A Cayley–Hamilton trace identity for \(2 \times 2\) matrices over Lie-solvable rings

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**Abstract**

First we construct an algebra satisfying the polynomial identity
\[
[[x, y], [u, v]] = 0,
\]
but none of the stronger identities
\[
[x, y][u, v] = 0 \quad \text{and} \quad [[x, y], z] = 0.
\]
Then we exhibit a Cayley–Hamilton trace identity for \(2 \times 2\) matrices with entries in a ring \(R\) satisfying
\[
[[x, y], [x, z]] = 0 \quad \text{and} \quad \frac{1}{2} \in R.
\]

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1. Introduction

The Cayley–Hamilton theorem and the corresponding trace identity play a fundamental role in proving classical results about the polynomial and trace identities of the $n \times n$ matrix algebra $M_n(K)$ over a field $K$ (see [2, 3]). In case of $\text{char}(K) = 0$, Kemer’s pioneering work (see [5]) on the $T$-ideals of associative algebras revealed the importance of the identities satisfied by the $n \times n$ matrices over the Grassmann (exterior) algebra

$$E = K \left\{ v_1, v_2, \ldots, v_r, \ldots \mid v_iv_j + v_jv_i = 0 \text{ for all } 1 \leq i \leq j \right\}$$

generated by the infinite sequence of anticommutative indeterminates $(v_i)_{i \geq 1}$.

For $n \times n$ matrices over a Lie-nilpotent ring $R$ satisfying the polynomial identity

$$[[[\ldots [[x_1, x_2], x_3], \ldots], x_m], x_{m+1}] = 0$$

(with $[x, y] = xy - yx$), a Cayley–Hamilton identity of degree $n^m$ (with left- or right-sided scalar coefficients) was found in [6]. Since $E$ is Lie-nilpotent with $m = 2$, the above mentioned Cayley–Hamilton identity for a matrix $A \in M_n(E)$ is of degree $n^2$.

In [1] Domokos presented a slightly modified version of this identity in which the coefficients are invariant under the conjugation action of $GL_n(K)$. For a matrix $A \in M_3(E)$ he obtained the trace identity

$$A^4 - 2\text{tr}(A)A^3 + \left( 2\text{tr}^2(A) - \text{tr}(A^2) \right) A^2 + \left( \frac{1}{2} \text{tr}(A)\text{tr}(A^2) + \frac{1}{2} \text{tr}(A^2)\text{tr}(A) - \text{tr}^3(A) \right) A$$

$$+ \frac{1}{4} \left( \text{tr}^4(A) + \text{tr}^2(A^2) - \frac{5}{2} \text{tr}^2(A)\text{tr}(A^2) + \frac{1}{2} \text{tr}(A^2)\text{tr}^2(A) \right) I = 0,$$

where $I$ is the identity matrix and $\text{tr}(A)$ denotes the sum of the diagonal entries of $A$. A similar identity with right coefficients also holds for $A$. Here $E$ can be replaced by any ring $R$ which is Lie-nilpotent of index 2.

The identity $[x, y][x, z] = 0$ is a consequence of Lie-nilpotency of index 2 (see [4]), as is obviously $[[x, y], [x, z]] = 0$. The first aim of the present paper is to provide an example of an algebra satisfying $[[[x, y], [u, v]], [u, v]] = 0$, but neither $[x, y][u, v] = 0$ nor $[[x, y], z] = 0$. Since the above mentioned trace identity cannot be used for matrices over such an algebra, our second purpose is to exhibit a new trace identity of the same kind (of degree 4 in $A$) for a matrix $A$ in $M_2(R)$, where $R$ is any ring satisfying the identity

$$[[[x, y], [u, v]], [u, v]] = 0$$

and $\frac{1}{2} \in R$. We note that a ring satisfying $[[x, y], [u, v]] = 0$ is called Lie-solvable of index 2.

From now onward $R$ and $S$ are rings with 1. In Section 2 we consider the ring $U^*_3(R)$ of upper triangular $3 \times 3$ matrices with equal diagonal entries over $R$. First we observe that $U^*_3(R)$ is never commutative. We prove that if $R$ is commutative then the algebra $U^*_3(R)$ satisfies the identities $[x, y][u, v] = 0$ and $[[x, y], z] = 0$. However, for a non-commutative $R$ we show that the ring $U^*_3(R)$ never satisfies any of the identities $[x, y][u, v] = 0$ and $[[x, y], z] = 0$.

The main result in Section 2 states that if $S$ satisfies the identities $[x, y][u, v] = 0$ and $[[x, y], z] = 0$, then the matrix ring $U^*_3(S)$ is Lie-solvable of index 2. It follows that if $R$ is commutative, then $U^*_3(U^*_3(R))$ is an example of an algebra satisfying $[[x, y], [u, v]] = 0$, but neither $[x, y][u, v] = 0$ nor $[[x, y], z] = 0$.

Section 3 is entirely devoted to the construction of our Cayley–Hamilton trace identity.

2. A particular Lie-solvable matrix algebra

Since

$$E_{1,2}, E_{2,3} \in U^*_3(R) = \left\{ \begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} \mid a, b, c, d \in R \right\}$$

and $E_{1,2}E_{2,3} = E_{1,3} \neq 0 = E_{2,3}E_{1,2}$, the ring $U^*_2(R)$ is never commutative. Any element of $U^*_2(R)$ can be written as $al + X$, where $X$ is strictly upper triangular. We note that $XYZ = 0$ for strictly upper triangular $3 \times 3$ matrices. If $R$ is commutative, then $al$ is central in $U^*_2(R)$ (of course, also in $M_3(R)$), $[al + X, bl + Y] = [X, Y]$ for all $a, b \in R$ and so $U^*_2(R)$ satisfies all polynomial identities in which each summand is a product of certain (possibly iterated) commutators. For example, 

$$[x, y][u, v] = 0 \quad \text{and} \quad [[x, y], z] = 0$$

are typical such identities for $U^*_2(R)$. If $R$ is non-commutative, say $[r, s] \neq 0$ for some $r, s \in R$, then for $x = rl, y = sE_{1,2}, u = E_{2,2}, v = z = E_{2,3}$ in $U^*_2(R)$ we have 

$$[x, y][u, v] = [[x, y], z] = [r, s]E_{1,3} \neq 0.$$ 

**Theorem 2.1.** If $S$ satisfies $[x, y][u, v] = 0$ and $[[x, y], z] = 0$, then $U^*_2(S)$ satisfies $[[x, y], [u, v]] = 0$.

**Proof.** Using the matrices

$$x = \begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} e & f & g \\ 0 & e & h \\ 0 & 0 & e \end{bmatrix}$$

in $U^*_2(S)$, a straightforward calculation gives that

$$[x, y] = \begin{bmatrix} [a, e] & [a, f] + [b, e] & [a, g] + [c, e] + (bh - fd) \\ 0 & [a, e] & [a, h] + [d, e] \\ 0 & 0 & [a, e] \end{bmatrix} = [a, e]I + C + \alpha E_{1,3},$$

where $\alpha = bh - fd$ and $C$ is a strictly upper triangular matrix with entries in $[S, S]$ (the additive subgroup of $S$ generated by all commutators). Now $[[a, e], s] = 0$ for all $s \in S$, hence $[a, e]I$ is central in $U^*_2(S)$ (also in $M_3(S)$). Thus we have

$$[[x, y], [u, v]] = [[a, e]I + C + \alpha E_{1,3}, [a', e']I + C' + \alpha' E_{1,3}] = [C + \alpha E_{1,3}, C' + \alpha' E_{1,3}] = 0$$

because of $(C + \alpha E_{1,3})(C' + \alpha' E_{1,3}) = (C' + \alpha' E_{1,3})(C + \alpha E_{1,3}) = 0$. Indeed, $CC' = C'C = 0$ is a consequence of $C, C' \in M_3([S, S])$ and of $[x, y][u, v] = 0$ in $S$, and $CE_{1,3} = E_{1,3}C = C'E_{1,3} = E_{1,3}C'$ follows from the fact that $C$ and $C'$ are strictly upper triangular. □

**Corollary 2.2.** If $R$ is commutative, then the algebra $U^*_2(U^*_2(R))$ satisfies $[[x, y], [u, v]] = 0$, but neither $[x, y][u, v] = 0$ nor $[[x, y], z] = 0$.

3. Matrices with commutator entries

The following can be considered as the “real” $2 \times 2$ Cayley–Hamilton trace identity.

**Proposition 3.1.** If $\frac{1}{2} \in R$ and $A = [a_{ij}] \in M_2(R)$, then

$$A^2 - \operatorname{tr}(A)A + \frac{1}{2}(\operatorname{tr}^2(A) - \operatorname{tr}(A^2))I = \begin{bmatrix} \frac{1}{2}[a_{11}, a_{22}] + \frac{1}{2}[a_{12}, a_{21}] & [a_{12}, a_{22}] \\ [a_{21}, a_{11}] & -\frac{1}{2}[a_{11}, a_{22}] - \frac{1}{2}[a_{12}, a_{21}] \end{bmatrix}.$$

**Proof.** A straightforward computation suffices. □

**Corollary 3.2.** If $\frac{1}{2} \in R$ and $B = [b_{ij}] \in M_2(R)$ with $\operatorname{tr}(B) = 0$, then

$$B^2 - \frac{1}{2}\operatorname{tr}(B^2)I = \begin{bmatrix} \frac{1}{2}[b_{12}, b_{21}] & -[b_{12}, b_{11}] \\ [b_{21}, b_{11}] & -\frac{1}{2}[b_{12}, b_{21}] \end{bmatrix}.$$
**Proof.** Since \( b_{22} = -b_{11} \), we have \([b_{11}, b_{22}] = 0\) and \([b_{12}, b_{22}] = -[b_{12}, b_{11}] \). Thus the formula in Proposition 3.1 immediately gives the identity for \( B \). \( \square \)

**Theorem 3.3.** If \( \frac{1}{2} \in R \) and \( R \) satisfies \([x, y], [x, z] = 0\), then
\[
\left( C^2 - \frac{1}{2} \text{tr}(C^2)I \right)^2 - \frac{1}{2} \text{tr}\left( \left( C^2 - \frac{1}{2} \text{tr}(C^2)I \right)^2 \right) I = 0
\]
for all \( C \in M_2(R) \) with \( \text{tr}(C) = 0 \).

**Proof.** Take \( C = [c_{ij}] \). In view of Corollary 3.2 we have
\[
C^2 - \frac{1}{2} \text{tr}(C^2)I = \begin{bmatrix}
\frac{1}{2}[c_{12}, c_{21}] & -[c_{12}, c_{11}] \\
[c_{21}, c_{11}] & -\frac{1}{2}[c_{12}, c_{21}]
\end{bmatrix}.
\]
Since \( \text{tr}(C^2 - \frac{1}{2} \text{tr}(C^2)I) = 0 \), the repeated application of Corollary 3.2 to \( B = C^2 - \frac{1}{2} \text{tr}(C^2)I \) gives that
\[
\left( C^2 - \frac{1}{2} \text{tr}(C^2)I \right)^2 - \frac{1}{2} \text{tr}\left( \left( C^2 - \frac{1}{2} \text{tr}(C^2)I \right)^2 \right) I = \frac{1}{2} \begin{bmatrix}
-[[c_{12}, c_{11}], [c_{21}, c_{11}]] & [[c_{12}, c_{11}], [c_{12}, c_{21}]] \\
[[c_{21}, c_{11}], [c_{12}, c_{21}]] & [[c_{12}, c_{11}], [c_{21}, c_{11}]]
\end{bmatrix}.
\]
Now we have
\[
[[c_{12}, c_{11}], [c_{21}, c_{11}]] = [[c_{11}, c_{12}], [c_{11}, c_{21}]]
\]
and
\[
[[c_{21}, c_{11}], [c_{12}, c_{21}]] = -[[c_{21}, c_{11}], [c_{21}, c_{12}]].
\]
Thus each entry of the above \( 2 \times 2 \) matrix is of the form \( \pm [[x, y], [x, z]] = 0 \) and the desired identity follows. \( \square \)

In Corollaries 3.4 and 3.5 we assume that \( \frac{1}{2} \in R \) and \( R \) satisfies \([x, y], [x, z] = 0\).

**Corollary 3.4.** If \( C \in M_2(R) \) with \( \text{tr}(C) = \text{tr}(C^2) = \text{tr}(C^4) = 0 \), then \( C^4 = 0 \).

**Proof.** Expanding the left hand side of the identity in Theorem 3.3, we get
\[
C^4 - \frac{1}{2} \text{tr}(C^2)C^2 - \frac{1}{2} C^2 \text{tr}(C^2) + \frac{1}{2} \left( \text{tr}^2(C^2) - \text{tr}(C^4) \right) I = 0,
\]
whose all terms but \( C^4 \) contain a factor \( \text{tr}(C^2) \) or \( \text{tr}(C^4) \). \( \square \)

**Corollary 3.5.** If \( \frac{1}{2} \in R \) and \( R \) is a ring satisfying \([x, y], [x, z] = 0\), then for all \( A \in M_2(R) \) we have
\[
A^4 - \frac{1}{2} A^2 \text{tr}(A)A - \frac{1}{2} \text{Atr}(A)A^2 - \frac{1}{2} A^3 \text{tr}(A) + \frac{1}{2} \text{tr}(A)A^3 + \frac{1}{2} A^2 \text{tr}^2(A) + \frac{1}{2} \text{tr}(A)A^2
\]
\[
- \frac{1}{2} A^2 \text{tr}(A^2) - \frac{1}{2} \text{tr}(A^2)A^2 + \frac{1}{4} \text{Atr}(A) \text{Atr}(A) + \frac{1}{4} \text{tr}(A) \text{Atr}(A)A
\]
\[
+ \frac{1}{4} \text{tr}(A)A^2 \text{tr}(A) + \frac{1}{4} \text{Atr}^2(A)A - \frac{1}{4} \text{tr}(A)A \text{tr}^2(A) - \frac{1}{4} \text{tr}^2(A)A \text{tr}(A)
\]
\[
+ \frac{1}{4} \text{tr}(A)A \text{tr}(A^2) + \frac{1}{4} \text{tr}(A^2)A \text{tr}(A) - \frac{1}{4} \text{Atr}^3(A) - \frac{1}{4} \text{tr}^3(A)A
\]
\[
+ \frac{1}{2} \text{Atr}(A) \text{tr}(A^2) + \frac{1}{4} \text{tr}(A^2) \text{tr}(A)A - \frac{1}{2} \text{tr}^2(A) \text{tr}(A^2)I - \frac{1}{2} \text{tr}(A^2) \text{tr}^2(A)I
\]
\[
+ \frac{1}{2} \text{tr}^2(A^2)I + \frac{1}{4} \text{tr}(A^2) \text{tr}(A^2)I + \frac{1}{4} \text{tr}(A^2) \text{tr}(A)I + \frac{1}{4} \text{tr}(A) \text{tr}^3(A)I
\]
\[
- \frac{1}{8} \text{tr}(A) \text{tr}^2(A)A I - \frac{1}{8} \text{tr}(A) \text{tr}(A^2)A I - \frac{1}{8} \text{tr}(A^2) \text{tr}^2(A)I - \frac{1}{8} \text{tr}(A) \text{tr}(A^2) \text{tr}(A)
\]
\[
+ \frac{1}{2} \text{tr}^4(A)I - \frac{1}{2} \text{tr}^3(A)I = 0.
\]

**Proof.** Apply Theorem 3.3 for \( C = A - \frac{1}{2} \text{tr}(A)I \); using linearity of tr(−), we get the identity above. \( \square \)

We note that the trace identity in Corollary 3.5 is different from the trace identity given by Domokos [1] in the following respect: in the latter in each term a power of \( A \) is multiplied from the left by a trace expression, whereas in our identity terms like \( A^2 \text{tr}(A)A \) appear.

Throughout this section we have used the identity \([x, y], [x, z] = 0\). The referee pointed out that this identity implies the “seemingly stronger” identity \([x, y], [u, v] = 0\) of Lie solvability, which plays an important role in Section 2.

Starting with a matrix \( C \in M_2(R) \) such that \( \text{tr}(C) = 0 \), define the sequence \( (C_k)_{k \geq 0} \) by the following recursion: \( C_0 = C \) and

\[
C_{k+1} = C_k^2 - \frac{1}{2} \text{tr}(C_k^2)I.
\]

Clearly, \( \text{tr}(C_k) = 0 \) for all \( k \geq 0 \) and \( C_k \) is a trace polynomial expression of \( C \). In view of Corollary 3.2, the entries of \( C_1 \) are of the form \([x_1, x_2]\). The repeated application of Corollary 3.2 (as it can be seen in the proof of Theorem 3.3) and a straightforward induction show that the (four) entries of \( C_k \) are all of the form \([x_1, x_2, \ldots, x_{2^k}]_{\text{solv}}\), where \([x_1, x_2]_{\text{solv}} = [x_1, x_2]\) and for \( i \geq 1 \) we take the Lie brackets as

\[
[x_1, x_2, \ldots, x_{2^{i+1}}]_{\text{solv}} = [x_1, x_2, \ldots, x_{2^i}]_{\text{solv}}, [x_{2^i+1}, x_{2^i+2}, \ldots, x_{2^{i+1}}]_{\text{solv}}.
\]

If \( R \) satisfies the general identity

\[
[x_1, x_2, \ldots, x_{2^k}]_{\text{solv}} = 0
\]

of Lie solvability, then \( C_k = 0 \), whence we can derive a trace identity for \( C \). Thus the substitution \( C = A - \frac{1}{2} \text{tr}(A)I \) gives a trace identity for an arbitrary \( A \in M_2(R) \).

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**References**