} r_1 & m_1 \\ n_1 & s_1 \end{bmatrix}\right)\phi\left(\begin{bmatrix} r_2 & m_2 \\ n_2 & s_2 \end{bmatrix}\right) \quad \text{and} \quad \phi\left(\begin{bmatrix} r_1r_2 + [m_1, n_2] & r_1m_2 + m_1s_2 \\ n_1r_2 + s_1n_2 & (n_1, m_2) + s_1s_2 \end{bmatrix}\right)$$

in position $(1, 2)$ are equal, since the equality in the other positions can be checked in a similar way. These are, respectively,

$$\gamma(r_1)\left(\gamma(r_2)m_0' - m_0'\delta(s_2) + u(m_2)\right) + \left(\gamma(r_1)m_0' - m_0'\delta(s_1) + u(m_1)\right)\delta(s_2)$$

and

$$\gamma(r_1)\left(r_1r_2 + [m_1, n_2]\right)m_0' - m_0'\delta\left((n_1, m_2) + s_1s_2\right) + u(r_1m_2 + m_1s_2).$$

Using the hypotheses of the proposition, both these expressions simplify to

$$\gamma(r_1)\gamma(r_2)m_0' + \gamma(r_1)u(m_2) - m_0'\delta(s_1)\delta(s_2) + u(m_1)\delta(s_2).$$

To see that ϕ is onto, let $\left[\begin{smallmatrix} r' & m' \\ n' & s' \end{smallmatrix}\right] \in T'$. The surjectivity of γ, δ, u and v implies that there are $r \in R$, $s \in S$, $m \in M$ and $n \in N$ such that $\gamma(r) = r'$, $\delta(s) = s'$, $u(m) = m' - \gamma(r)m_0' + m_0'\delta(s)$ and $v(n) = n' - n_0'\gamma(r) + \delta(s)n_0'$, and so

$$\begin{aligned}
&\phi\left(\begin{bmatrix} r & m \\ n & s \end{bmatrix}\right) \\
&= \begin{bmatrix} \gamma(r) & \gamma(r)m_0' - m_0'\delta(s) + (m' - \gamma(r)m_0' + m_0'\delta(s)) \\ n_0'\gamma(r) - \delta(s)n_0' + (n' - n_0'\gamma(r) + \delta(s)n_0') & \delta(s) \end{bmatrix} \\
&= \begin{bmatrix} r' & m' \\ n' & s' \end{bmatrix},
\end{aligned}$$

which concludes the proof. ■

Proposition 2.2 *Let $(\rho, \sigma, \mu, \nu, m_*', n_*')$ be a 6-tuple with $\rho : R \to S'$ and $\sigma : S \to R'$ ring isomorphisms, $\mu : (M, +) \to (N', +)$ and $\nu : (N, +) \to (M', +)$ group isomorphisms such that $\mu(rms) = \rho(r)\mu(m)\sigma(s)$ and $\nu(snr) = \sigma(s)\nu(n)\rho(r)$ for all $m \in M$, $n \in N$, $r \in R$, $s \in S$, and $m_*' \in M'$ and $n_*' \in N'$ are fixed elements, such that the following properties are satisfied*:

(i) $[m_*', N'] = 0$ *and* $(N', m_*') = 0$,
(ii) $[M', n_*'] = 0$ *and* $(n_*', M') = 0$,
(iii) $(\mu(m), \nu(n)) = \rho([m, n])$ *and* $[\nu(n), \mu(m)] = \sigma((n, m))$ *for all* $m \in M$, $n \in N$.

Then the map $\psi : T \to T'$ defined by

$$\psi\left(\begin{bmatrix} r & m \\ n & s \end{bmatrix}\right) = \begin{bmatrix} \sigma(s) & m_*'\rho(r) - \sigma(s)m_*' + \nu(n) \\ \rho(r)n_*' - n_*'\sigma(s) + \mu(m) & \rho(r) \end{bmatrix}$$

is a ring isomorphism.

Proof Similar to the proof of Proposition 2.1. ■

We denote by $\mathrm{Iso}_0^0(T, T')$ and $\mathrm{Iso}_0^1(T, T')$ the sets of ring isomorphisms defined in Propositions 2.1 and 2.2 respectively. Of course, $\mathrm{Iso}_0^0(T, T')$ and $\mathrm{Iso}_0^1(T, T')$ may be

empty; for instance, in the case where the ring R is isomorphic neither to R' nor to S', both these sets are empty. In any case we have that $\mathrm{Iso}_0^0(T,T')\cap \mathrm{Iso}_0^1(T,T')=\emptyset$. We denote

$$\mathrm{Iso}_0(T,T')=\mathrm{Iso}_0^0(T,T')\cup \mathrm{Iso}_0^1(T,T').$$

In the particular case where $T'=T$, the isomorphisms defined in Propositions 2.1 and 2.2 are automorphisms of the ring T. We denote $\mathrm{Aut}_0^0(T)=\mathrm{Iso}_0^0(T,T)$, $\mathrm{Aut}_0^1(T)=\mathrm{Iso}_0^1(T,T)$ and $\mathrm{Aut}_0(T)=\mathrm{Iso}_0(T,T)$. Clearly $\mathrm{Aut}_0^0(T)\neq\emptyset$, since it always contains the identity morphism, while $\mathrm{Aut}_0^1(T)$ may be empty (e.g. in the case where R and S are not isomorphic).

Direct verification yields the following two results.

PROPOSITION 2.3 *Let T,T',T'' be the Morita context rings associated with the Morita contexts (R,S,M,N,f,g), (R',S',M',N',f',g'), $(R'',S'',M'',N'',f'',g'')$, respectively. Let $\phi\in\mathrm{Iso}_0^0(T,T')$ and $\phi'\in\mathrm{Iso}_0^0(T',T'')$, with ϕ and ϕ' corresponding to the 6-tuples $(\gamma,\delta,u,v,m_0',n_0')$ and $(\gamma',\delta',u',v',m_0'',n_0'')$, respectively, and let $\psi\in\mathrm{Iso}_0^1(T,T')$ and $\psi'\in\mathrm{Iso}_0^1(T',T'')$, with ψ and ψ' corresponding to the 6-tuples $(\rho,\sigma,\mu,\nu,m_*',n_*')$ and $(\rho',\sigma',\mu',\nu',m_*'',n_*'')$, respectively. Then*

(i) $\phi'\circ\phi,\psi'\circ\psi\in\mathrm{Iso}_0^0(T,T''),\phi^{-1}\in\mathrm{Iso}_0^0(T',T)$, $\quad\phi'\circ\psi,\psi'\circ\phi\in\mathrm{Iso}_0^1(T,T'')$, $\psi^{-1}\in\mathrm{Iso}_0^1(T',T)$.

(ii) *The six isomorphisms in* (i) *correspond to the following 6-tuples respectively*:

$$\begin{aligned}
\phi'\circ\phi &\leftrightarrow (\gamma'\circ\gamma,\delta'\circ\delta,u'\circ u,v'\circ v,m_0''+u'(m_0'),n_0''+v'(n_0')),\\
\psi'\circ\psi &\leftrightarrow (\sigma'\circ\rho,\rho'\circ\sigma,\nu'\circ\mu,\mu'\circ\nu,\nu'(n_*')-m_*'',\mu'(m_*')-n_*''),\\
\phi^{-1} &\leftrightarrow (\gamma^{-1},\delta^{-1},u^{-1},v^{-1},u^{-1}(-m_0'),v^{-1}(-n_0')),\\
\phi'\circ\psi &\leftrightarrow (\delta'\circ\rho,\gamma'\circ\sigma,v'\circ\mu,u'\circ\nu,u'(m_*')-m_0'',v'(n_*')-n_0''),\\
\psi'\circ\phi &\leftrightarrow (\rho'\circ\gamma,\sigma'\circ\delta,\mu'\circ u,\nu'\circ v,\nu'(n_0')+m_*'',\mu'(m_0')+n_*''),\\
\psi^{-1} &\leftrightarrow (\sigma^{-1},\rho^{-1},\nu^{-1},\mu^{-1},\mu^{-1}(n_*'),\nu^{-1}(m_*')).
\end{aligned}$$

COROLLARY 2.4 *$\mathrm{Aut}_0(T)$ is a subgroup of $\mathrm{Aut}(T)$ and $\mathrm{Aut}_0^0(T)$ is a normal subgroup of $\mathrm{Aut}_0(T)$.*

Let us note that if (R,S,M,N,f,g) is a Morita context, and $T=\left[\begin{smallmatrix}R & M\\ N & S\end{smallmatrix}\right]$ is the associated Morita context ring, then (S,R,N,M,g,f) is also a Morita context, and we denote by $T^{-1}=\left[\begin{smallmatrix}S & N\\ M & R\end{smallmatrix}\right]$ its associated Morita context ring. Clearly the map

$$\alpha_T: T\to T^{-1},\quad \alpha_T\left(\begin{bmatrix}r & m\\ n & s\end{bmatrix}\right)=\begin{bmatrix}s & n\\ m & r\end{bmatrix}$$

is a ring isomorphism. In fact, $\alpha_T\in\mathrm{Iso}_0^1(T,T^{-1})$. Moreover, we have that $\alpha_T^{-1}=\alpha_{T^{-1}}$. Hence, it is easy to see from Proposition 2.3 that

$$\mathrm{Iso}_0^1(T,T')=\mathrm{Iso}_0^0(T^{-1},T')\alpha_T=\alpha_{T'}^{-1}\mathrm{Iso}_0^0(T,(T')^{-1}) \tag{2}$$

The following shows that in general $\mathrm{Iso}_0(T,T')\neq\mathrm{Iso}(T,T')$.

Remark 2.5 Let (R,S,M,N,f,g) and (R',S',M',N',f',g') be Morita contexts and consider the associated Morita context rings T and T'. Assume that $T\simeq T'$. We show

that if $\mathrm{Iso}_0(T,T')=\mathrm{Iso}(T,T')$, the set of all ring isomorphisms from T to T', then necessarily f,g,f' and g' are 0 (thus all the maps $[\,,\,]$ and $(\,,\,)$ from the two contexts are zero). Indeed, we also have that $\mathrm{Iso}_0(T',T)=\mathrm{Iso}(T',T)$. Since $T\simeq T'$, we have that

$$\begin{aligned}\mathrm{Aut}(T) &= \{\phi_1\circ\phi_2 \mid \phi_1\in\mathrm{Iso}(T',T),\phi_2\in\mathrm{Iso}(T,T')\}\\ &= \{\phi_1\circ\phi_2 \mid \phi_1\in\mathrm{Iso}_0(T',T),\phi_2\in\mathrm{Iso}_0(T,T')\}\\ &\subseteq \mathrm{Aut}_0(T),\end{aligned}$$

and so $\mathrm{Aut}_0(T)=\mathrm{Aut}(T)$.

Now let $m\in M$. Then $\left[\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\right]\in U(T)$, and let $\eta\in\mathrm{Aut}(T)$ be the associated inner automorphism. Then

$$\begin{aligned}\eta\left(\begin{bmatrix}0 & 0\\ n & 0\end{bmatrix}\right) &= \begin{bmatrix}1 & m\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & 0\\ n & 0\end{bmatrix}\begin{bmatrix}1 & -m\\ 0 & 1\end{bmatrix}\\ &= \begin{bmatrix}[m,n] & -[m,n]m\\ n & -(n,m)\end{bmatrix}\end{aligned}$$

for every $n\in N$. As $\mathrm{Aut}_0(T)=\mathrm{Aut}(T)$, η must be either in $\mathrm{Aut}_0^0(T)$ or in $\mathrm{Aut}_0^1(T)$, and so $\eta\left(\left[\begin{smallmatrix}0 & 0\\ N & 0\end{smallmatrix}\right]\right)$ is either contained in $\left[\begin{smallmatrix}0 & 0\\ N & 0\end{smallmatrix}\right]$ or in $\left[\begin{smallmatrix}0 & M\\ 0 & 0\end{smallmatrix}\right]$. This shows that $[m,n]=0$ and $(n,m)=0$ for all $m\in M$ and all $n\in N$. Thus $f=0$ and $g=0$. By symmetry one gets also that $f'=0$ and $g'=0$.

3. Isomorphisms associated with the graded structure of Morita context rings

We keep the notation as in Section 2. For basic concepts about graded rings we refer to [16]. A ring A is $\mathbb{Z}$-graded if $A=\oplus_{i\in\mathbb{Z}}A_i$, a direct sum of additive subgroups, such that $A_iA_j\subseteq A_{i+j}$ for any $i,j\in\mathbb{Z}$.

The Morita context ring $T=\left[\begin{smallmatrix}R & M\\ N & S\end{smallmatrix}\right]$ has a structure of a $\mathbb{Z}$-graded ring with homogeneous components

$$T_{-1}=\begin{bmatrix}0 & 0\\ N & 0\end{bmatrix},\quad T_0=\begin{bmatrix}R & 0\\ 0 & S\end{bmatrix},\quad T_1=\begin{bmatrix}0 & M\\ 0 & 0\end{bmatrix}$$

and $T_i=0$ for every $i\notin\{-1,0,1\}$. Therefore if T' is another Morita context ring, we can consider two classes of ring isomorphisms from T to T', related to the graded structure, as follows:

- *Graded isomorphisms*, which are isomorphisms $\phi: T\to T'$ such that $\phi(T_i)\subseteq T'_i$ for all $i\in\mathbb{Z}$. We denote by $\mathrm{Iso}_g^+(T,T')$ the set of all such isomorphisms.
- *Anti-graded isomorphisms*, which are isomorphisms $\phi: T\to T'$ such that $\phi(T_i)\subseteq T'_{-i}$ for all $i\in\mathbb{Z}$. We denote by $\mathrm{Iso}_g^-(T,T')$ the set of all such isomorphisms.

We denote by $\mathrm{Iso}_g(T,T')=\mathrm{Iso}_g^+(T,T')\cup\mathrm{Iso}_g^-(T,T')$. It is clear that if $M\neq 0$ or $N\neq 0$, then the sets $\mathrm{Iso}_g^+(T,T')$ and $\mathrm{Iso}_g^-(T,T')$ are disjoint. If $M=0$ and $N=0$, we have that $\mathrm{Iso}_g^+(T,T')=\mathrm{Iso}_g^-(T,T')$. Obviously, $\mathrm{Iso}_g(T,T')$ can be empty, for instance in the case where the rings T and T' are not isomorphic.

In the particular case where $T'=T$, we use the notation $\mathrm{Aut}_g^+(T)=\mathrm{Iso}_g^+(T,T)$, $\mathrm{Aut}_g^-(T)=\mathrm{Iso}_g^-(T,T)$ and $\mathrm{Aut}_g(T)=\mathrm{Aut}_g^+(T)\cup\mathrm{Aut}_g^-(T)$. We note that $\mathrm{Aut}_g^+(T)$

is always non-empty, since it contains the identity morphism, while $\mathrm{Aut}_g^-(T)$ may be empty. It is easy to see that $\mathrm{Aut}_g(T)$ is a subgroup of $\mathrm{Aut}(T)$, and $\mathrm{Aut}_g^+(T)$ is a normal subgroup of $\mathrm{Aut}_g(T)$.

We recall that a Morita context (R, S, M, N, f, g) is strict if f and g are surjective, and this implies that f and g are isomorphisms. In this case, the bilinear maps $[\,,\,]$ and $(\,,\,)$ are left and right non-degenerate (i.e. for example if $[m, N] = 0$ for some $m \in M$, then $m = 0$). Let us note that if $\mathrm{Iso}_0(T, T') \neq \emptyset$, then the Morita context with which T is associated is strict if and only if so is the Morita context with which T' is associated. Indeed, if there exists $\phi \in \mathrm{Iso}_0^0(T, T')$, associated with the 6-tuple $(\gamma, \delta, u, v, m_0', n_0')$, then the relations $[u(m), v(n)] = \gamma([m, n])$ and $(v(n), u(m)) = \delta((n, m))$, combined with the surjectivity of u, v, γ, δ, show that f and g are surjective if and only if f' and g' are surjective. Similarly in the case where there exists $\phi \in \mathrm{Iso}_0^1(T, T')$.

PROPOSITION 3.1 *If the Morita context with which T (or T') is associated is strict, then* $\mathrm{Iso}_0(T, T') \subseteq \mathrm{Iso}_g(T)$.

Proof If $\mathrm{Iso}_0(T, T') = \emptyset$, the result is clear. If $\mathrm{Iso}_0(T, T') \neq \emptyset$, then by the remark preceding this proposition, the Morita context with which T is associated is strict if and only if so is the Morita context with which T' is associated. Therefore we can assume that both these contexts are strict.

Let $\phi \in \mathrm{Iso}_0^0(T, T')$ be associated with the 6-tuple $(\gamma, \delta, u, v, m_0', n_0')$. Since the Morita context is strict, the condition $[m_0', N'] = 0$ implies that $m_0' = 0$. Similarly, $n_0' = 0$. Then $\phi\left(\left[\begin{smallmatrix} r & m \\ n & s \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} \gamma(r) & u(m) \\ v(n) & \delta(s) \end{smallmatrix}\right]$, and so $\phi \in \mathrm{Iso}_g^+(T, T')$. Similarly, any $\phi \in \mathrm{Iso}_0^1(T, T')$ is of the form $\phi\left(\left[\begin{smallmatrix} r & m \\ n & s \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} \sigma(s) & v(n) \\ \mu(m) & \rho(r) \end{smallmatrix}\right]$ for some ρ, σ, μ, ν, and so $\phi \in \mathrm{Iso}_g^-(T, T')$. ■

We recall that a ring A is called indecomposable if it is not isomorphic to a direct product of two rings with identity; this is equivalent to the fact that the only central idempotents of A are 0 and 1. We will need the following simple fact.

LEMMA 3.2 *Let $T = \left[\begin{smallmatrix} R & M \\ N & S \end{smallmatrix}\right]$ be a Morita context ring. Then the central idempotent elements of T are the matrices of the form $\left[\begin{smallmatrix} r & 0 \\ 0 & s \end{smallmatrix}\right]$, where r and s are central idempotents of R and S respectively, such that $rm = ms$ for all $m \in M$, and $sn = nr$ for all $n \in N$.*

Proof By looking at the commutation relations with $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, we see that a central element of T must be of the form $X = \left[\begin{smallmatrix} r & 0 \\ 0 & s \end{smallmatrix}\right]$, and it is clear that r must be central in R, and s must be central in S. If moreover X is an idempotent, then r and s are central idempotents in R and S, respectively. Looking at the commutation relations with an elements of the form $\left[\begin{smallmatrix} 0 & m \\ n & 0 \end{smallmatrix}\right]$, we get that $rm = ms$ for all $m \in M$, and $sn = nr$ for all $n \in N$. Clearly any such matrix $\left[\begin{smallmatrix} r & 0 \\ 0 & s \end{smallmatrix}\right]$ is a central idempotent of T. ■

An immediate consequence is the following.

COROLLARY 3.3 *Let $T = \left[\begin{smallmatrix} R & M \\ N & S \end{smallmatrix}\right]$ be a Morita context ring such that R and S are indecomposable rings and not both M and N are zero. Then T is an indecomposable ring.*

Another consequence of Lemma 3.2 shows that T can be indecomposable without both R and S being so.

Example 3.4 Let R be a ring which is not indecomposable, and $S = \mathbb{Z}$. Let $M = R$, regarded as an R–$\mathbb{Z}$-bimodule, and $N = 0$. Then using Lemma 3.2, it is easy to see that T does not have non-trivial central idempotents, and so it is indecomposable.

At this point we introduce two new types of isomorphisms related to the graded structure of T and T'.

Definition 3.5 Let $\phi: T \to T'$ be an isomorphism between the Morita context rings T and T'. Then ϕ is called

- a semigraded isomorphism if $\phi(T_i) \subseteq T'_i$ for all $i \in \{-1, 1\}$,
- an anti-semigraded isomorphism if $\phi(T_i) \subseteq T'_{-i}$ for all $i \in \{-1, 1\}$.

Now we can prove the main result of this section.

THEOREM 3.6 *Let* $T = \left[\begin{smallmatrix} R & M \\ N & S \end{smallmatrix}\right]$ *and* $T' = \left[\begin{smallmatrix} R' & M' \\ N' & S' \end{smallmatrix}\right]$ *be Morita context rings such that* R' *and* S' *are indecomposable rings, and at least one of* M' *and* N' *is nonzero. Then the following assertions hold*:

- $\mathrm{Iso}_0^0(T, T')$ *is the set of all semigraded isomorphisms from* T *to* T'.
- $\mathrm{Iso}_0^1(T, T')$ *is the set of all anti-semigraded isomorphisms from* T *to* T'.

In particular, $\mathrm{Iso}_g(T, T') \subseteq \mathrm{Iso}_0(T, T')$.

Proof (1) It is clear that any $\phi \in \mathrm{Iso}_0^0(T, T')$ is a semigraded isomorphism.

Let ϕ be a semigraded isomorphism from T to T'. Then there exist $u: M \to M'$ and $v: N \to N'$ such that

$$\phi\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & u(m) \\ 0 & 0 \end{bmatrix} \tag{3}$$

for every $m \in M$, and

$$\phi\left(\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ v(n) & 0 \end{bmatrix} \tag{4}$$

for every $n \in N$. Clearly u and v are injective additive morphisms.

Let $\phi\left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} r'_0 & m'_0 \\ n'_0 & s'_0 \end{smallmatrix}\right]$. Since $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is an idempotent in T, we have that $\left[\begin{smallmatrix} r'_0 & m'_0 \\ n'_0 & s'_0 \end{smallmatrix}\right]$ is an idempotent in T', which in particular implies that

$$(r'_0)^2 + [m'_0, n'_0] = r'_0 \tag{5}$$

and

$$(s'_0)^2 + (n'_0, m'_0) = s'_0. \tag{6}$$

Since $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] T \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & M \\ 0 & 0 \end{smallmatrix}\right]$, by applying ϕ we get that

$$\begin{bmatrix} r'_0 & m'_0 \\ n'_0 & s'_0 \end{bmatrix} T' \begin{bmatrix} 1 - r'_0 & -m'_0 \\ -n'_0 & 1 - s'_0 \end{bmatrix} \subseteq \begin{bmatrix} 0 & M' \\ 0 & 0 \end{bmatrix}. \tag{7}$$

Equation (7) also shows that for every $m' \in M'$,

$$\begin{bmatrix} r'_0 & m'_0 \\ n'_0 & s'_0 \end{bmatrix} \begin{bmatrix} 0 & m' \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 - r'_0 & -m'_0 \\ -n'_0 & 1 - s'_0 \end{bmatrix} = \begin{bmatrix} -r'_0[m', n'_0] & r'_0 m'(1 - s'_0) \\ -(n'_0, m')n'_0 & (n'_0, m')(1 - s'_0) \end{bmatrix} \in \begin{bmatrix} 0 & M' \\ 0 & 0 \end{bmatrix},$$

implying that

$$r_0'[m', n_0'] = 0 \tag{8}$$

and

$$(n_0', m')(1 - s_0') = 0. \tag{9}$$

Also, we have that for every $n' \in N'$,

$$\begin{bmatrix} r_0' & m_0' \\ n_0' & s_0' \end{bmatrix}\begin{bmatrix} 0 & 0 \\ n' & 0 \end{bmatrix}\begin{bmatrix} 1 - r_0' & -m_0' \\ -n_0' & 1 - s_0' \end{bmatrix} = \begin{bmatrix} [m_0', n'](1 - r_0') & -[m_0', n']m_0' \\ s_0'n'(1 - r_0') & -s_0'(n', m_0') \end{bmatrix} \in \begin{bmatrix} 0 & M' \\ 0 & 0 \end{bmatrix},$$

implying that

$$[m_0', n'](1 - r_0') = 0 \tag{10}$$

and

$$s_0'(n', m_0') = 0. \tag{11}$$

Similarly, by using the fact that

$$\begin{bmatrix} 1 - r_0' & -m_0' \\ -n_0' & 1 - s_0' \end{bmatrix} T' \begin{bmatrix} r_0' & m_0' \\ n_0' & s_0' \end{bmatrix} \subseteq \begin{bmatrix} 0 & 0 \\ N' & 0 \end{bmatrix},$$

we obtain that

$$[m_0', n']r_0' = 0 \tag{12}$$

and

$$(1 - s_0')(n', m_0') = 0, \tag{13}$$

as well as that

$$(1 - r_0')[m', n_0'] = 0 \tag{14}$$

and

$$(n_0', m')s_0' = 0 \tag{15}$$

for every $m' \in M'$ and $n' \in N'$.

Adding in pairs relations (8) and (14), (9) and (15), (10) and (12), (11) and (13), we see that

$$[m', n_0'] = 0, \quad (n_0', m') = 0, \quad [m_0', n'] = 0, \quad (n', m_0') = 0, \tag{16}$$

for every $m' \in M'$ and $n' \in N'$.

Now if we use (16) in (5) and (6), we find that r_0' is an idempotent of R', and s_0' is an idempotent of S'.

If we apply ϕ to the relation $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & m \\ 0 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & m \\ 0 & 0 \end{smallmatrix}\right]$, where $m \in M$, we obtain that

$$r_0'u(m) = u(m) \tag{17}$$

for every $m \in M$.

Similarly, if we apply ϕ to the relation $\left[\begin{smallmatrix}0 & m\\ 0 & 0\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right] = 0$, we obtain that

$$u(m)s_0' = 0 \tag{18}$$

for every $m \in M$.

Similarly, we get that

$$v(n)r_0' = v(n) \tag{19}$$

and

$$s_0'v(n) = 0 \tag{20}$$

for every $n \in N$.

Now fix some $a' \in R'$, $b' \in S'$. Since ϕ is surjective, there exists $\left[\begin{smallmatrix}r & p\\ q & s\end{smallmatrix}\right] \in T$ such that $\phi\left(\left[\begin{smallmatrix}r & p\\ q & s\end{smallmatrix}\right]\right) = \left[\begin{smallmatrix}a' & 0\\ 0 & b'\end{smallmatrix}\right]$. Then $\phi\left(\left[\begin{smallmatrix}r & 0\\ 0 & s\end{smallmatrix}\right]\right) = \left[\begin{smallmatrix}a' & -u(p)\\ -v(q) & b'\end{smallmatrix}\right]$. Apply ϕ to the relation

$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}r & 0\\ 0 & s\end{bmatrix} = \begin{bmatrix}r & 0\\ 0 & s\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$$

and get

$$\begin{bmatrix}r_0' & m_0'\\ n_0' & s_0'\end{bmatrix}\begin{bmatrix}a' & -u(p)\\ -v(q) & b'\end{bmatrix} = \begin{bmatrix}a' & -u(p)\\ -v(q) & b'\end{bmatrix}\begin{bmatrix}r_0' & m_0'\\ n_0' & s_0'\end{bmatrix}.$$

In particular, this implies that

$$r_0'a' - [m_0', v(q)] = a'r_0' - [u(p), n_0'],$$

which in view of (16) shows that $r_0'a' = a'r_0'$. Also we get that

$$-(n_0', u(p)) + s_0'b' = -(v(q), m_0') + b's_0',$$

which shows that $s_0'b' = b's_0'$. Thus r_0' is a central idempotent in R' and s_0' is a central idempotent in S'. Since R' and S' are indecomposable rings, we conclude that r_0' must be either 0 or 1 and s_0' must be either 0 or 1.

On the other hand, T' is an indecomposable ring by Corollary 3.3. This implies that not both M and N are zero, otherwise $T \simeq R \times S$, which is not indecomposable.

Since $r_0'u(m) = u(m)$ for all $m \in M$ by (17), $v(n)r_0' = v(n)$ for all $n \in N$ by (19), and at least one of M and N is nonzero, we cannot have $r_0' = 0$ (otherwise u or v could not be injective); therefore, $r_0' = 1$. Similarly, since $u(m)s_0' = 0$ for all $m \in M$ by (18), and $s_0'v(n) = 0$ for all $n \in N$ by (20), we must have $s_0' = 0$. We have obtained that

$$\phi\left(\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\right) = \begin{bmatrix}1 & m_0'\\ n_0' & 0\end{bmatrix}.$$

Next, let $r \in R$. Then

$$\phi\left(\begin{bmatrix}r & 0\\ 0 & 0\end{bmatrix}\right) = \begin{bmatrix}r_1' & m_1'\\ n_1' & s_1'\end{bmatrix}$$

for some $r_1' \in R', m_1' \in M', n_1' \in N'$ and $s_1' \in S'$. Apply ϕ to the relations

$$\begin{bmatrix}r & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}r & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}r & 0\\ 0 & 0\end{bmatrix},$$

and obtain that

$$\begin{bmatrix} r_1' & m_1' \\ n_1' & s_1' \end{bmatrix} \begin{bmatrix} 1 & m_0' \\ n_0' & 0 \end{bmatrix} = \begin{bmatrix} r_1' & m_1' \\ n_1' & s_1' \end{bmatrix} = \begin{bmatrix} 1 & m_0' \\ n_0' & 0 \end{bmatrix} \begin{bmatrix} r_1' & m_1' \\ n_1' & s_1' \end{bmatrix},$$

and so, using (16), we obtain that

$$\begin{bmatrix} r_1' & r_1'm_0' \\ n_1' + s_1'n_0' & 0 \end{bmatrix} = \begin{bmatrix} r_1' & m_1' \\ n_1' & s_1' \end{bmatrix} = \begin{bmatrix} r_1' & m_1' + m_0's_1' \\ n_0'r_1' & 0 \end{bmatrix}.$$

This shows that

$$m_1' = r_1'm_0', \quad s_1' = 0, \quad n_1' = n_0'r_1',$$

and so

$$\phi\left(\begin{bmatrix} r & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} r_1' & r_1'm_0' \\ n_0'r_1' & 0 \end{bmatrix}.$$

Consequently, if we define $\gamma : R \to R'$ by $\gamma(r) = r_1'$, then

$$\phi\left(\begin{bmatrix} r & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} \gamma(r) & \gamma(r)m_0' \\ n_0'\gamma(r) & 0 \end{bmatrix} \tag{21}$$

for all $r \in R$. Moreover, it is clear from (21) that γ is additive and injective.

Applying ϕ to

$$\begin{bmatrix} r_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r_2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} r_1r_2 & 0 \\ 0 & 0 \end{bmatrix}, \quad r_1, r_2 \in R,$$

we get

$$\begin{bmatrix} \gamma(r_1) & \gamma(r_1)m_0' \\ n_0'\gamma(r_1) & 0 \end{bmatrix} \begin{bmatrix} \gamma(r_2) & \gamma(r_2)m_0' \\ n_0'\gamma(r_2) & 0 \end{bmatrix} = \begin{bmatrix} \gamma(r_1r_2) & \gamma(r_1r_2)m_0' \\ n_0'\gamma(r_1r_2) & 0 \end{bmatrix}.$$

Looking at position (1,1), and taking into account that $[\gamma(r_1)m_0', n_0'\gamma(r_2)] = \gamma(r_1)[m_0', n_0']\gamma(r_2) = 0$, we find that γ is multiplicative. Therefore γ is an injective ring morphism.

Now

$$\phi\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & -m_0' \\ -n_0' & 1 \end{bmatrix},$$

and so similar arguments as above provide a map $\delta : S \to S'$ such that

$$\phi\left(\begin{bmatrix} 0 & 0 \\ 0 & s \end{bmatrix}\right) = \begin{bmatrix} 0 & -m_0'\delta(s) \\ -\delta(s)n_0' & \delta(s) \end{bmatrix} \tag{22}$$

for all $s \in S$. Also, we obtain in a similar way to γ that δ is an injective ring morphism.

We conclude from (3), (4), (21) and (22) that

$$\phi\left(\begin{bmatrix} r & m \\ n & s \end{bmatrix}\right) = \begin{bmatrix} \gamma(r) & \gamma(r)m_0' - m_0'\delta(s) + u(m) \\ n_0'\gamma(r) - \delta(s)n_0' + v(n) & \delta(s) \end{bmatrix} \tag{23}$$

for all $r \in R$, $s \in S$, $m \in M$, $n \in N$.

We have that γ and δ are surjective. Indeed, let $r' \in R'$, $s' \in S'$. Then there exists $\left[\begin{smallmatrix} r & m \\ n & s \end{smallmatrix}\right] \in T$ such that

$$\begin{bmatrix} r' & 0 \\ 0 & s' \end{bmatrix} = \phi\left(\begin{bmatrix} r & m \\ n & s \end{bmatrix}\right) = \begin{bmatrix} \gamma(r) & \gamma(r)m_0' - m_0'\delta(s) + u(m) \\ n_0'\gamma(r) - \delta(s)n_0' + v(n) & \delta(s) \end{bmatrix},$$

and so $\gamma(r) = r'$ and $\delta(s) = s'$. Therefore $\gamma : R \to R'$ and $\delta : S \to S'$ are ring isomorphisms.

We also have that u and v are surjective. Indeed, for $m' \in M'$ and $n' \in N'$ there is $\left[\begin{smallmatrix} r & m \\ n & s \end{smallmatrix}\right] \in T$ such that

$$\begin{bmatrix} 0 & m' \\ n' & 0 \end{bmatrix} = \phi\left(\begin{bmatrix} r & m \\ n & s \end{bmatrix}\right) = \begin{bmatrix} \gamma(r) & \gamma(r)m_0' - m_0'\delta(s) + u(m) \\ n_0'\gamma(r) - \delta(s)n_0' + v(n) & \delta(s) \end{bmatrix}.$$

Since γ and δ are injective, we get that $r = 0$ and $s = 0$, and then $m' = u(m)$ and $n' = v(n)$.

Next we show that u is a γ–δ-bimodule isomorphism, i.e. $u(rms) = \gamma(r)u(m)\delta(s)$ for all $r \in R$, $m \in M$, $s \in S$, and v is a δ–γ-bimodule isomorphism. Let $r \in R$, and let $m \in M$. If we apply ϕ to

$$\begin{bmatrix} r & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & rm \\ 0 & 0 \end{bmatrix},$$

then by (23)

$$\begin{bmatrix} \gamma(r) & \gamma(r)m_0' \\ n_0'\gamma(r) & 0 \end{bmatrix}\begin{bmatrix} 0 & u(m) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & u(rm) \\ 0 & 0 \end{bmatrix},$$

and so it follows from position $(1, 2)$ that

$$u(rm) = \gamma(r)u(m).$$

Similarly,

$$u(ms) = u(m)\delta(s), \quad v(sn) = \delta(s)v(n) \quad \text{and} \quad v(nr) = v(n)\gamma(r)$$

for all $r \in R$, $s \in S$, $m \in M$, $n \in N$.

Finally we show that (iii) in Proposition 2.1 is satisfied. Let $m \in M$, $n \in N$. Applying ϕ to

$$\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix} = \begin{bmatrix} [m, n] & 0 \\ 0 & 0 \end{bmatrix},$$

we obtain that

$$\begin{bmatrix} 0 & u(m) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ v(n) & 0 \end{bmatrix} = \begin{bmatrix} \gamma([m, n]) & \gamma([m, n])m_0' \\ n_0'\gamma([m, n]) & 0 \end{bmatrix},$$

and so $[u(m), v(n)] = \gamma([m, n])$. Similarly, $(v(n), u(m)) = \delta((n, m))$. We thus conclude that $\phi \in \mathrm{Iso}_0^0(T, T')$.

(2) It is clear that any $\phi \in \mathrm{Iso}_0^1(T, T')$ is an anti-semigraded isomorphism. Conversely, let ϕ be an anti-semigraded isomorphism from T to T'. If $\alpha_T : T \to T^{-1}$ is the canonical isomorphism, then $\phi\alpha_T^{-1} : T^{-1} \to T'$ is a semigraded isomorphism,

so by the first part of the theorem we have that $\phi\alpha_T^{-1} \in \mathrm{Iso}_0^0(T^{-1}, T')$. Then $\phi \in \mathrm{Iso}_0^0(T^{-1}, T')\alpha_T = \mathrm{Iso}_0^1(T, T')$ by the first equality in equation (2). ■

Remark 3.7 Assume that R, S, R' and S' are indecomposable rings, and all of M, N, M' and N' are zero. Then identifying $T = R \times S$ and $T' = R' \times S'$, it is easy to see that any isomorphism $\phi: T \to T'$ is either of the form $\phi(r, s) = (\gamma(r), \delta(s))$ for some isomorphisms $\gamma: R \to R'$ and $\delta: S \to S'$, or of the form $\phi(r, s) = (\sigma(s), \rho(r))$ for some isomorphisms $\rho: R \to S'$ *and* $\sigma: S \to R'$. Therefore we have that $\mathrm{Iso}(T, T') = \mathrm{Iso}_g(T, T') = \mathrm{Iso}_g^+(T, T') = \mathrm{Iso}_g^-(T, T')$, and this is the set of all semigraded isomorphisms from T to T' (and also the set of all anti-semigraded isomorphisms from T to T'). Also it is the disjoint union of $\mathrm{Iso}_0^0(T, T')$ and $\mathrm{Iso}_0^1(T, T')$, so $\mathrm{Iso}(T, T') = \mathrm{Iso}_g(T, T') = \mathrm{Iso}_0(T, T')$.

COROLLARY 3.8 *If R, S, R' and S' are indecomposable, then* $\mathrm{Iso}_g(T, T') \subseteq \mathrm{Iso}_0(T, T')$.

Proof The result is clear if $\mathrm{Iso}_g(T, T') = \emptyset$. Assume now that $\mathrm{Iso}_g(T, T')$ is non-empty, implying that T and T' are isomorphic. If not both M' and N' are zero, then the result follows from Theorem 3.6. If $M' = 0$ and $N' = 0$, then we also must have $M = 0$ and $N = 0$ (otherwise T is indecomposable by Corollary 3.3, while T' is not so), and the result follows from Remark 3.7. ■

COROLLARY 3.9 *If R, S, R' and S' are indecomposable rings and the Morita context with which T (or T') is associated is strict, then* $\mathrm{Iso}_g(T, T') = \mathrm{Iso}_0(T, T')$.

Proof It follows from Proposition 3.1 and Corollary 3.8. ■

Remark 3.10 If R and S are not indecomposable, then $\mathrm{Aut}_g(T)$ is not necessarily contained in $\mathrm{Aut}_0(T)$. Indeed, consider, for example, the case where $R = S \times S$, $M = 0$ and $N = 0$. Then we can identify T with $S \times S \times S$, and the map $\phi: T \to T$, $\phi(s_1, s_2, s_3) = (s_1, s_3, s_2)$, $s_1, s_2, s_3 \in S$, is an automorphism of T, which is clearly graded, since $T = T_0$, but it is easy to see that $\phi \notin \mathrm{Aut}_0(T)$.

4. Automorphisms in the case of zero Morita maps

Let $T = \left[\begin{smallmatrix} R & M \\ N & S \end{smallmatrix}\right]$ and $T' = \left[\begin{smallmatrix} R' & M' \\ N' & S' \end{smallmatrix}\right]$ be Morita context rings. We have seen that in general $\mathrm{Iso}_0(T, T')$ is smaller than $\mathrm{Iso}(T, T')$. Remark 2.5 shows that if $\mathrm{Iso}_0(T, T') = \mathrm{Iso}(T, T') \neq \emptyset$, then necessarily all the Morita maps in both Morita contexts are zero. In this section we show that the converse also holds, provided that the rings R and S have only 0 and 1 as idempotents.

Throughout this section T and T' will be Morita context rings as above such that R, S, R' and S' have only 0 and 1 as idempotents, and the Morita maps are zero in each of the two contexts with which T and T' are associated.

First note that a matrix $\left[\begin{smallmatrix} r & m \\ n & s \end{smallmatrix}\right]$ in T is idempotent if and only if

$$r^2 = r, \quad s^2 = s, \quad rm + ms = m \quad \text{and} \quad nr + sn = n.$$

Therefore, apart from the trivial idempotents $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, the other idempotents of T are of the form

$$\begin{bmatrix} 1 & m \\ n & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & m \\ n & 1 \end{bmatrix}. \tag{24}$$

The two types of idempotents in (24) will henceforth be called type 1 idempotents and type 2 idempotents, respectively.

We look at the action of an isomorphism $\phi: T \to T'$ on an idempotent $E \in T$ of the form

$$\begin{bmatrix} 1 & m \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ n & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & m \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 \\ n & 1 \end{bmatrix}.$$

Note that the first two are type 1 idempotents, and the last two are type 2 idempotents. Since ϕ is injective and $\phi(E)$ is also an idempotent, we have that $\phi(E)$ has one of the forms in (24) inside the ring T'.

LEMMA 4.1 *If there exists $m_1 \in M$ such that ϕ maps the type* 1 *idempotent $\left[\begin{smallmatrix} 1 & m_1 \\ 0 & 0 \end{smallmatrix}\right]$ to a type* 2 *idempotent, then*

(a) *ϕ maps every type* 1 *idempotent of the form $\left[\begin{smallmatrix} 1 & m \\ 0 & 0 \end{smallmatrix}\right]$ to a type* 2 *idempotent; and*
(b) *there is a fixed element $a_2' \in M'$ and a map $v_2: M \to N'$ such that, for all $m \in M$,*

$$\phi\left(\begin{bmatrix} 1 & m \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & a_2' \\ v_2(m) & 1 \end{bmatrix}.$$

Proof Let $\phi\left(\left[\begin{smallmatrix} 1 & m_1 \\ 0 & 0 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 0 & m_1' \\ n_1' & 1 \end{smallmatrix}\right]$ for some $m_1' \in M'$, $n_1' \in N'$.

(a) Suppose there is an $m \in M$ such that $\phi\left(\left[\begin{smallmatrix} 1 & m \\ 0 & 0 \end{smallmatrix}\right]\right)$ is a type 1 idempotent, i.e. $\phi\left(\left[\begin{smallmatrix} 1 & m \\ 0 & 0 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 1 & m' \\ n' & 0 \end{smallmatrix}\right]$ for some $m' \in M'$, $n' \in N'$. Since $\left[\begin{smallmatrix} 1 & m_1 \\ 0 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & m \\ 0 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & m \\ 0 & 0 \end{smallmatrix}\right]$, applying ϕ yields

$$\begin{bmatrix} 0 & m_1' \\ n_1' & 1 \end{bmatrix}\begin{bmatrix} 1 & m' \\ n' & 0 \end{bmatrix} = \begin{bmatrix} 1 & m' \\ n' & 0 \end{bmatrix}. \tag{25}$$

Equating the entries in position $(1,1)$ in (25), we get that $0 = 1$, a contradiction. Hence, ϕ maps every type 1 idempotent of the form $\left[\begin{smallmatrix} 1 & m \\ 0 & 0 \end{smallmatrix}\right]$ to a type 2 idempotent.

(b) By (a), there are maps $\alpha_2: M \to M'$ and $v_2: M \to N'$ such that

$$\phi\left(\begin{bmatrix} 1 & m \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & \alpha_2(m) \\ v_2(m) & 1 \end{bmatrix}$$

for all $m \in M$. Let m and m_0 be arbitrary elements of M. Applying ϕ to $\left[\begin{smallmatrix} 1 & m_0 \\ 0 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & m \\ 0 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & m \\ 0 & 0 \end{smallmatrix}\right]$, we obtain that

$$\begin{bmatrix} 0 & \alpha_2(m_0) \\ v_2(m_0) & 1 \end{bmatrix}\begin{bmatrix} 0 & \alpha_2(m) \\ v_2(m) & 1 \end{bmatrix} = \begin{bmatrix} 0 & \alpha_2(m) \\ v_2(m) & 1 \end{bmatrix}.$$

Position $(1,2)$ shows that $\alpha_2(m) = \alpha_2(m_0)$. Consequently, α_2 is a constant function, say $\alpha_2(m) = a_2'$ for some fixed $a_2' \in M'$, which concludes the proof. ■

COROLLARY 4.2 *If there is an $m \in M$ such that ϕ maps the type* 1 *idempotent $\left[\begin{smallmatrix} 1 & m \\ 0 & 0 \end{smallmatrix}\right]$ to a type* 1 *idempotent, then*

(a) *ϕ maps every type* 1 *idempotent of the form $\left[\begin{smallmatrix} 1 & m \\ 0 & 0 \end{smallmatrix}\right]$ to a type* 1 *idempotent; and*
(b) *there is a fixed element $b_1' \in N'$ and a map $u_1: M \to M'$ such that for all $m \in M$,*

$$\phi\left(\begin{bmatrix} 1 & m \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & u_1(m) \\ b_1' & 0 \end{bmatrix}.$$

Proof (a) This is an immediate consequence of Lemma 4.1(a).

(b) By (a), there are maps $u_1\colon M \to M'$ and $\beta_1\colon M \to N'$ such that

$$\phi\left(\begin{bmatrix} 1 & m \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & u_1(m) \\ \beta_1(m) & 0 \end{bmatrix}$$

for any $m \in M$. Proceeding as in the proof of Lemma 4.1(b), but using position $(2,1)$ instead of position $(1,2)$, the desired result follows. ■

The foregoing results are summarized in:

PROPOSITION 4.3 *Either there is a map $u_1\colon M \to M'$ and a fixed element $b_1' \in N'$ such that ϕ maps every type 1 idempotent of the form $\left[\begin{smallmatrix} 1 & m \\ 0 & 0 \end{smallmatrix}\right]$ to a type 1 idempotent of the form $\left[\begin{smallmatrix} 1 & u_1(m) \\ b_1' & 0 \end{smallmatrix}\right]$, or there is a map $v_2\colon M \to N'$ and a fixed element $a_2' \in M'$ such that ϕ maps every type 1 idempotent of the form $\left[\begin{smallmatrix} 1 & m \\ 0 & 0 \end{smallmatrix}\right]$ to a type 2 idempotent of the form $\left[\begin{smallmatrix} 0 & a_2' \\ v_2(m) & 1 \end{smallmatrix}\right]$.*

Similar arguments yield the following result for the action of ϕ on type 1 idempotents of the form $\left[\begin{smallmatrix} 1 & 0 \\ n & 0 \end{smallmatrix}\right]$.

PROPOSITION 4.4 *For all $n \in N$, either $\phi\left(\left[\begin{smallmatrix} 1 & 0 \\ n & 0 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 1 & a_1' \\ v_1(n) & 0 \end{smallmatrix}\right]$ for some fixed $a_1' \in M'$ and some map $v_1\colon N \to N'$, or $\phi\left(\left[\begin{smallmatrix} 1 & 0 \\ n & 0 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 0 & u_2(n) \\ b_2' & 1 \end{smallmatrix}\right]$ for some fixed $b_2' \in N'$ and some map $u_2\colon N \to M'$.*

Setting $m = 0$ in $\left[\begin{smallmatrix} 1 & m \\ 0 & 0 \end{smallmatrix}\right]$ in Proposition 4.3 and setting $n = 0$ in $\left[\begin{smallmatrix} 1 & 0 \\ n & 0 \end{smallmatrix}\right]$ in Proposition 4.4, we obtain in both cases the type 1 idempotent $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, and so if we consider positions $(1,1)$ and $(2,2)$ of the possible actions of ϕ on $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ as described in these two propositions, we arrive at:

COROLLARY 4.5 *For all $m \in M$ and all $n \in N$, either*

$$(\mathrm{I}')\quad \phi\left(\begin{bmatrix} 1 & m \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & u_1(m) \\ v_1(0) & 0 \end{bmatrix} \quad \text{and} \quad \phi\left(\begin{bmatrix} 1 & 0 \\ n & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & u_1(0) \\ v_1(n) & 0 \end{bmatrix}$$

for some maps $u_1\colon M \to M'$ and $v_1 : N \to N'$, or

$$(\mathrm{I}'')\quad \phi\left(\begin{bmatrix} 1 & m \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & u_2(0) \\ v_2(m) & 1 \end{bmatrix} \quad \text{and} \quad \phi\left(\begin{bmatrix} 1 & 0 \\ n & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & u_2(n) \\ v_2(0) & 1 \end{bmatrix}$$

for some maps $u_2\colon N \to M'$ and $v_2\colon M \to N'$.

We set

$$m_0' := u_1(0) \quad \text{and} \quad n_0' := v_1(0),$$

where u_1 and v_1 are as in Corollary 4.5.

Turning our attention to type 2 idempotents of the form $\left[\begin{smallmatrix} 0 & n \\ n & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & 0 \\ n & 1 \end{smallmatrix}\right]$, it can be verified as above that

COROLLARY 4.6 *For all $m \in M$ and all $n \in N$, either*

$$(\mathrm{II}')\quad \phi\left(\begin{bmatrix} 0 & m \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & h_1(m) \\ p_1(0) & 1 \end{bmatrix} \quad \text{and} \quad \phi\left(\begin{bmatrix} 0 & 0 \\ n & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & h_1(0) \\ p_1(n) & 1 \end{bmatrix}$$

for some maps $h_1\colon M \to M'$ and $p_1\colon N \to N'$, or

(II″) $\phi\left(\begin{bmatrix} 0 & m \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & h_2(0) \\ p_2(m) & 0 \end{bmatrix}$ *and* $\phi\left(\begin{bmatrix} 0 & 0 \\ n & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & h_2(n) \\ p_2(0) & 0 \end{bmatrix}$

for some maps $h_2\colon N \to M'$ *and* $p_2\colon M \to N'$.

THEOREM 4.7 *Let* R, S, R' *and* S' *be rings having only trivial idempotents,* M *an* R–S*-bimodule,* N *an* S–R*-bimodule,* M' *an* R'–S'*-bimodule and* N' *an* S'–R'*-bimodule. Let* $T = \left[\begin{smallmatrix} R & M \\ N & S \end{smallmatrix}\right]$ *and* $T' = \left[\begin{smallmatrix} R' & M' \\ N' & S' \end{smallmatrix}\right]$ *be the associated Morita context rings, where the contexts are considered with both Morita maps equal to zero. Then* $\mathrm{Iso}_0(T, T') = \mathrm{Iso}(T, T')$.

Proof The result is clear if $\mathrm{lso}(T, T') = \emptyset$. Assume that $\mathrm{Iso}(T, T')$ is non-empty, i.e. T and T' are isomorphic. If both M and N are zero, then $T \simeq R \times S$ is not indecomposable. This forces M' and N' to be zero, otherwise T' would be indecomposable by Corollary 3.3. Now $\mathrm{Iso}_0(T, T') = \mathrm{Iso}(T, T')$ by Remark 3.7.

Assume now that at least one of M and N is non-zero. Then as above we cannot have both M' and N' equal to zero. Let $\phi\colon T \to T'$ be a ring isomorphism. Since $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, it follows from Corollaries 4.5–4.6, by considering position (2,2), that (I′) and (II″) cannot simultaneously hold; neither can (I″) and (II′) simultaneously hold. Therefore, either (I′) and (II′) are true, or (I″) and (II″) are true.

We first consider

(I′) and (II′): For every $m \in M$,

$$\begin{aligned} \phi\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right) &= \phi\left(\begin{bmatrix} 1 & m \\ 0 & 0 \end{bmatrix}\right) + \phi\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) - \phi\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\ &= \begin{bmatrix} 1 & u_1(m) \\ v_1(0) & 0 \end{bmatrix} + \begin{bmatrix} 0 & h_1(0) \\ p_1(0) & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & u_1(m) + h_1(0) \\ v_1(0) + p_1(0) & 0 \end{bmatrix}, \end{aligned}$$

and setting $m = 0$, we conclude that

$$h_1(0) = -m_0'.$$

Hence, defining $u\colon M \to M'$ by $u(m) = u_1(m) - m_0'$, it follows that

$$\phi\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & u(m) \\ 0 & 0 \end{bmatrix} \tag{26}$$

for all $m \in M$.

Similarly, considering the equality

$$\phi\left(\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}\right) = \phi\left(\begin{bmatrix} 1 & 0 \\ n & 0 \end{bmatrix}\right) + \phi\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) - \phi\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

and defining $v\colon N \to N'$ by $v(n) = v_1(n) - v_1(0)$, it follows as above that

$$\phi\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ v(n) & 0 \end{bmatrix}, \tag{27}$$

for all all $n \in N$.

Equations (26) and (27) show that $\phi(T_1) \subseteq T_1$ and $\phi(T_{-1}) \subseteq T_{-1}$. Thus $\phi: T \to T'$ is a semigraded isomorphism, hence $\phi \in \mathrm{Iso}_0^0(T, T')$ by Theorem 3.6.

The other possible situation is

(I″) and (II″): Then $\psi = \phi\alpha_T^{-1} : T^{-1} \to T'$ is a ring isomorphism between the Morita context rings T^{-1} and T', and it is easy to see that ψ acts on the particular idempotents (of T^{-1}) that we consider according to the case (I′) and (II′). Then applying the result in the first part of the proof, we find that $\psi \in \mathrm{Iso}_0^0(T^{-1}, T')$. Now using (2) we get that $\phi = \psi\alpha_T \in \mathrm{Iso}_0^1(T, T')$. ■

Remark 4.8 (1) We first note that any ring A which has a non-trivial idempotent e is isomorphic to a Morita context ring. To see this, we consider the Peirce decomposition associated with the complete system of orthogonal idempotents $\{e, 1-e\}$; more precisely,

$$A = eAe \oplus eA(1-e) \oplus (1-e)Ae \oplus (1-e)A(1-e)$$

as additive groups. Moreover, eAe is a ring with identity element e, $(1-e)A(1-e)$ is a ring with identity element $1-e$, $eA(1-e)$ is a left eAe, right $(1-e)A(1-e)$-bimodule with actions defined by the multiplication of the ring A, and similarly $(1-e)Ae$ is a left $(1-e)A(1-e)$, right eAe-bimodule. We have a Morita context $(eAe, (1-e)A(1-e), eA(1-e), (1-e)Ae, f, g)$, where f and g are induced by the multiplication of A. It is easy to see that the above decomposition as additive groups induces in fact an isomorphism of rings

$$A \simeq \begin{bmatrix} eAe & eA(1-e) \\ (1-e)Ae & (1-e)a(1-e) \end{bmatrix}.$$

A ring $T = \left[\begin{smallmatrix} R & M \\ N & S \end{smallmatrix}\right]$ associated with a general Morita context is an instance of a ring with non-trivial idempotents, since $e := \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is such an idempotent. It is clear that

$$eTe = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \simeq R, \quad (1-e)T(1-e) = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix} \simeq S,$$

$$eT(1-e) = \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} \simeq M, \quad (1-e)Te = \begin{bmatrix} 0 & 0 \\ N & 0 \end{bmatrix} \simeq N.$$

We conclude that a ring has non-trivial idempotents if and only if it is isomorphic to a Morita context ring.

(2) In [1], a ring R is called strongly indecomposable if it is not isomorphic to a ring of the form $\left[\begin{smallmatrix} A & X \\ 0 & B \end{smallmatrix}\right]$, where A and B are rings, and X is a nonzero left A, right B-bimodule. The automorphism group of a generalized triangular matrix ring $\left[\begin{smallmatrix} R & {}_RM_S \\ 0 & S \end{smallmatrix}\right]$ over strongly indecomposable rings R and S is computed in [1].

As we explained in the first part of the remark, a ring does not have non-trivial idempotents if and only if it is not isomorphic to a Morita context ring, so this is a 4-corner version of the 'strongly indecomposable ring' concept of [1]. Thus our Theorem 4.7 can be seen as a result similar to Theorem 3.2 of [1] for 4-corners generalized matrix rings.

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