Smooth exponential objects and smooth distributions

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Abstract. First we obtain some Whitney type embeddings and then show that for $J$ an infinite set the product $\mathbb{R}^J$ taken in the category DIFF of differential spaces is same as that taken in the category of Frölicher spaces FRL. Then, we introduce some objects useful for homotopy theory in DIFF. For a differential space $X$, we produce a bifunctor $\otimes_d : \text{DIFF} \times \text{DIFF} \to \text{DIFF}$ with respect to which $X$ is exponential. Differential and Frölicher measures with respect to $\otimes_d$ and $\otimes_f$ lead to corresponding notions of differential and Frölicher distributions.

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1 Introduction

For clarity, we sometimes refer to a map in the category DIFF of differential spaces as d-smooth and a map in the category FRL of Frölicher spaces as $\mathbb{F}$-smooth. Like the category of topological spaces, both DIFF and FRL have initial and final structures. We also label a product $\mathbb{R}^J$ taken in DIFF using $\mathbb{R}^J_d$ and taken in FRL using $\mathbb{R}^J_\mathbb{F}$. A product of two differential spaces $X$ and $Y$ in DIFF will be called a d-product and denoted $X \times_d Y$. Similarly, a subset of a differential space acquires a differential subspace or d-subspace structure. In the same way, we talk about $\mathbb{F}$-products $X \times_\mathbb{F} Y$ and $\mathbb{F}$-subspaces for Frölicher spaces. The topology of a differential (resp., Frölicher) space is the initial topology defined by its structure functions. An open subset of a differential space is referred to as d-open.

Let $\mathcal{C}$ be a concrete category (i.e., a category of sets with structure and structure preserving morphisms). The categories FRL and DIFF are examples of concrete categories. We suppose that the product of two objects $X$ and $Y$ of $\mathcal{C}$, as in FRL and DIFF, has as underlying set the set product of the underlying sets of $X$ and $Y$. Suppose that there are bifunctors

\[ \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \quad \text{and} \quad \mathbb{H} : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \]

where

1. $\otimes$ is covariant in each variable and, for objects $X, Y \in \mathcal{C}$, the underlying set of $X \otimes Y$ is the same as the underlying set of the product $X \times Y$ in $\mathcal{C}$.

2. $\mathbb{H}$ is contravariant in the first variable, covariant in the second variable and the underlying set of $\mathbb{H}(X, Y)$ is $\text{Hom}_{\mathcal{C}}(X, Y)$.

Definition 1.1 An object $X$ in the category $\mathcal{C}$ is exponential for a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ if there is a bifunctor $\mathbb{H} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}'$, both $\otimes$ and $\mathbb{H}$ as described above, and then a natural equivalence

\[ \text{Hom}_{\mathcal{C}}(A \otimes X, Y) \equiv \text{Hom}_{\mathcal{C}}(A, \mathbb{H}^X(Y)), \]
where $H^X(Y) = \mathbb{H}(X,Y)$, in $A,X$ and $Y$ exists. If $\otimes$ is the product on $C$, then $X$ is said to be a Cartesian object. We often write $H^X(Y) = Y^X$.

We first show that a Hausdorff differential space can be embedded in a d-product of $\mathbb{R}$. A similar result for Frölicher spaces is also given. We then show that d-products of copies of $\mathbb{R}$ are the same as $\mathbb{F}$-products. We would like to show that the closed unit interval $I$ is a Cartesian object in DIFF. We can demonstrate the results needed up to a point. However, the argument fails because certain evaluations are not d-smooth. Let $X$ be a differential space. We add maps d-smooth in each variable to the structure functions on d-products $Z \times_d X$ and obtain a bifunctor $\otimes_d : DIFF \times DIFF \to DIFF$ sending $(Z, X)$ to $Z \otimes_d X$ such that, in DIFF, $X$ is exponential for $\otimes_d$. We thus obtain some tools useful for developing homotopy theory in DIFF similar to those in Cherenack [2].

The Whitney topology is compared to the topology on $Y^X_\mathbb{R}$ and $Y^X_{\mathbb{F}}$, correspondingly. Measures related to $\otimes_d$ and $\otimes_{\mathbb{F}}$ are introduced and their implication for exponentiation developed. Finally, we consider the corresponding notions of d-distribution and $\mathbb{F}$-distribution. We see that the differential equation $\frac{dy}{dx} = \delta(x)$, where $\delta(x)$ is the Dirac delta function is solvable in $\mathbb{F}RL$. For basic facts about differential spaces, see [10]. For basic facts about Frölicher spaces see [6] where they are called smooth spaces.

## 2 Infinite products

### 2.1 Whitney type embeddings

First we show:

**Lemma 2.1** Every Hausdorff differential space $(X, \mathcal{F})$ can be embedded in an infinite product in DIFF, each factor being $\mathbb{R}$. More precisely, let $\mathcal{F}_0$ generate $\mathcal{F}$ and separate the points of $X$. Consider $\mathcal{F}_0$ as a set.
Then the inclusion
\[ \iota : X \to \mathbb{R}^{F_0}_d, \]
sending \( x \) to \( (f(x))_{f \in F_0} \) is \( d \)-smooth and identifies \( X \) with a \( d \)-subspace of \( \mathbb{R}^{F_0}_d \).

**Proof:** Clearly \( \iota \) is one-one. Let \( \pi_f : \mathbb{R}^{F_0}_d \to \mathbb{R} \), for \( f \in F_0 \), be the natural projection sending \( (x_f)_{f \in F_0} \) to \( x_f \). The \( \pi_f \), for \( f \in F_0 \), generate the differential structure on \( \mathbb{R}^{F_0}_d \). The subspace structure on \( X \) is generated by \( \{ \pi_f \circ \iota \}_{f \in F_0} = \{ f \}_{f \in F_0} \) and hence is the same as the given structure on \( X \).

One can prove a similar result for Frölicher spaces. So, let \( X \) be a Frölicher space with \( F \) the set of structure functions on \( X \) and \( F_0 \subset F \). Since \( \text{FRL} \) is Cartesian closed, \( F = \text{Hom}_{\text{FRL}}(X, \mathbb{R}) \) has a natural Frölicher space structure and then \( F_0 \) has the induced subspace structure. Again, since \( \text{FRL} \) is Cartesian closed, \( \text{Hom}_{\text{FRL}}(F_0, \mathbb{R}) \) has a natural Frölicher space structure which we denote \( \mathbb{R}^{F_0}_F \).

**Lemma 2.2** Let \( X \) be a FRL space with a generating set \( F_0 \) of structure functions that separates points in \( X \). Then the inclusion
\[ \iota : X \to \mathbb{R}^{F_0}_F, \]
sending \( x \) to \( (f(x))_{f \in F_0} \) is \( F \)-smooth and identifies \( X \) with a \( F \)-subspace of \( \mathbb{R}^{F_0}_F \).

**Proof:** To see that \( \iota : X \to \mathbb{R}^{F_0}_F \) is smooth, let \( c : \mathbb{R} \to X \) be a structure curve. Then
\[ \iota \circ c : \mathbb{R} \to \mathbb{R}^{F_0}_F \]
is \( F \)-smooth if, and only if the map
\[ \iota \circ c : \mathbb{R} \times_F F_0 \to \mathbb{R}, \]
where \( \iota \circ c(t, g) = \iota(c(t))(g) = g(c(t)) \) is \( F \)-smooth. But, since
g\( (c(t)) = ev_{F_0} \circ c \times 1_{F_0} \)
where \( ev_{F_0} \) is the evaluation map and \( 1_{F_0} \) is the identity map, \( \iota \circ c \) is smooth.

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In the other direction, to see that \( \iota \) is an isomorphism onto its image, we need to show that if \( c : \mathbb{R} \to \iota(X) \) is a structure curve, where \( \iota(X) \) is a \( \mathbb{F} \)-subspace of \( \mathbb{R}^\mathbb{F}_0 \), then \( \iota^{-1} \circ c \) is a structure curve on \( X \).

To see this, assume that \( c \) is such a structure curve. Recall that by the Cartesian closedness of \( \text{FRL} \), \( c : \mathbb{R} \to \mathbb{R}^\mathbb{F}_0 \) is \( \mathbb{F} \)-smooth if and only if

\[
\bar{c} : \mathbb{R} \times \mathbb{F}_0 \to \mathbb{R},
\]

given by \( \bar{c}(t, g) = c(t)(g) \) is \( \mathbb{F} \)-smooth. But

\[
c(t)(g) = ((\iota \circ \iota^{-1})(c(t))(g)) = \iota(\iota^{-1}(c(t)))(g) = g(\iota^{-1}(c(t)))
\]
\[
= g \circ \iota^{-1}(c(t)).
\]

But then \( g \circ \iota^{-1} \circ c \) is \( \mathbb{F} \)-smooth for all \( g \in \mathbb{F}_0 \) and hence \( \iota^{-1} \circ c \) is \( \mathbb{F} \)-smooth. \( \Box \)

### 2.2 Infinite products in \( \text{FRL} \) and \( \text{DIFF} \) compared

We let \( \mathcal{B} \) be the basis for the topology on \( \mathbb{R}^J_d \) consisting of open sets of the form \( U = \times_{j \in J} U_j \) where \( U_j = (a_j - \epsilon_j, a_j + \epsilon_j) \) where \( (a_j, \epsilon_j) \in \mathbb{R} \) and \( \epsilon_j > 0 \) for finitely many \( j \) or \( \epsilon_j = \infty \) (i.e., \( U_j = \mathbb{R} \)) for \( j \in J \).

**Definition 2.1** A function \( f : \mathbb{R}^J \to \mathbb{R} \) depends locally at \( P \in \mathbb{R}^J \) on finitely many variables if there is an open subset \( U \in \mathcal{B} \) of \( \mathbb{R}^J_d \) containing \( P \) and a finite subset \( \{j_1, \ldots, j_m\} \subset J \) such that

\[
f|_U((x_j)_{j \in J}) = g(x_{j_1}, \ldots, x_{j_m})
\]

for some smooth function

\[
g : \{(x_{j_1}, \ldots, x_{j_m})|(x_j)_{j \in J} \in U\} \to \mathbb{R}.
\]

A map \( g : \mathbb{R}^J \to \mathbb{R} \) is said to be \( d \)-smooth (resp., \( \mathbb{F} \)-smooth) if it is a structure function on \( \mathbb{R}^J_d \) (resp., \( \mathbb{R}^J_\mathbb{F} \)).
Lemma 2.3 Every $\mathbb{F}$-smooth function $g : \mathbb{R}^J \to \mathbb{R}$ is $d$-smooth if and only if, for each $P \in \mathbb{R}^J$, it locally, for the topology on $\mathbb{R}^J_d$, depends on finitely many variables at $P$.

Proof: Suppose that $g : \mathbb{R}^J \to \mathbb{R}$ is $\mathbb{F}$-smooth and locally depends on finitely many variables at $P \in \mathbb{R}^J$. Let $P = (a_j)_{j \in J}$. Then, for some $U \in \mathcal{B}$ containing $P$ and finite subset $J_1 = \{j_1, \ldots, j_m\} \subset J$,

$$g|_U((x_j)_{j \in J}) = h(x_{j_1}, \ldots, x_{j_m})$$

for some smooth function

$$h : \{(x_{j_1}, \ldots, x_{j_m})|(x_j)_{j \in J} \in U\} \to \mathbb{R}.$$

By extending the variable set of $h$, one can assume that $U_j = \mathbb{R}$ if $j \not\in J_1$. Suppose that, for $j \in J_1$ and $U \in \mathcal{B}$, $\epsilon_j > 0$ is chosen so $(a_j - \epsilon_j, a_j - \epsilon_j) \subset U_j$. Let $\alpha_j : \mathbb{R} \to \mathbb{R}$ ($j \in J_1$) be a smooth function such that

1. $\alpha_j(t) = t$ for $t \in (a_j - \frac{\epsilon_j}{2}, a_j + \frac{\epsilon_j}{2})$.
2. $\alpha_j(t) = a_j - \frac{3\epsilon_j}{4}$ for $t < a_j - \frac{3\epsilon_j}{4}$.
3. $\alpha_j(t) = a_j + \frac{3\epsilon_j}{4}$ for $t > a_j + \frac{3\epsilon_j}{4}$.
4. $\alpha_j(t) \in (a_j - \epsilon_j, a_j + \epsilon_j)$.

Define $k : \mathbb{R}^J \to \mathbb{R}$ by setting

$$k((x_j)_{j \in J}) = h(\alpha_{j_1}(x_{j_1}), \ldots, \alpha_{j_m}(x_{j_m})).$$

Then, one readily sees that $k$ is a $d$-smooth map and $g|_{U_0} = k|_{U_0}$ where $U_0 = \times_{j \in J} U_j^0$, $U_j^0 = (a_j - \frac{\epsilon_j}{2}, a_j + \frac{\epsilon_j}{2})$ if $j \in J_1$ and $U_j^0 = U_j$ if $j \not\in J_1$. Since $P \in U_0^0 \in \mathcal{B}$, $g$ is $d$-smooth.

Suppose conversely that $g : \mathbb{R}^J \to \mathbb{R}$ is $d$-smooth. Then, from the defining properties of such $d$-smooth maps, locally and in particular on some open set $U \in \mathcal{B}$,

$$g|_U((x_j)_{j \in J}) = h(\pi_{j_1}((x_j)_{j \in J}), \ldots, \pi_{j_m}((x_j)_{j \in J})) = h(x_{j_1}, \ldots, x_{j_m})$$
for some finite subset \( \{ j_1, \cdots, j_m \} \subset J \) and smooth function \( h : \mathbb{R}^m \to \mathbb{R} \).

As a precursor to the next result, we provide an example. Let \( U_n = (n-1, n)^n \times \mathbb{R}^{n-(1, \cdots, n)} \). For each \( n \in \mathbb{N} \) there is a smooth map \( \alpha_n : (n-1, n) \to \mathbb{R} \) such that \( \alpha_n(t) = 0 \) if \( t \leq n - \frac{3}{4} \) or \( t \geq n - \frac{1}{4} \) and is not constant between \( n - \frac{3}{4} \) and \( n - \frac{1}{4} \). Define a \( d \)-smooth map \( \beta_n : U_n \to \mathbb{R} \) by setting \( \beta_n((x_m)_{m \in \mathbb{N}}) = \prod_{j=1}^n \alpha_n^j(x_j) \) where \( \alpha_n^j = \alpha_n \) for all \( n \) and \( j = 1, \cdots, n \). Since the \( U_n \) are piecewise disjoint, one can extend the \( \beta_n \) to a map \( \beta : \mathbb{R}^N \to \mathbb{R} \) by setting \( \beta(x) = \beta_n(x) \) if \( x \in U_n \) for some \( n \in \mathbb{N} \) and \( \beta(x) = 0 \) otherwise.

Suppose that \( c : \mathbb{R} \to \mathbb{R}^N \) is a \( F \)-smooth map. If \( c(a) \in U_n \), for some \( n \in \mathbb{N} \), there is an \( \epsilon > 0 \) such that \( c((a-\epsilon, a+\epsilon)) \subset U_n \). It follows that

\[
\beta \circ c\mid_{(a-\epsilon, a+\epsilon)}(t) = \prod_{j=1}^n \alpha_n^j(c_j(t))
\]

is smooth at \( t = a \). If \( c(a) \notin U_n \) for any \( n \in \mathbb{N} \), there is an \( \epsilon > 0 \) such that \( \beta \circ c((a-\epsilon, a+\epsilon)) = \{0\} \) and \( \beta \circ c \) is again smooth at \( a \). Hence, \( \beta \circ c \) is smooth and thus \( \beta : \mathbb{R}^N \to \mathbb{R} \) is a \( F \)-smooth map.

On the other hand, one cannot write

\[
\beta((x_j)_{j \in \mathbb{N}}) = g(x_{j_1}, \cdots, x_{j_m})
\]

for some smooth \( g : \{(x_{j_1}, \cdots, x_{j_m}) | (x_j)_{j \in \mathbb{N}} \in \mathbb{R}^N \} \to \mathbb{R} \) since clearly \( \beta \) still depends on the coordinate \( x_i \) where \( i > \max\{j_1, \cdots, j_m\} \). Thus, \( \beta \) globally depends on infinitely many coordinates.

One can now show:

**Theorem 2.1** The countable product \( \mathbb{R}^N \) has, as a differential space, the same structure functions that it has as a Frölicher space.

**Proof:** Suppose that \( f : \mathbb{R}^N \to \mathbb{R} \) is \( F \)-smooth but does not depend on finitely many variables near a point \( P \). We may suppose that
$P = O$ is the zero vector in $\mathbb{R}^N$. Because $f$ depends on infinitely many variables, it is not difficult to show that, for some $n_1 \in \mathbb{N}$ and $i_1 \in \mathbb{N} \cup \{0\}$, $f$ depends on $x_{n_1}$ on the subset

$$W_1 = \mathbb{R}^{n_1-1} \times J_{n_1} \times \mathbb{R}^{N-\{1,2,\cdots,n_1\}}$$

of $\mathbb{R}^N$ where $J_{n_1} = (\frac{1}{i_1+1}, \frac{1}{i_1})$ or $(-\frac{1}{i_1}, -\frac{1}{i_1+1})$ and $\frac{1}{\theta}$ is interpreted as some sufficiently large number. By sending $x_{n_1}$ to $-x_{n_1}$, one can assume that $J_{n_1} = (\frac{1}{i_1+1}, \frac{1}{i_1})$.

Let now

$$U_1 = I_{n_1} \times \mathbb{R}^{N-\{1,2,\cdots,n_1\}}$$

where $I_1$ is an open interval containing $0$ and $I_1 \subset (-\infty, \frac{1}{i_1+1})$. The first $n_1$ coordinates of $U_1$ can be made arbitrarily small by choosing $I_1$ arbitrarily small. Furthermore, $f$ restricted to $U_1$ is $\mathbb{F}$-smooth and depends on infinitely many variables as $O \in U_1$ and $U_1$ is $d$-open in $\mathbb{R}^N$. As above, for some $n_2 \in \mathbb{N}$, $n_2 > n_1$, there is an $i_2 \in \mathbb{N} \cup \{0\}$ such that $f$ depends on $x_{n_2}$ on

$$W_2 = I_{n_1} \times \mathbb{R}^{n_2-n_1-1} \times J_{n_2} \times \mathbb{R}^{N-\{1,2,\cdots,n_2\}}$$

where $J_{n_2} = (\frac{1}{i_2+1}, \frac{1}{i_2})$. The choice of $I_1$ implies that $W_1 \cap W_2 = \emptyset$.

One supposes, for $j = 1, 2, \cdots, \infty$ and for some $n_j \in \mathbb{N}$ with $n_j > n_{j-1}$, that

$$U_j = I_{n_j} \times \mathbb{R}^{N-\{1,2,\cdots,n_j\}}$$

where the first $n_j$ coordinates of $U_j$ can be made arbitrarily small on choosing the open interval $I_j$ as an arbitrarily small open interval containing $0$. We suppose that $n_j$ and $i_j$, for $j = 1, 2, \cdots, k-1$, are fixed. In addition, $f$ restricted to $U_j$ is $\mathbb{F}$-smooth and depends on infinitely many variables as $O \in U_j$ and $U_j$ is $d$-open in $\mathbb{R}^N$. Also, letting

$$W_{j+1} = I_{n_j} \times \mathbb{R}^{n_{j+1}-n_j-1} \times J_{n_{j+1}} \times \mathbb{R}^{N-\{1,2,\cdots,n_{j+1}\}}$$

where $J_{n_{j+1}} = (\frac{1}{i_{j+1}+1}, \frac{1}{i_{j+1}})$ and $i_{j+1} \in \mathbb{N} \cup \{0\}$ for $j = 1, 2, \cdots, k-1$, one makes the sets $W_1, W_2, \cdots, W_k$ pairwise disjoint on choosing
Let \( P = (\text{certain rate}) \), then, for fixed \( c \), that \( a \) for each \( j = 1, 2, \ldots, k - 1 \), \( f \) restricted to \( W_{j+1} \) depends on \( x_{n_{j+1}} \).

We extend from \( k \) to \( k + 1 \). Since for some \( n_{k+1} \in \mathbb{N}, n_{k+1} > n_k \), there is an \( i_{k+1} \in \mathbb{N} \cup \{0\} \) such that \( f \) depends on \( x_{n_{k+1}} \) on

\[
A = \mathbb{I}_{k}^{n_k} \times \mathbb{R}^{n_{k+1}-n_k-1} \times J_{n_{k+1}} \times \mathbb{R}^{\mathbb{N}-(1,2,\ldots,n_{k+1})}
\]

where \( J_{n_{k+1}} = (\frac{1}{i_{k+1}+1}, \frac{1}{i_{k+1}+1}) \). Define, inductively, \( W_{k+1} = A \).

Inductively, the interval \( I_{n_k} \) is chosen so that \( I_{n_k} \) is a sufficiently small open interval containing 0 and \( I_{n_k} \subset (-\infty, \frac{1}{i_{m+1}}) \) for \( m \leq k \). Such a choice insures that the \( W_1, W_2, \ldots, W_{k+1} \) are pairwise disjoint.

Let \( P_k = (P_j^k)_{j \in \mathbb{N}} \in W_k \). If the lengths of the \( I_k \) converge to 0 at a certain rate, then, for fixed \( j \), the \( P_k^j \) converge to 0 at the same rate as \( k \to \infty \). One can assume that \( P_k \) is chosen so that \( \mu_k = \frac{\partial f}{\partial x_n_k}(P_k) \neq 0 \). Otherwise, \( f \) being \( \mathbb{F} \)-smooth on \( W_k \) doesn’t depend on \( x_{n_k} \) on \( W_k \).

Choose \( \psi_k \) so that \( \psi_k \mu_k > k \) for each \( k \in \mathbb{N} \). Let \((a_k)_{k \in \mathbb{N}} \) be a strictly decreasing sequence of positive real numbers. For \( a_{k+1} \leq t \leq a_k \) and \( j \neq n_k \), let \( c_j(t) \) be a piecewise linear continuous real valued function such that

\[
c_j(t) = \begin{cases} 
t \quad & a_{k+1} \leq t \leq \frac{1}{5}a_k + \frac{4}{5}a_{k+1} \\
t \quad & \frac{4}{5}a_k + \frac{1}{2}a_{k+1} \leq t \leq a_k \\
P_k^j \quad & \frac{2}{5}a_k + \frac{3}{5}a_{k+1} \leq t \leq \frac{2}{5}a_k + \frac{3}{5}a_{k+1} \end{cases}
\]

Furthermore, suppose that the vertices of the graph of such a \( c_j(t) \) on \( a_{k+1} \leq t \leq a_k \) are

\[
(\frac{1}{5}a_k + \frac{4}{5}a_{k+1}, \frac{1}{5}a_k + \frac{1}{2}a_{k+1}), (\frac{2}{5}a_k + \frac{3}{5}a_{k+1}, P_k^j), (\frac{3}{5}a_k + \frac{3}{5}a_{k+1}, P_k^j), (\frac{4}{5}a_k + \frac{3}{5}a_{k+1}, \frac{4}{5}a_k + \frac{1}{2}a_{k+1}).
\]

For \( a_{k+1} \leq t \leq a_k \), let \( c_{n_k}(t) \) be a smooth real valued function such that \( c_{n_k}(t) = \)

\[
\begin{cases} 
t \quad & a_{k+1} \leq t \leq \frac{1}{5}a_k + \frac{4}{5}a_{k+1} \\
t \quad & \frac{4}{5}a_k + \frac{1}{2}a_{k+1} \leq t \leq a_k \\
P_k^j + \psi_k(t - \frac{a_k+a_{k+1}}{2}) \quad & \frac{2}{5}a_k + \frac{3}{5}a_{k+1} \leq t \leq \frac{3}{5}a_k + \frac{2}{5}a_k \end{cases}
\]

Note that \( |t - \frac{a_k+a_{k+1}}{2}| \leq \frac{1}{5}(a_k - a_{k+1}) \) and that to guarantee \( c_{n_k}(t) \in J_{n_k} \) for \( a_{k+1} \leq t \leq a_k \), the number \( \psi_k(t - \frac{a_k+a_{k+1}}{2}) \) must be
sufficiently small. To this end, we place two numbers $a^*_{k+1}$ and $a^*_k$ between $a_{k+1}$ and $a_k$ so that

1. $a^*_{k+1} \leq a^*_k$.

2. $P^k_{n_k}$ is the middle point of $(a^*_k, a^*_{k+1})$.

3. $\psi_k(t - \frac{a^*_k + a^*_{k+1}}{2})$ is sufficiently small.

We redefine $(c_j(t))_{j \in \mathbb{N}}$ on the interval $(a^*_k, a^*_{k+1})$ as it was defined on $(a_{k+1}, a_k)$ above. Then, for $a^*_k$ and $a^*_{k+1}$ chosen sufficiently close and on choosing the sequence $(a_k)_{k \in \mathbb{N}}$ initially to have sufficiently small terms, one has $(c_j(t))_{j \in \mathbb{N}} \in W_k$ for $\frac{2}{5}a^*_k + \frac{2}{5}a^*_{k+1} \leq t \leq \frac{3}{5}a^*_k + \frac{2}{5}a^*_{k+1}$. Note that $W_k$ is a product of open intervals. On the set $[a_{k+1}, a^*_{k+1}] \cup (a^*_k, a_k]$, one lets $c_j(t) = t$ for all $j \in \mathbb{N}$.

To apply the Special Curve Lemma in [8], one needs to know that the $c_j(t)$ define polygonal curves for $t$ large, as we have seen above, and that, if $(b_i, c_j(b_i))$ are the successive vertices of this polygonal curve, then $c_j(b_i) \to 0$ at a sufficiently quick rate. The last fact can be obtained by

1. choosing the lengths of the $I_k$ and hence the size of the $P^k_j$ to converge to 0 as $k \to \infty$ at a sufficiently quick rate.

2. choosing the size of the $a_k (k \in \mathbb{N})$ to converge to 0 as $k \to \infty$ at a sufficiently quick rate.

Thus, applying Special Curve Lemma and changing $c_j(t)$ for $t$ sufficiently large, one can assume, for $j \in \mathbb{N}$, that

**SCLA:** $c_j(t) = P^k_j$ for $\frac{2}{5}a^*_k + \frac{3}{5}a^*_{k+1} \leq t \leq \frac{3}{5}a^*_k + \frac{2}{5}a^*_{k+1}$ and $j \neq n_k$.

**SCLB:** $c_j(t)$ is smooth for all $t \in \mathbb{R}$ (it may be necessary to smooth $c_j(t)$ at finitely many points to which the Special Curve Lemma is not applied).
SCLC: for \( a_k^* + 1 \leq t \leq a_k^* \), \( c_{nk}(t) \) is as defined in (!!) above.

Set \( g(t) = f(c(t)) \). For \( k \in \mathbb{N} \) and using SCLA-SCLC,
\[
g\left( \frac{a_k^* + a_{k+1}^*}{2} \right) = \frac{\partial f}{\partial x_{nk}}(P_k)c'_n \left( \frac{a_k^* + a_{k+1}^*}{2} \right) = \psi_k \mu_k \text{ by the chain rule.} \]
It follows that \( \lim_{k \to \infty} g\left( \frac{a_k^* + a_{k+1}^*}{2} \right) = \infty \). But \( \frac{a_k^* + a_{k+1}^*}{2} \to 0 \) as \( k \to \infty \). As \( g(t) \) is smooth, \( g'(t) \) is continuous. But then
\[
g'(0) = \lim_{t \to 0} g'(t) = \lim_{n \to \infty} g\left( \frac{a_k^* + a_{k+1}^*}{2} \right),
\]
a contradiction.

\[\square\]

Theorem 2.1 clearly extends to the following result.

**Corollary 2.1** Let \( J \) be an infinite set. The countable product \( \mathbb{R}^J \) has,
as a differential space, the same structure functions that it has as a Frölicher space.

We let \( I_d \) (resp., \( I_\bar{\mathbb{R}} \)) denote the closed unit interval viewed as a differential (resp., Frölicher) subspace of \( \mathbb{R} \). Note that \( I_\bar{\mathbb{R}} \) without its structure curves is the same as \( I_d \) (see [4]). Hence, as differential spaces, \( I_d \) and \( I_\bar{\mathbb{R}} \) are isomorphic. For this reason, we will usually write \( I \) instead of \( I_d \) or \( I_\bar{\mathbb{R}} \).

One proves in the same way as one proves Theorem 2.1:

**Corollary 2.2** The differential space \( \mathbb{R}^d_x \times_\bar{\mathbb{R}} I \) is isomorphic to the \( \mathbb{R}^d_x \times_d I \) in \( \text{DIFF} \).

### 3 Exponential objects in \( \text{DIFF} \) and \( \text{FRL} \)

On the set level there is a natural bijection
\[
v_{XYZ} : \text{Hom}_{\text{SETS}}(Z \times X, Y) \to \text{Hom}_{\text{SETS}}(Z, Y^X)
\]
where \( Y^{X} = \text{Hom}_{\text{SETS}}(X, Y) \) and \( v_{XYZ}(f)(z)(x) = f(z, x) \). We write \( \hat{f} = v_{XYZ}(f) \). If \( \gamma_{XYZ} \) is the inverse of \( v_{XYZ} \), we write \( \bar{g} = \gamma_{XYZ}(g) \).

There is a set embedding \((I\ as\ defined\ before\ Corollary\ 2.2)\)

\[ \iota : \text{Hom}_{\text{FRL}}(I, \mathbb{R}) \to \mathbb{R}^I_f \]
sending \( f \) to \((f(t))_{t \in I}\). One gives \( \text{Hom}_{\text{FRL}}(I, \mathbb{R}) \) the Frölicher subspace structure and denotes the result by \( \mathbb{R}^I_{f} \). The structure on \( \mathbb{R}^I_{f} \) is thus generated by the set of \( \text{ev}_s(s \in I) \) where \( \text{ev}_s(f) = \pi_s((f(t))_{t \in I}) = f(s) \). With this structure, using the Uniform Boundedness Principle in Frölicher and Kriegl [6], one sees that \( I \) is a Cartesian object in \( \text{FRL} \) for which \( \mathbb{R}^I_{f} \) possesses the required structure on \( \text{Hom}_{\text{FRL}}(I, \mathbb{R}) \). One can extend the Uniform Boundedness Principle referred to above in the following way:

**Theorem 3.1** Let \( X \) and \( Y \) be Frölicher spaces. Suppose that \( F_Y \), the set of structure functions on \( Y \), separates points. The structure on \( Y^X_f \) is induced by the evaluation maps \( \text{ev}_x : \text{Hom}_{\text{FRL}}(X, Y) \to Y(x \in X) \) where \( \text{ev}_x(F) = f(x) \).

**Proof:** If \( Y = \mathbb{R} \), the theorem follows from [6], as just mentioned. Since \((\mathbb{R}^X_f)^I_f = (\mathbb{R}^X_f)_f^I \) and the projection maps \( \pi_j : (\mathbb{R}^X_f)_f^I \to \mathbb{R}^X_f \) generate the structure of \((\mathbb{R}^X_f)_f^I \), the result is clear if \( Y = (\mathbb{R}^X_f)_f^I \).

Applying Lemma 2.2, there is an embedding \( \iota : Y \to \mathbb{R}^F_Y \). The inclusion \( j : \mathbb{R}^F_Y \to \mathbb{R}^F_Y \) sending \( F \in \mathbb{R}^F_Y \) to \((F(g))_{g \in F_Y} \) identifies \( \mathbb{R}^F_Y \) with a subspace of \( \mathbb{R}^F_Y \) since the projection maps on \( \mathbb{R}^F_Y \) restrict to evaluation maps on \( \mathbb{R}^F_Y \) which generate the structure of \( \mathbb{R}^F_Y \). Thus \( j \circ \iota : Y \to \mathbb{R}^F_Y \) identifies \( Y \) with a \( F \)-subspace of \( \mathbb{R}^F_Y \).

Now \( c : \mathbb{R} \to \mathbb{Y}^X_f \) is \( F \)-smooth if and only if \( \hat{c} : \mathbb{R} \times F X \to Y \), defined by setting \( \hat{c}(t, x) = c(t)(x) \) is \( F \)-smooth and in turn next if and only if \( \hat{c} : \mathbb{R} \times F x \to \mathbb{R}^F_Y \) is \( F \)-smooth and \( \hat{c}(\mathbb{R} \times X) \subset Y \) and then finally if and only if \( c : \mathbb{R} \to \mathbb{R}^F_Y \) is \( F \)-smooth and \( c(\mathbb{R}) \subset \text{Hom}_{\text{FRL}}(X, Y) \). Since the evaluation maps define the \( F \)-structure on \((\mathbb{R}^F_Y)^X_f \), they clearly also define the structure on \( Y^X_f \). \( \square \)
Motivated by the above results, we let $R_d^I$ denote the differential space whose underlying set is $\text{Hom}_{\text{DIFF}}(I, \mathbb{R}) = \text{Hom}_{\text{FRL}}(I, \mathbb{R})$ and whose structure is generated in $\text{DIFF}$ this time again by the set of $ev_s(s \in I)$.

Corollary 2.1 shows that, as differential spaces, $\mathbb{R}_/x46^I$ is the same as $R_d^I$; hence one might expect $R_d^I = \mathbb{R}_/x46^I$.

We work up to the fact that $I = I_d$ satisfies up to a point the requirements for it to be a Cartesian object in $\text{DIFF}$. One shows first:

**Lemma 3.1** The map $\gamma_{I_\mathbb{R}} : \text{Hom}_{\text{DIFF}}(\mathbb{R}, R_d^I) \rightarrow \text{Hom}_{\text{DIFF}}(\mathbb{R} \times_d I, \mathbb{R})$ sending $f$ to $\bar{f}$ exists and is a bijection.

**Proof:** Since $\mathbb{R}_/x46^I$ and $\mathbb{R}_/x46^I$ have the same set of generating functions, the identity map $\text{Hom}_{\text{DIFF}}(\mathbb{R}, \mathbb{R}_/x46^I) \rightarrow \text{Hom}_{\text{FRL}}(\mathbb{R}, \mathbb{R}_/x46^I)$ sending $f$ to $f$ exists and is a bijection.

Since $\text{FRL}$ is Cartesian closed, there is a bijection

$$\text{Hom}_{\text{FRL}}(\mathbb{R}, \mathbb{R}_/x46^I) \rightarrow \text{Hom}_{\text{FRL}}(\mathbb{R} \times_{\mathbb{F}} I, \mathbb{R}) = \text{Hom}_{\text{DIFF}}(\mathbb{R} \times_d I, \mathbb{R})$$

sending $f$ to $\bar{f}$, with the equality a consequence of Corollary 2.2. □

Let now $J$ be an infinite set and $f \in \text{Hom}_{\text{DIFF}}(\mathbb{R}^J, R_d^I)$. Then, $ev_t \circ f$ is smooth on $\mathbb{R}_/x46^I$ and then $ev_t \circ f|_U = g_t|_U$ where $g_t : \mathbb{R}^J \rightarrow \mathbb{R}$ is a $d$-smooth map such that $g_t((x_j)_{j \in J}) = h_t(x_{i_1}, \cdots, x_{i_m})(i_1, \cdots, i_m \in J)$ with $h_t : \mathbb{R}^m \rightarrow \mathbb{R}$ smooth.

**Lemma 3.2** Using our above notation, the map $\bar{f} : \mathbb{R}_/x46^J \times_d I \rightarrow \mathbb{R}$ is smooth.

**Proof:** Let $f \in \text{Hom}_{\text{DIFF}}(\mathbb{R}_/x46^J, R_d^I)$. Then, for each $t \in I$,

$ev_t \circ f \in \text{Hom}_{\text{DIFF}}(\mathbb{R}_/x46^J, \mathbb{R})$ and thus $ev_t \circ f : \mathbb{R}_/x46^J \rightarrow \mathbb{R}$ is a $F$-smooth map. Hence, $\bar{f} : \mathbb{R}_/x46^J \rightarrow \mathbb{R}_/x46^I$ is $F$-smooth and there is then a $F$-smooth map $\bar{f} : \mathbb{R}_/x46^J \times_{\mathbb{F}} I \rightarrow \mathbb{R}$ arising from the Cartesian closedness of $\text{FRL}$. But, by Corollary 2.2,

$$\mathbb{R}_/x46^J \times_{\mathbb{F}} I \simeq \mathbb{R}_/x46^J \times_d I$$

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in \( \text{DIFF} \). Hence, \( \bar{f} : \mathbb{R}^d \times_d I \to \mathbb{R} \) is \( d \)-smooth. \( \square \)

In the other direction, let \( g : \mathbb{R}^d \times_d I \to \mathbb{R} \) be a \( d \)-smooth map

**Lemma 3.3** Using our above notation, the map \( \hat{g} : \mathbb{R}^d \to \mathbb{R}^d \) is \( d \)-smooth.

**Proof:** The map \( \hat{g} \) is \( d \)-smooth if and only if, for each \( s \in I \), \( ev_s \circ \hat{g} \) is \( d \)-smooth if and only if, for each \( s \in I \), \( ev_s \circ \hat{g}((x_j)_{j \in J}) = g((x_j)_{j \in J}, s) \) is \( d \)-smooth, which follows immediately. \( \square \)

Putting the last two lemmas together one obtains:

**Proposition 3.1** The map \( \gamma_{I, \mathbb{R}^d} : \text{Hom}_{\text{DIFF}}(\mathbb{R}^d, \mathbb{R}^d) \to \text{Hom}_{\text{DIFF}}(\mathbb{R}^d \times_d I, \mathbb{R}) \) sending \( f \) to \( \bar{f} \) exists and is a bijection.

We would have liked to show that \( I \) is a Cartesian object in \( \text{DIFF} \) using the structure on \( \mathbb{R}^d \). For this to work the image of the identity map \( 1 : \mathbb{R}^d \to \mathbb{R}^d \) under \( \gamma_{I, \mathbb{R}^d} \) must be \( d \)-smooth. However, \( \gamma_{I, \mathbb{R}^d}(1) = ev : \mathbb{R}^d \times_d I \to \mathbb{R} \) is the evaluation map. For \( ev \) to be \( d \)-smooth, for \( Q = (R_t)_{t \in I} \times_d \{s\} \in \mathbb{R}^d \times_d I \), there must exist an open neighborhood \( V \times T \) of \( Q \) for the differential space structure on \( \mathbb{R}^d \times_d I \) such that \( V \in \mathcal{B} \) as above, \( T \) is an open subset of \( I \) containing \( s \) and \( ev|_{V \times T}((R_t)_{t \in I}, v) = h((R_t)_{t \in I}, v) \) where \( I_1 \) is a finite subset of \( I \) and \( h : \times_{t \in I_1} V_t \times T \to \mathbb{R} \) is smooth. But this means that every smooth map \( d : I \to \mathbb{R} \) such that \( d(t_0) \in V_{t_0} \) for \( t_0 \in I_1 \) has its values at \( t \in T \) determined by its values at those \( t_0 \) where \( t_0 \in I_1 \). But, clearly this is not the case and \( \mathbb{R}^d \) does thus not have the right structure to make \( I \) into a Cartesian object.

One can define \( \mathbb{R}^d_X \) for a general differential space \( X \) as one defined \( \mathbb{R}^d \) above. Thus, the structure on \( \mathbb{R}^d_X \) is generated by the maps \( ev_x : \text{Hom}_{\text{DIFF}}(X, R) \to \mathbb{R} \) where \( ev_x(f) = f(x) \) and \( x \in X \). For a map \( h : W \times X \to U \), \( x \in X \) and \( w \in W \), we let \( h_x : W \to Y \) be defined by setting \( h_x(w) = h(w, x) \) and we let \( h_w : X \to Y \) be defined by setting
$h_w(x) = h(w, x)$. Let now, for a differential space $W$,

$\mathcal{M}_{W \times X} = \{ h : W \times_d X \to \mathbb{R} : h_x \text{ and } h_w \text{ are } d\text{-smooth for } x \in X, w \in W \}$.

We let $(W \otimes_d X, F_{W \otimes_d X})$ be the differential space generated by $\mathcal{M}_{W \times X}$.

Let $h : W \to U$ be a $d$-smooth map. As $h^* F_U \subset F_W$ and $(h \times 1_X)^* \mathcal{M}_{U \times X} \subset \mathcal{M}_{W \times X}$, there is an induced map $h \otimes_d 1_X : W \otimes_d X \to U \otimes_d X$. It is clear then that the assignment

$$h \mapsto h \otimes_d 1_X$$

defines a functor $- \otimes_d X : \text{DIFF} \to \text{DIFF}$. Similarly, the assignment $k \mapsto 1_W \otimes_d k$ defines a functor $W \otimes_d - : \text{DIFF} \to \text{DIFF}$ and one in this way obtains a bifunctor $\otimes_d : \text{DIFF} \times \text{DIFF} \to \text{DIFF}$. The construction of $W \otimes_d X$ is symmetric because one can compose a map $h : W \times_d X \to \mathbb{R}$ as it appears in the definition of $\mathcal{M}_{W \times X}$ with a $d$-smooth map which interchanges coordinates. Hence, $W \otimes_d X$ is $d$-isomorphic to $X \otimes_d W$.

We now show:

**Proposition 3.2** The map

$$\gamma_{X \otimes W} : \text{Hom}_{\text{DIFF}}(W, \mathbb{R}^X_d) \to \text{Hom}_{\text{DIFF}}(W \otimes_d X, \mathbb{R})$$

sending $f$ to $\bar{f}$ is a bijection.

**Proof:** Let $f \in \text{Hom}_{\text{DIFF}}(W, \mathbb{R}^X_d)$. Then, $\gamma_{X \otimes W}(f) = ev_X \circ (f \otimes_d 1_X)$ is $d$-smooth since $- \otimes_d X$ is a functor on $\text{DIFF}$ and $ev_X : \mathbb{R}^X_d \otimes_d X \to \mathbb{R}$ is $d$-smooth since

- $(ev_X)_x = ev_x$ is $d$-smooth
- $(ev_X)_f = f$ and $f$ is $d$-smooth

and thus by definition $ev_X$ is a structure function on $\mathbb{R}^X_d \otimes_d X$. 

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Conversely, let \( h \in \text{Hom}_{\text{DIFF}}(W \otimes_d X, \mathbb{R}) \). One may suppose that \( h \in \mathcal{M}_{W \times X} \).

Then, the map \( h_w : \{w\} \times X \to \mathbb{R} \) is \( d \)-smooth. It follows that the assignment sending \( w \) to \( \hat{h}(w) \) maps \( W \) to \( \mathbb{R}^X_d \). However, for \( x \in X \), \( e_x \circ \hat{h}(w) = h(w, x) \) and because \( h(w, x) \) is a smooth function of \( w \), \( \hat{h} \) is a \( d \)-smooth function \( W \to \mathbb{R}^X_d \).

Let now \( (\mathbb{R}^J_d)^X = (\mathbb{R}^d)^J \). Then,
\[
\text{Hom}_{\text{DIFF}}(W, (\mathbb{R}^J_d)^X) = \text{Hom}_{\text{DIFF}}(W, (\mathbb{R}^d)^J) = \text{Hom}_{\text{DIFF}}(W, \mathbb{R}^X_d)^J
\]
\[
= \text{Hom}_{\text{DIFF}}(W \otimes_d X, \mathbb{R})^J = \text{Hom}_{\text{DIFF}}(W \otimes_d X, \mathbb{R}^d).\]

Let \( (Y, \mathcal{F}_Y) \) be a differential space, \( F = \mathcal{F}_Y \) and \( \iota : Y \to \mathbb{R}^d_F \) be the map sending \( y \) to \( (f(y))_{f \in F} \). Consider the map \( \iota^X = \text{Hom}_{\text{DIFF}}(X, \iota) : \text{Hom}_{\text{DIFF}}(X, Y) \to (\mathbb{R}^d)^X_d \) where \( J = F \). Via this map, \( \text{Hom}_{\text{DIFF}}(X, Y) \) acquires an induced \( d \)-subspace structure denoted \( Y^X_d \).

**Lemma 3.4** The assignment \((X, Y) \mapsto Y^X_d\) induces a bifunctor \( \text{DIFF} \times \text{DIFF} \to \text{DIFF} \).

**Proof:** Let \( Z \) be a differential space with \( G = \mathcal{F}_Z \) the set of structure functions on \( Z \). Let \( h : Y \to Z \) be \( d \)-smooth. One has a diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{h} & Z \\
\downarrow \iota_Y & & \downarrow \iota_Z \\
\mathbb{R}^F_d & \xrightarrow{H} & \mathbb{R}^G_d
\end{array}
\]

where \( \iota_Y(y) = (f(y))_{f \in F} \), \( \iota_Z(z) = (g(y))_{g \in G} \) and
\( H((x_f)_{f \in F}) = (x_{h^*g})_{g \in G} \). Since \( \pi_g \circ H = \pi_{h^*g} \), \( H \) is \( d \)-smooth. Also,
\( H \circ \iota_Y(y) = H((f(y))_{f \in F}) = (h_{*g}(y))_{g \in G} \) and \( \iota_Z \circ h(y) = (g(h(y)))_{g \in G} \).
Hence, the above diagram is commutative. The above commutative diagram induces, by applying the functor \( \text{Hom}_{\text{DIFF}}(X, -) \), the
following commutative diagram:

$$
\begin{array}{ccc}
\text{Hom}_{\text{DIFF}}(X, Y) & \xrightarrow{h^*} & \text{Hom}_{\text{DIFF}}(X, Z) \\
\downarrow{\iota Y^*} & & \downarrow{\iota Z^*} \\
\text{Hom}_{\text{DIFF}}(X, \mathbb{R}^F_d) & \xrightarrow{H^*} & \text{Hom}_{\text{DIFF}}(X, \mathbb{R}^G).
\end{array}
$$

Suppose that $H^*$, as a map $H^* : (\mathbb{R}^F)_d^X \to (\mathbb{R}^G)_d^X$ is d-smooth. There is then an induced smooth map $Y^X_d \to Z^X_d$.

One has $H^*(m) = H \circ m = H((\pi f^m)_{f \in F}) = (\pi h^* g^m)_{g \in G}$. But $\pi_g H^* = \pi h^* g$ is d-smooth for each $g \in G$ and thus $H^*$ is d-smooth as envisaged.

In the other variable, let $h : X \to U$ be d-smooth. One has a commutative diagram

$$
\begin{array}{ccc}
\text{Hom}_{\text{DIFF}}(U, \mathbb{R}) & \xrightarrow{h^*} & \text{Hom}_{\text{DIFF}}(X, \mathbb{R}) \\
\downarrow{u^*} & & \downarrow{\iota X} \\
\mathbb{R}^X_d & \xrightarrow{h^*} & \mathbb{R}^X_d
\end{array}
$$

where for instance $\iota_X(g) = (g(x))_{x \in X}$ and $h^*(a_u)_{u \in U} = (a_{h(x)})_{x \in X}$. Hence, there is an induced d-smooth map $h^* : \mathbb{R}^U_d \to \mathbb{R}^X_d$ and then, for any set $J$, a d-smooth map $(h^*)_J : (\mathbb{R}^U_d)_d^J \to (\mathbb{R}^X_d)_d^J$. Finally, since, as we have seen above, a differential space $Y$ acquires its structure via a map $\iota : Y \to \mathbb{R}^F$ sending $y$ to $(f(y))_{f \in F}$, there is a canonical d-smooth map $Y^U_d \to Y^X_d$. Since the hom-set $\text{Hom}_{\text{SETS}}(X, Y)$ defines a bifunctor on $\text{SETS}$, the assignment $(X, Y) \to Y^X_d$ defines a bifunctor on $\text{DIFF}$.

We now conclude the argument begun before Lemma 3.4.

Let $g \in \text{Hom}_{\text{DIFF}}(W, Y^X_d)$ and $h = \iota^X \circ g : W \to (\mathbb{R}^F)_d^X$. As $h$ is d-smooth, as we have just shown before Lemma 3.4, $\bar{h} : W \otimes_d X \to \mathbb{R}^F$ is d-smooth. But $\bar{h}(x, t) = h(x)(t) = (\iota^X \circ g)(x)(t) = \iota^X(g(x))(t) = $
\( \iota(g(x)(t)) \). The maps \( \pi_f : \mathbb{R}_d^F \to \mathbb{R}(f \in F) \) generate the structure on \( \mathbb{R}_d^F \). The structure on \( Y \) induced by \( \iota \) is generated by \( \{ \pi_f \circ \iota^X | f \in F \} = F \) and thus just the given structure on \( Y \). It follows that \( \hat{g} \) is smooth. Conversely and in a similar way, if \( g \in \text{Hom}_{\text{DIFF}}(W \otimes_d X, Y) \), then \( \hat{g} \in \text{Hom}_{\text{DIFF}}(W, Y_d^X) \).

Putting the above facts together, we have shown:

**Theorem 3.2** Let \( X, Y \) and \( W \) be differential spaces. The map \( \gamma_{XYW} \) induces a natural equivalence

\[
\text{Hom}_{\text{DIFF}}(W \otimes_d X, Y) \to \text{Hom}_{\text{DIFF}}(W, Y_d^X)
\]

in \( W \) and \( Y \). Hence, any differential space \( X \) is exponential for \( \otimes_d \) in \( \text{DIFF} \).

**Remark:** It is unclear whether \( \mathbb{R}_d^I \) is a Cartesian object in \( \text{DIFF} \).

We provide an example in homotopy theory of one use of Theorem 3.2. Let \( X \) and \( Y \) belong to \( \text{DIFF}^* \), the category of based differential spaces. All base points will be denoted by \( * \). The \( d \)-cone on \( X \) is the quotient of \( X \otimes_d I \) by the equivalence relation \( \sim \) which identifies \((P, 0)\) with \((Q, 0)\) for \( P, Q \in X \). Its basepoint is the equivalence class \([P, 0]\) where \( P \in X \).

Let \( P(Y) = \{ f \in Y^I | f(0) = * \} \) be a \( d \)-subspace of \( Y_d^I \) called the \( d \)-path space of \( Y \). Then, as for topological spaces, one has:

**Proposition 3.3** The map \( \gamma_{IYX} \) induces a bijection

\[
\text{Hom}_{\text{DIFF}^*}(C(X), Y) \to \text{Hom}_{\text{DIFF}^*}(X, P(Y)).
\]

4 **Whitney, \( d \) and \( F \) topologies**

The \( d \)-topology (resp., \( F \)-topology) on \( \text{Hom}_{\text{DIFF}}(X, Y) \) (resp., \( \text{Hom}_{\text{FRL}}(X, Y) \)) is just the underlying topology of \( Y_d^X \) (resp., \( Y_R^X \)). Let \( M \) and \( N \) be
smooth differentiable manifolds viewed as Frölicher spaces. It has been shown in Cherenack [2] that the Whitney topology on $\text{Hom}_{\text{FRL}}(M, N)$ is courser than the corresponding $\mathbb{F}$-topology. Consider $M$ and $N$ as differential spaces. One requires:

**Lemma 4.1** The differential structure on $N^M_d$ is defined by the evaluation maps $ev_a : N^M_d \to N(a \in M)$ where $ev_a(f) = f(a)$.

**Proof:** If $N = \mathbb{R}$, the result follows from definition. Let $N = \mathbb{R}^d_J$. Then, $N^M_d = (\mathbb{R}^M_d)^J_d$ and its structure is defined first by the projection maps $P_j : (\mathbb{R}^M_d)^J_d \to \mathbb{R}^M_d$ sending $(z_j)_{j \in J}$ to $z_j$ and then in turn by the maps $ev_a \circ P_j : (\mathbb{R}^M_d)^J_d \to \mathbb{R}(a \in M)$ sending $f \in (\mathbb{R}^M_d)^J_d$ to $(P_j \circ f)(a) = P_j(ev_a(f))$. Thus, the structure on $(\mathbb{R}^M_d)^J_d$ is defined by the $ev_a(a \in M)$ since the $P_j(j \in J)$ define the product structure on $\mathbb{R}^d_J$.

Let $N$ be an arbitrary differential space. Since the structure on $N^M_d$ is induced from the structure on $(\mathbb{R}^d_J)^M_d$ for a suitable set $J$, the structure on $N^M_d$ is induced by $ev_a(a \in M)$ on $(\mathbb{R}^d_J)^M_d$ restricted to $N^M_d$.

Since

1. the map $ev_a : N^M_d \to N$ factors through the jet map $j^k_a : N^M_d \to J^k(M, N)$ ($a \in M, k \geq 1$),
2. the k-th jet space $J^k(M, N)$ is a manifold,
3. the Whitney topology is the topology initial for the maps $j^k_a(k \geq 0, a \in M)$ and
4. the d-topology on $N^M_d$ is defined by the $ev_a(a \in M)$, as determined in Lemma 4.1,

one has:

**Proposition 4.1** The Whitney topology on $\text{Hom}_{\text{DIFF}}(M, N)$ is finer than the d-topology but courser than the $\mathbb{F}$-topology.
5 Admissible and proper structures

Let $X, Y$ and $Z$ be differential spaces, $- \otimes : \text{DIFF} \to \text{DIFF}$ any functor where $Y \otimes X$ has underlying set $Y \times X$ and $Y^X = \text{Hom}_{\text{DIFF}}(X, Y)$. A differential structure $\Xi$ on $Y^X$ is called $\otimes$-proper if using the notation of section 3, $f : Z \otimes X \to Y$ d-smooth implies $\hat{f} : Z \to (Y^X, \Xi)$ d-smooth. In the other direction, if $h : Z \to (Y^X, \Xi)$ d-smooth implies $\hat{h} : Z \otimes X \to Y$ d-smooth, then $\Xi$ is called $\otimes$-admissible.

**Remark** Let $Y^X_d$ and $- \otimes_d X$ be the differential spaces referred to in section 3. Then, we have seen there:

1. the structure on $Y^X_d$ is $\otimes_d$-admissible but not $\times_d$ admissible.
2. the structure on $Y^X_d$ is $\otimes_d$ and $\times_d$-proper.

Our main reference for the next part of the paper is Choquet-Bruhat, DeWitt-Morette [5]. Let $(S, F_S)$ be a differential space and $F_S^c$ the set of $f \in F_S$ having compact support.

**Definition 5.1** We call a measure $m$ on $S$ a $d$-measure with respect to $\otimes_d$ if, for every compact set $K \subset S$ and every $f \in F^c_S$ with support in $K$,

1. $\int_S f(s) dm(s)$ exists,
2. $|\int_S f(s) dm(s)| \leq C(K) \sup_{s \in K} |f(s)|$ and
3. for all d-smooth maps $H : U \otimes_d S \to \mathbb{R}$ and $f \in F^c_S$, the map $\mu : U \to \mathbb{R}$ defined by setting

   $$\mu(u) = \int_S f(s) \cdot H(u, s) dm(s)$$

   is d-smooth.

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In the definition of $\mu$, $fdm$ is usually viewed as the metric for integration.

**Proposition 5.1** Let $m$ be a d-measure on a Frölicher space. Then, condition 3 of definition 5.1 is extraneous.

**Proof:** Note that $\otimes_F = \times_F$ for Frölicher spaces. Let $c : \mathbb{R} \to U$ be $F$-smooth. Let $f$ have support contained in $K$. We show that $\mu \circ c(t)$ is smooth at each $a \in \mathbb{R}$ and it follows that $\mu$ is a $F$-smooth map. One has

$$\frac{\mu \circ c(t) - \mu \circ c(a)}{t-a} = \int_S f(s)\frac{H(c(t), s) - H(c(a), s)}{t-a}dm(s)(t \neq a).$$

Note that (for $t \neq a$) both $f(s)\frac{\partial H}{\partial t}(c(t), s)$ and $f(s)\frac{H(c(t), s) - H(c(a), s)}{t-a}$ are $F$-smooth as functions of $s$ and $t$ and, as functions in $s$, have compact support in $K$. One has

$$\left|\int_S f(s)\frac{H(c(t), s) - H(c(a), s)}{t-a} - \frac{\partial H}{\partial t}(c(a), s)\right| dm(s) - \frac{\partial H}{\partial t}(c(a), s)|$$

which tends to 0 as $t \to a$. Hence

$$(\mu \circ c)'(a) = \lim_{t \to a} \int_S f(s)\frac{\partial H}{\partial t}(c(a), s)dm(s).$$

One also has

$$\left|\int_S f(s)\frac{\partial H}{\partial t}(c(t), s)dm(s) - \int_S f(s)\frac{\partial H}{\partial t}(c(a), s)dm(s)\right| \leq$$

$$C(K)\sup_{s \in K}|f(s)|\max_{s \in K}\left|\frac{\partial H}{\partial t}(c(t), s) - \frac{\partial H}{\partial t}(c(a), s)\right| \to 0$$

as $t \to a$. Thus, $(\mu \circ c)'(t) = \int_S f(s)\frac{\partial H}{\partial t}(c(t), s)dm(s)$. Continuing in this way, one finds that $(\mu \circ c)(t)$ is infinitely differentiable, hence $\mu \circ c$ is smooth and finally $\mu$ is $F$-smooth.

Thus, in Frölicher spaces the class of useful measures is larger.
Let $m$ be a $d$-measure on $X$ with respect to $\otimes_d$. We set

$$\Theta_{m_{\alpha \beta}}(f) = \int_X \alpha(x) \cdot (\beta \circ f(x))dm(x)$$

for $\alpha \in F^c_X$, $\beta \in F_Y$ and $f \in Y^X_d$. Suppose that $F_{m_{0}}$ is the differential structure on $Y^X_d$ generated by the set $\Gamma_{m_{0}} = \{ \Theta_{m_{\alpha \beta}} : \alpha \in F^c_X, \beta \in F_X \}$ together with the set $F_{Y^X_d}$ of structure functions of $Y^X_d$, defined in section 3. In the case where $m = \{ m_t \}_{t \in T}$ is a family of $d$-measures on $X$ with respect to $\otimes_d$, we let $F_{m_{0}}$ be the differential structure on $Y^X_d$ generated by the set $\bigcup_{t \in T} \Gamma_{m_{0}} \cup F_{Y^X_d}$. We prove the following result:

**Proposition 5.2** Let $m = \{ m_t \}_{t \in T}$ be a family of $d$-measures on $X$ with respect to $\times_d$. The differential structure $F_{m_{0}}$ is $\times_d$ proper.

**Proof:** Let $Z$ be a differential space and consider a $d$-smooth map $Z \times_d X \to Y$. Then, for any $z \in Z$, we obtain

$$\Theta_{m_t {\alpha \beta}} \circ \hat{h}(z) = \int_X \alpha(x) \cdot (\beta \circ \hat{h}(z)(x))dm_t(x) = \int_X \alpha(x) \cdot (\beta(h(z, x)))dm_t(x).$$

Because $\beta$ and $h$ are smooth and because $m_t$ is a $d$-measure with respect to $\times_d$ for each $t \in T$, it follows that the assignment

$$z \mapsto \int_X \alpha(x) \cdot (\beta \circ \hat{h}(z)(x))dm_t(x) = \int_X \alpha(x) \cdot (\beta(h(z, x)))dm_t(x)$$

is $d$-smooth for each $t \in T$. Since $\times_d$ is proper for the usual structure on $Y^X_d$, $\hat{h}$ is $d$-smooth and hence $F_{m_{0}}$ is $\times_d$ proper. \qed

An analogous result follows if one lets $F_{m_{nn}}$, for $n = 1, 2, \cdots$ and a family $\{ m_t \}_{t \in T}$ of $d$-measures on $X$ with respect to $\times_d$, be the differential structure on $Hom_{\text{DIFF}}(X, Y)$ generated by the structure functions on $Y^X_d$ together with the maps $\Theta_{m_t {\alpha \beta}}^X$ where

$$\Theta_{m_t {\alpha \beta}}^X(f) = \int_X \alpha(x) \cdot X(x)(\beta \circ f)dm_t(x).$$
and where \( \alpha \) ranges over \( \mathcal{F}_X \), \( \beta \) ranges over \( \mathcal{F}_Y \), \( t \in T \) and \( X \) ranges over all smooth vector fields of \( n \)-th order. See [9].

Since \( \otimes_d \) is then proper for the usual structure on \( Y_d^X \), one can show, as in Proposition 5.2:

**Proposition 5.3** Let \( Y_d^X \) denote \( Y_d^X \) but with structure defined by \( \mathcal{F}_{m0} \) where \( m = \{ m_t \}_{t \in T} \) is a family of \( d \)-measures with respect to \( \otimes_d \). The differential structure \( \mathcal{F}_{m0} \) is \( \otimes_d \) proper.

**Proof:** Since \( \otimes_d \) is proper for the usual structure on \( Y_d^X \), one need only show that, for \( g : W \otimes_d X \to Y \) \( d \)-smooth and \( \hat{g} : W \to \hat{Y}_d^X \) that the map

\[
\vartheta = \Theta_{m_t \alpha \beta} \circ \hat{g} : W \to \mathbb{R}
\]

is \( d \)-smooth. Let \( w \in W \). Then, \( \Theta_{m_t \alpha \beta} \circ \hat{g}(w) = \int_X \alpha(x) \cdot \beta(\hat{g}(w)(x)) dm_t(x) = \int_X \alpha(x) \cdot \beta(g(w,x)) dm_t(x) \). Since \( g : W \otimes_d X \to Y \) is \( d \)-smooth and \( \{ m_t \}_{t \in T} \) is a family of \( d \)-measures with respect to \( \otimes_d \), \( \vartheta \) is smooth. \( \square \)

Finally, one shows:

**Theorem 5.1** Let \( X, Y \) and \( Z \) be differential spaces. There is then a natural bijection

\[
\text{Hom}_{\text{DIFF}}(Z \otimes_d X, Y) \to \text{Hom}_{\text{DIFF}}(Z, Y_d^X)
\]

induced by \( \gamma_{XYZ} \). Hence, \( Y_d^X \) is isomorphic to \( Y_d^X \) in \( \text{DIFF} \).

**Proof:** Let \( h : Z \to Y_d^X \) be \( d \)-smooth. Then, \( h : Z \to Y_d^X \) is \( d \)-smooth as \( Y_d^X \) possesses more structure functions than \( Y_d^X \). Hence, as we have seen, \( h : Z \otimes_d X \to Y \) is \( d \)-smooth. The other direction follows from Proposition 5.3.

Here, \( Y^X \) is isomorphic to \( Y^X_d \) since the functors defined by sending \( Z \to \text{Hom}_{\text{DIFF}}(Z, Y_d^X) \) and \( Z \to \text{Hom}_{\text{DIFF}}(Z, Y^X_d) \) are naturally isomorphic. \( \square \)
Note that the last result depends on the special choice of d-measures.

Since evaluation maps in the category \( \mathbf{FRL} \) are d-smooth, \( \otimes_d = \times_F \) in Frölicher spaces. Suppose now that \( X, Y \) and \( Z \) are Frölicher spaces. Let \( Y^X_F \) denote the object in \( \mathbf{FRL} \) whose underlying set is \( \text{Hom}_{\mathbf{FRL}}(X, Y) \) and which arises out of the Cartesian closedness of \( \mathbf{FRL} \). Let \( \mathcal{F}_{m_0} \) be the set consisting of the structure functions on \( Y^X_F \) together with the functions in \( \bigcup_{t \in T} \Gamma_{m_t} \). The above theorem then becomes:

**Theorem 5.2** Let \( X, Y \) and \( Z \) be Frölicher spaces. Let \( Y^X_F \) denote \( Y^X_F \) but with structure defined by \( \mathcal{F}_{m_0} \) where \( m = \{ m_t \}_{t \in T} \) is a family of d-measures with respect to \( \times_F \). There is then a natural bijection

\[
\text{Hom}_{\mathbf{FRL}}(Z \times_F X, Y) \to \text{Hom}_{\mathbf{DIFF}}(Z, Y^X_F)
\]

induced by \( \gamma_{XYZ} \). Hence, \( Y^X_F \) is isomorphic to \( Y^X_F \) in \( \mathbf{FRL} \).

The proof is similar to that of the preceeding theorem.

### 6 d and \( \mathbb{F} \) distributions

Let \( X \) be a differential space. We let \( \mathbb{R}^X_{dc} \) denote the d-subspace of \( \mathbb{R}^X_d \) consisting of those \( f \in \mathbb{R}^X_d \) having compact support. For a Frölicher space \( X \), we let \( \mathbb{R}^X_{dc} \) denote the subset of \( \mathbb{R}^X_d \) consisting of those \( f \in \mathbb{R}^X_d \) having compact support. The Frölicher space \( \mathbb{R}^X_{dc} \) is defined by specifying a set of generating curves. Thus, a curve \( c : \mathbb{R} \to \mathbb{R}^X_{dc} \) is a generating curve if and only if the associated curve \( \hat{c} : \mathbb{R} \times X \to \mathbb{R} \) defined by setting \( \hat{c}(s, x) = c(s)(x) \) is smooth and, for each \( t_0 \in \mathbb{R} \), there is a compact subset \( K \) of \( X \) and a \( \epsilon > 0 \) such that, for \( t_0 - \epsilon \leq t \leq t_0 + \epsilon \), \( \hat{c}(t, x) = 0 \) if \( x \not\in K \). It seems likely that \( \mathbb{R}^X_{dc} \) is not a \( \mathbb{F} \)-subspace of \( \mathbb{R}^X_d \). However, the example below motivates our choice of the structure on \( \mathbb{R}^X_{dc} \). Clearly, any element of \( \mathbb{R}^X_{dc} \) or \( \mathbb{R}^X_{dc} \) takes on a maximum value. A differential (resp., Frölicher) space \( X \) which is at the same time a real vector space is a d-vector space (resp., \( \mathbb{F} \)-vector space) if the addition
and scalar multiplication of $X$ as a vector space are $d$-smooth (resp., $F$-smooth). Since the differential or Frölicher space structure on $\mathbb{R}^X_{df}$ is defined by evaluation maps, it is easy to see that pointwise addition makes $\mathbb{R}^X_{df}$ into a $d$-vector space or $F$-vector space, respectively.

**Definition 6.1** A $d$-distribution (resp., $F$-distribution) $D$ on $\mathbb{R}^X_{df}$ (resp., $\mathbb{R}^X_{df}$) with pointwise addition of functions is a $d$-smooth (resp., $F$-smooth) linear map $D : \mathbb{R}^X_{df} \to \mathbb{R}$ (resp., $D : \mathbb{R}^X_{df} \to \mathbb{R}$) such that, for each $g \in \mathbb{R}^X_{df}$ (resp., $g \in \mathbb{R}^X_{df}$) with compact support $K$,

$$|D(g)| \leq C(K) \max_{x \in K} |g(x)|.$$ 

Of course, one might hope that the smoothness of $D$ above implies the last inequality.

We provide two examples of $d$-distributions. Let $h \in \mathbb{R}^X_{df}$. If $m$ is a $d$-measure with respect to $\times_d$, as defined in section 3, it is easy to see that the mapping $D_{mh} : \mathbb{R}^X_{df} \to \mathbb{R}$ sending $g$ to $\int_X ghdm$ is a $d$-distribution. Thus, for a smooth curve $c : \mathbb{R} \to \mathbb{R}^X_{df}$, $D_{mh} \circ c$ is smooth if the associated map sending $t$ to $\int_X f(x)c(t, x)dm(x)$ is smooth and this is the case because $m$ is a $d$-measure with respect to $\times_d$. The Dirac delta function $\delta_{x_0} : \mathbb{Y}^R_{df} \to \mathbb{R}$ defined by setting $\delta_{x_0}(f) = f(x_0)$ is readily seen to be a $d$-distribution. Notice the interesting fact that the $\delta_{x_0} = ev_{x_0}$ for $x_0 \in X$ define the differential structure on $\mathbb{R}^X_{df}$. Let $m_{x_0}$ be the measure on $X$ defined by setting $m_{x_0}(A) = 1$ if $x_0 \in A$ and $m_{x_0}(A) = 0$ if $x_0 \notin A$. Then, $\delta_{x_0}(f) = \int_X fdm_{x_0}$ and thus $m_{x_0}$ is a $d$-measure. See [5]

**Example** We provide an example to show how one begins to solve differential equations in FRL. See [5] for background. Consider the ordinary differential equation $\frac{dy}{dx} = \delta(x) = \delta_0(x)$ with $x \in \mathbb{R}$ where one has the condition $y(1) = 1$. Since $\delta(x)$ is not a function, we convert it into a distribution. In this way, one must solve

$$\int_{\mathbb{R}} g(x) \frac{dy}{dx} dx = \int_{\mathbb{R}} g(x)\delta(x) dx = g(0) \quad (1)$$

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where \( g(x) \) is any smooth function on \( \mathbb{R} \) with compact support. We assume that a solution exists. Let

\[
  g_\epsilon(x) = \begin{cases} 
  1 & 0 < \epsilon \leq x \leq L \\
  \leq 1 \text{ and } \geq 0 & 0 \leq x \leq \epsilon \text{ or } L \leq x \leq L + \epsilon \\
  0 & x \leq 0 \text{ or } L + \epsilon \leq x.
\end{cases}
\]

where \( \epsilon > 0 \). Then equation 1 becomes

\[
  \int_{0}^{\epsilon} g_\epsilon(x) \frac{dy}{dx} \, dx + \int_{0}^{L} \frac{dy}{dx} \, dx + \int_{L}^{L+\epsilon} g_\epsilon(x) \frac{dy}{dx} \, dx = 0
\]

or, making the substitution \( \frac{dy}{dx} = \delta(x) \),

\[
  g_\epsilon(0) + (y(L) - y(\epsilon)) + 0 = 0.
\]

Letting \( \epsilon \to 0 \), one obtains \( y(L) = \lim_{\epsilon \to 0} = y_+ \) and thus, as \( L \) is arbitrary, \( y(x) \) is constant for \( x > 0 \). Hence, \( y(x) = y(1) = 1 \) for \( x > 0 \). Similarly \( y(x) \) is constant for \( x < 0 \). Let now

\[
  h_\epsilon(x) = \begin{cases} 
  1 & -1 \leq x \leq 1 \\
  \leq 1 \text{ and } \geq 0 & -1 - \epsilon \leq x \leq -1 \text{ or } 1 \leq x \leq 1 + \epsilon \\
  0 & x \geq 1 + \epsilon \text{ or } x \leq -1 - \epsilon.
\end{cases}
\]

where \( \epsilon > 0 \). Then using equation 1, one obtains

\[
  \int_{-1-\epsilon}^{-1} h_\epsilon(x) \frac{dy}{dx} \, dx + \int_{-1}^{1} \frac{dy}{dx} \, dx + \int_{1}^{1+\epsilon} h_\epsilon(x) \frac{dy}{dx} \, dx = h_\epsilon(0) = 1 \quad \text{or}
\]

\[
  0 + \int_{-1}^{1} \frac{dy}{dx} \, dx + 0 = 1.
\]

Letting \( \epsilon \to 0 \), one sees that \( y(1) - y(-1) = 1 \). Hence

\[
  y(x) = \begin{cases} 
  1 & x > 0 \\
  0 & x < 0
\end{cases}
\]

which is a Heaviside type function.
We justify now the above calculation of \( y(x) \). To represent \( y(x) \) as a \( F \)-distribution, one must show that the map \( \Delta : \mathbb{R}_{\mathcal{F}}^K \to \mathbb{R} \) sending \( g \) to \( \int_{\mathbb{R}} g(x)y(x)dx \) is a \( F \)-distribution. It is easy to show that

\[
|\Delta(g)| \leq C(K) \max_{x \in K} |g(x)|
\]

for some constant \( C(K) \) depending only on \( K \) when \( g \) has compact support in \( K \). Let \( c : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}^K \) be a \( F \)-smooth curve. As \( \mathcal{F} \)RL is Cartesian closed, the associated map \( \bar{c} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is smooth. Substituting \( y(x) \), one obtains \( \Delta \circ c(t) = \int_\mathbb{R} \bar{c}(t, x)y(x)dx = \int_0^\infty \frac{\partial h}{\partial x}(t, x)dx \) where \( h(t, x) = \int_0^x \bar{c}(t, y)dy \) is a smooth map on \( \mathbb{R} \times \mathbb{R} \). Thus, \( \Delta \circ c(t) = \lim_{M \to -\infty} h(t, x)|^M_0 = \lim_{M \to -\infty} h(t, M) \). Let \( t_0 \in \mathbb{R} \). Since \( \bar{c} \) is a structure curve for \( \mathbb{R}_{\mathcal{F}}^X \), there is a \( C > 0 \) and a \( \epsilon > 0 \) such that, for \( t_0 - \epsilon \leq t \leq t_0 + \epsilon \), \( \bar{c}(t, x) = 0 \) if \( |x| > C \). Hence, for \( t_0 - \epsilon \leq t \leq t_0 + \epsilon \), \( \Delta \circ c(t) = h(t, C) \). It follows that \( \Delta \circ c(t) \) is smooth near \( t_0 \) and, since \( t_0 \) is arbitrary, smooth. Hence \( \Delta \) is \( F \)-smooth and thus a \( F \)-distribution. Note however that \( \Delta \) is not d-smooth since its values would need to depend on the values of any \( g \) at certain finitely many real numbers (locally).

The derivative \( \Delta'(g) \) of \( \Delta(g) \) according to [5] satisfies

\[
\Delta'(g) = -\int_{\mathbb{R}} g'(x)y(x)dx.
\]

Hence \( \Delta'(g) = -\int_0^\infty g'(x)dx = g(0) \). Hence, as \( F \)-distributions, one has \( \Delta'(g) = \delta(g) \).

This example shows that it is probably reasonable to develop the theory of distributions in \( \mathcal{F} \)RL. For \( \mathcal{D} \)IFF one needs to decide which structure functions \( \Theta_{\alpha\beta} \) one needs to add to the structure functions on \( Y^X \). Finally, to demonstrate one key advantage of Frölicher spaces over differential spaces, we show:

**Lemma 6.1** The maps 1) \( D : \mathbb{R}_{\mathcal{F}}^K \to \mathbb{R}_{\mathcal{F}}^K \) and 2) \( D_c : \mathbb{R}_{\mathcal{F}c}^K \to \mathbb{R}_{\mathcal{F}c}^K \) sending a function \( f \) to its derivative \( f' \) are both smooth.
**Proof:** We consider only the first case which is similar to the second. Let \( c : \mathbb{R} \to \mathbb{R}^\mathbb{R} \) be smooth. Then, \( D \circ c \) is smooth if the associated map \( \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) sending \((t, s)\) to \((c(t))'(s) = \frac{\partial c(t, s)}{\partial s}\) is smooth. But this is clear as \( c \) smooth implies that \( \tilde{c} \) is smooth. \( \square \)

The corresponding result does not hold for differential spaces since even locally the values of \( f' \) do not depend on the values of \( f' \) at a finite number of points.

### References


