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# A NOTE ON SEMI-HOMOMORPHISMS OF RINGS

Y. FONG AND L. VAN WYK

Huq presented a general study of semi-homomorphisms of rings, following, amongst others, Kaplansky's study of semi-automorphisms of rings and Herstein's study of semi-homomorphisms of groups. Huq gave several "sufficient" conditions for a semi-homomorphism and a semi-monomorphism of rings to be a homomorphism and a monomorphism respectively. In this note we introduce semi-subgroups of groups, provide counterexamples to four of Huq's assertions and show how a minor, albeit forced, change to one of the conditions of the fourth assertion turns it into a special case of another theorem of Huq's.

## 1. PRELIMINARY RESULTS

Herstein [**2**] calls a mapping $\varphi\colon G \to H$ between two groups (written additively) a semi-homomorphism if

$$\varphi(a+b+a) = \varphi(a) + \varphi(b) + \varphi(a) \tag{1}$$

for all $a, b \in G$. Any homomorphism or anti-homomorphism is a semi-homomorphism, but the converse need not be true in general.

We call a subset $K$ of a group $A$ a *semi-subgroup* of $A$ if $h+k+h \in K$ for all $h, k \in K$. The subset $\{k + a \mid k \in K\}$ of $A$, for some $a \in A$, will be denoted by $K + a$. The singleton $\{a\}$ is a semi-subgroup of $A$ which is not a subgroup of $A$, for every $a \in A$ of order 2, and the image of every semi-homomorphism $\varphi\colon G \to H$ is a semi-subgroup of $H$. However, in the next paragraph we shall be interested in the subsets

$$H_\varphi = \{\varphi(a+b) - \varphi(a) - \varphi(b) - \varphi(0) \mid a, b \in G\} \text{ and } H_\varphi + \varphi(0) \text{ of } H.$$

The result in the first part of the "proof" of [**3**, Lemma 4] will be used frequently in the sequel; so we state it as

LEMMA 1.1. *If $\varphi\colon G \to H$ is a semi-homomorphism of abelian groups, then $2\varphi(a+b) = 2\varphi(a) + 2\varphi(b)$ for all $a, b \in G$.*

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## 2. Semi-homomorphisms and homomorphisms

We show that the condition,

$$\varphi(a+b)=\varphi(0)+\varphi(a)+\varphi(b) \tag{2}$$

for all $a,b \in G$, is stronger than (1) in general, but equivalent to (1) in the case where $G$ and $H$ are abelian and the semi-subgroup $H_\varphi$ of $H$ (see Lemma 2.2) contains no elements of order $2$. It is also shown that if $G$ and $H$ are abelian, then a semi-homomorphism $\varphi\colon G \to H$ is a homomorphism if and only if the semi-subgroup $H_\varphi + \varphi(0)$ of $H$ contains no elements of order $2$.

Lemma 2.1. *If a mapping $\varphi\colon G \to H$ between groups satisfies (2), then $\varphi$ is a semi-homomorphism.*

Proof: It follows from (2) that $2\varphi(0)=0$, and so $\varphi(a+b+a)=\varphi(0)+\varphi(a+b)+\varphi(a)=\varphi(0)+\varphi(0)+\varphi(a)+\varphi(b)+\varphi(a)=\varphi(a)+\varphi(b)+\varphi(a)$. ∎

Henceforth $G$ and $H$ will be abelian groups.

Lemma 2.2. *If $\varphi\colon G \to H$ is a semi-homomorphism, then $H_\varphi$ and $H_\varphi+\varphi(0)$ are semi-subgroups of $H$.*

Proof: By Lemma 1.1 and the fact that $2\varphi(0)=0$. ∎

Proposition 2.3. *Let $\varphi\colon G \to H$ be a semi-homomorphism. If $H_\varphi$ contains no elements of order $2$, then $\varphi$ satisfies (2).*

Proof: The result follows immediately since $2H_\varphi=0$. ∎

In order to show that (2) is stronger than (1) in general, we consider

**Example 2.4.** We shift for a brief moment from additive to multiplicative notation (composition of functions) in defining $\varphi\colon S_3 \to S_3 \times S_3$ by $\varphi(\alpha) = ((12)\alpha(12), (12)\alpha^{-1}(12))$ for every $\alpha \in S_3$, the symmetric group of degree $3$. It is a routine check that $\varphi$ is a semi-homomorphism; in fact, if $\pi_i$ denotes the $i$th coordinate projection, $i=1,2$, then $\pi_1\varphi\colon S_3 \to S_3$ is a homomorphism and $\pi_2\varphi\colon S_3 \to S_3$ is an anti-homomorphism. Furthermore, $\varphi(1)=1$, where $1$ denotes the identity of $S_3$, and so it is easy to see that the condition,

$$\varphi(\alpha\beta)=\varphi(1)\varphi(\alpha)\varphi(\beta)$$

for all $\alpha,\beta \in S_3$, is not satisfied.

Theorem 2.5. *A semi-homomorphism $\varphi\colon G \to H$ is a homomorphism if and only if the semi-subgroup $H_\varphi+\varphi(0)$ of $H$ contains no elements of order $2$.*

Proof: The result follows immediately as in Proposition 2.3, since $2(H_\varphi+\varphi(0))=0$. ∎

## 3. Counterexamples to assertions in [3]

Huq calls a mapping $\varphi\colon R \to R'$ between two rings a semi-homomorphism if

$$\varphi\colon (R, +) \to (R', +) \text{ is a semi-homomorphism of groups}$$

and

$$\varphi(aba) = \varphi(a)\varphi(b)\varphi(a) \tag{3}$$

for all $a, b \in R$, that is $\varphi\colon (R, \cdot) \to (R', \cdot)$ is a semi-homomorphism of semigroups. Note that Ancochea [1] calls an additive automorphism $\varphi\colon R \to R$ satisfying

$$\varphi(ab) + \varphi(ba) = \varphi(a)\varphi(b) + \varphi(b)\varphi(a) \tag{4}$$

for all $a, b \in R$, a semi-automorphism of $R$. Kaplansky [4] proved that if $R$ is a simple algebra of characteristic different from 2, then (3) is equivalent to (4), and otherwise stronger. In this paper we stick to Huq's definition of a semi-homomorphism of rings.

The first example in this section is a counterexample to [3, Lemma 4 and Corollary 5].

**Example 3.1.** Let $Z_6$ be the ring of integers modulo 6. Then $\varphi\colon Z_6 \to Z_6$, defined by $\varphi(x) = 3$ for all $x \in Z_6$, is easily seen to be a semi-homomorphism of rings. However, $\operatorname{char} Z_6 = 6 \neq 2$, and by Theorem 2.5 $\varphi$ is not a homomorphism of the underlying additive groups, since $(Z_6)_\varphi + \varphi(0) = \{3\}$ and $2 \cdot 3 = 0$, or equivalently, $\varphi(0) + \varphi(0) = 0 \neq 3 = \varphi(0 + 0)$. Also, $\varphi(-2 \cdot 0) = 3 \neq 0 = -2\varphi(0)$ (see [3, Corollary 5]).

Even if $\varphi\colon R \to R'$ is simultaneously a semi-monomorphism of rings and a homomorphism of the underlying multiplicative semigroups $(R, \cdot)$ and $(R', \cdot)$, and $\operatorname{char} R' \neq 2$, then [3, Lemma 4 and Corollary 5] need not be true, as seen in

**Example 3.2.** Consider the subring $\{0, 2, 4\}$ of $Z_6$, and define $\varphi\colon \{0, 2, 4\} \to Z_6$ by $\varphi(x) = \overline{4x + 3}$ for all $x \in \{0, 2, 4\}$, where $\overline{a}$ denotes the remainder of $a$ after division by 6. Then $\varphi(0) = 3$, and it can be easily verified that $\varphi$ is a semi-monomorphism of rings. In fact $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in \{0, 2, 4\}$, but by Theorem 2.5 $\varphi$ is not a homomorphism of the underlying additive groups.

It should be remarked that [3, Lemma 4 and Corollary 5] are true in case the codomain of the semi-homomorphism is a division ring $D$ (say), since if $\operatorname{char} D \neq 2$, then $D$ contains no elements of order 2.

By Theorem 2.5 correct versions of [3, Lemma 4 and Corollary 5] read as follows:

LEMMA 3.3. *A semi-homomorphism $\varphi\colon R \to R'$ of rings will be a homomorphism of the underlying additive groups if the semi-subgroup $R'_\varphi$ of $(R', +)$ contains no elements of order 2.*

COROLLARY 3.4. *For a semi-homomorphism $\varphi\colon R \to R'$ such that the semi-subgroup $R'_\varphi$ of $(R', +)$ contains no elements of order 2, we have $\varphi(-na) = -n\varphi(a)$ for every integer $n$ and every $a \in R$.*

By Lemma 1.1 and Theorem 2.5 the condition in Corollary 3.4 that $R'_\varphi$ contains no elements of order 2, can be replaced by the condition that the semi-subgroup $\{\varphi(2a) - 2\varphi(a) \mid a \in R\}$ of $R'$, which is contained in $R'_\varphi$, contains no elements of order 2.

The next example is a counterexample to [**3**, Theorem 11]:

**Example 3.5.** Let $\varphi\colon \mathsf{Z}_6 \to \mathsf{Z}_6$ be defined by $\varphi(x) = \overline{4x+3}$ for all $x \in \mathsf{Z}_6$. It is easy to verify that the conditions of [**3**, Theorem 11] are satisfied. In fact, $\varphi\colon (\mathsf{Z}_6, \cdot) \to (\mathsf{Z}_6, \cdot)$ is a homomorphism of semigroups as in Example 3.2. However, by Theorem 2.5 $\varphi$ is not a homomorphism. (The mentioning of an anti-homomorphism in [**3**, Theorem 11] is irrelevant, since $R$ and $R'$ are assumed to be commutative.)

A correct version of [**3**, Theorem 11] reads as follows:

THEOREM 3.6. *For commutative rings $R$ and $R'$ with identities, if $\varphi\colon R \to R'$ is an identity-preserving semi-homomorphism and the semi-subgroup $R'_\varphi$ of $R'$ contains no elements of order 2, then $\varphi$ is a homomorphism.*

We come now to [**3**, Theorem 10]. In order to exhibit a counterexample to this assertion, one needs, as will be shown shortly, a semi-monomorphism of rings with identities which maps 0 into 0, 1 into 1 and, above all, which is a homomorphism of the underlying multiplicative semigroups. (Note that in all the counterexamples so far 0 was not mapped into 0.)

**Example 3.7.** We consider the field $F := \mathsf{Z}_2[x]/(x^3 + x + 1)$ with 8 elements, that is the congruence classes in $\mathsf{Z}_2[x]$ modulo the ideal $(x^3 + x + 1)$. Define $\varphi\colon F \to F \times \mathsf{Z}_3$ by

$$\varphi(\beta) = \begin{cases} (0,0), & \text{if } \beta = 0 \\ (\beta^{-1}, 0), & \text{if } \beta \neq 0. \end{cases}$$

Then $\varphi$ is clearly a semi-homomorphism of the underlying additive groups, since $\operatorname{char} F = 2$. Moreover, setting $[x] =: \alpha$, where $[x]$ denotes the congruence class of $x$, we get $\varphi(1+\alpha) = \alpha^2 + \alpha \neq \alpha^2 = 1 + \alpha^2 + 1 = \varphi(1) + \varphi(\alpha)$, and so $\varphi$ is not a homomorphism of the underlying additive groups. It is eaily verified that $\varphi$ is a homomorphism of the underlying multiplicative semigroups, and so condition (iii) of [**3**, Theorem 10] is satisfied. Furthermore, char $F \times \mathbb{Z}_3 = 6 \neq 2$ and $\varphi(F) = F \times 0$ is a subfield of $F \times \mathbb{Z}_3$ (with identity $(1, 0)$). (It is clear from the "proof" of [**3**, Theorem 10] that Huq terms a division ring a skew field.) Finally, $\varphi$ is $1-1$, and so we have established a counterexample to [**3**, Theorem 10].

We are going to show that a minor, albeit forced, change to condition (i), together with conditions (ii) and (iii), of [**3**, Theorem 10], turn it into a special case of [**3**, Theorem 12]. A few preliminary consequences of conditions (ii) and (iii) are first needed:

LEMMA 3.8. *Let $\varphi\colon R \to R'$ be a semi-monomorphism of rings such that conditions (ii) and (iii) of* [**3**, Theorem 10] *are satisfied. Then $\varphi(0) = 0$.*

PROOF: Suppose that $\varphi(0) \neq 0$. Recall that $2\varphi(0) = 0$, since $\varphi$ is an additive semi-homomorphism. Therefore, $-\varphi(0) = \varphi(0)$, and so by condition (iii), with $y = 0$,

$$\varphi(0) = \varphi(0)[\varphi(0)]^{-1} = 1,$$

where 1 denotes the identity of the skew field $\varphi(R)$. However, $\varphi(a) = 0$ for some $a \in R$, since $0 \in \varphi(R)$, a skew field. But then

$$0 \neq \varphi(0) = \varphi(0a0) = \varphi(0)\varphi(a)\varphi(0) = 0,$$

which completes the proof. ∎

COROLLARY 3.9. *Let $\varphi$ satisfy the conditions of Lemma 3.8. Then $\varphi\colon (R \setminus \{0\}, \cdot) \to (\varphi(R) \setminus \{0\}, \cdot)$ is an isomorphism of groups, and so $R$ is a skew field.*

PROOF: It follows from condition (iii) that $\varphi(yz) = \varphi(yzy)[\varphi(y)]^{-1} = \varphi(y)\varphi(z)\varphi(y)[\varphi(y)]^{-1} = \varphi(y)\varphi(z)$ for all $y, z \in R \setminus \{0\}$, and so $\varphi$ is a homomorphism of semigroups. But $\varphi$ is $1-1$, and so $\varphi$ is an isomorphism, which implies that $(R \setminus \{0\}, \cdot)$ is a group, as $(\varphi(R) \setminus \{0\}, \cdot)$ is a group. Therefore $R$ is a skew field. ∎

It follows from Corollary 3.9 that $\varphi(1) = 1$, where 1 denotes the identities of the skew fields $R$ and $\varphi(R)$, and so we immediately get

PROPOSITION 3.10. *Let $\varphi$ satisfy the conditions of Lemma 3.8. Then $\varphi\colon R \to \varphi(R)$ is an identity-preserving semi-monomorphism of skew fields and $\varphi\colon (R \setminus \{0\}, \cdot) \to (\varphi(R) \setminus \{0\}, \cdot)$ is a homomorphism of groups.*

If we now change condition (i) of [**3**, Theorem 10] to the condition

$$\operatorname{char} \varphi(R) \neq 2,$$

then by Proposition 3.10 the following theorem, which is a correct version of [**3**, Theorem 10], is merely a special case of [**3**, Theorem 12]:

THEOREM 3.11. *A semi-monomorphism $\varphi\colon R \to R'$ of rings will be a monomorphism, if*

(i) $\operatorname{char} \varphi(R) \neq 2$

(ii) *$\varphi(R)$ is a skew subfield of $R'$ and*

(iii) $\varphi(2y + yz) - 2\varphi(y) = \varphi(yzy)[\varphi(y)]^{-1}$.

We conclude with a remark concerning semi-subgroups:
If a semi-subgroup $K$ of a group $A$ is not a subgroup of $A$, then we call $K$ a *non-subgroup* of $A$. Non-subgroups seem to have a very interesting structure, and we hope to give a characterisation of the non-subgroups of finite abelian groups in a forthcoming paper.

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Department of Mathematics
National Cheng Kung University
70102 Tainan, Taiwan
Republic of China

Department of Mathematics
University of Stellenbosch
7600 Stellenbosch
Republic of South Africa