The geometry of force

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Stellenbosch Theoretical Physics Seminar, 19 Sep 2013

1. Introduction

Maxwell's equations for electromagnetism are:

$$\frac{\partial \mathbf{E}}{\partial t} = -\nabla \times \mathbf{E} \qquad \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

They have some funky solutions:

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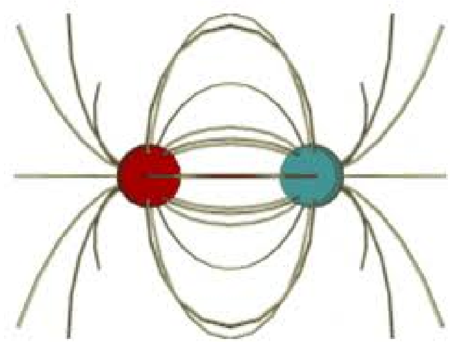
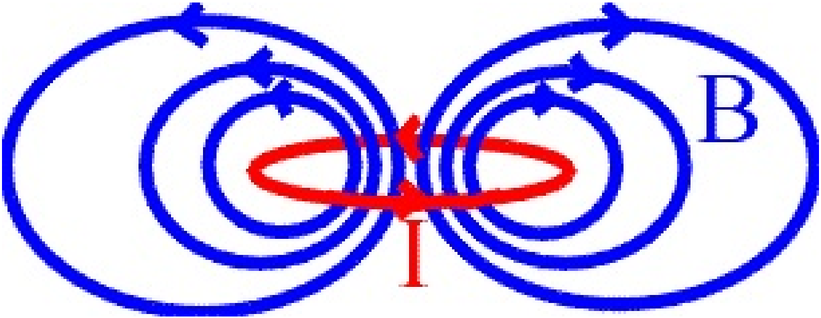
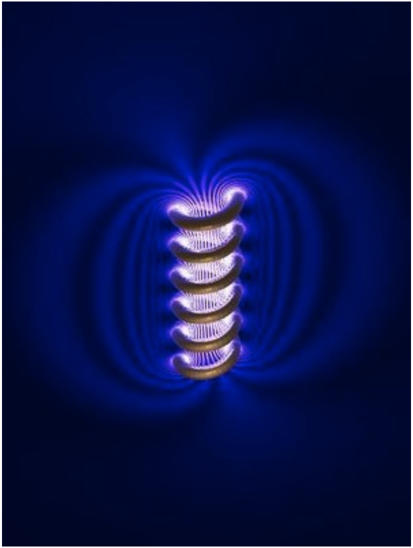
Electric field of a dipole:



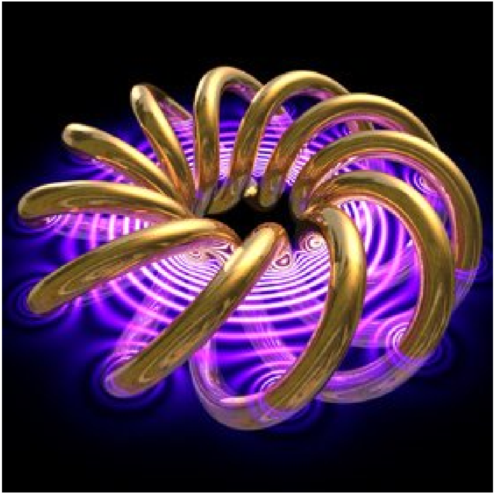
Magnetic field of a current-carrying loop:



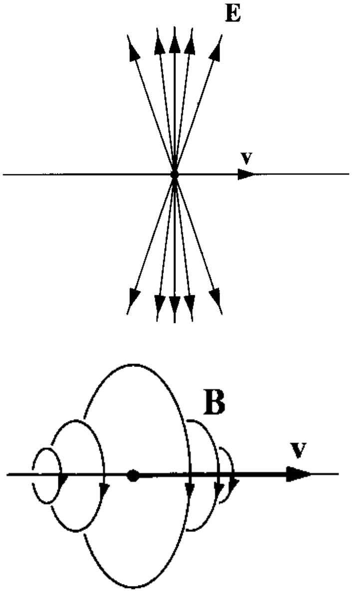
Magnetic field of a solenoid:



Magnetic field of a toroidal solenoid:



Electric and magnetic fields of a moving point charge:



...the electric and magnetic fields can even unfurl in a Hopf fibration!

LETTERS

Linked and knotted beams of light

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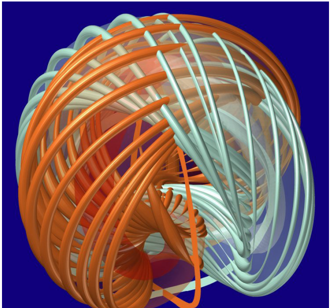
Maxwell's equations allow for curious solutions characterized by the property that all electric and magnetic field lines are closed loops with any two electric (or magnetic) field lines linked. These little known solutions, constructed by Rañada[1], are based on



The Lorentz force law describes how charged particles experience an electromagnetic field:

$$\mathbf{F} = q\mathbf{E} + \mathbf{v} \times \mathbf{B}.$$

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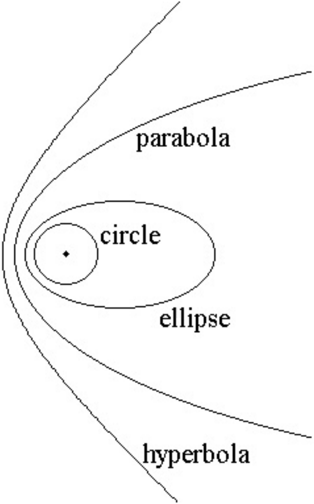
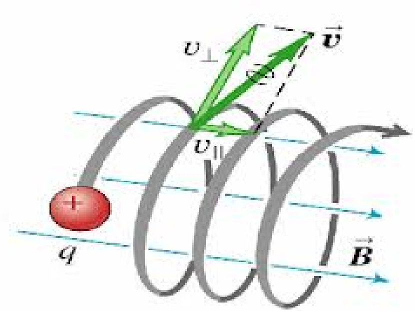
The resulting trajectories are also interesting, even for simple electromagnetic fields.

The particles can travel along the usual inverse square orbits...



... they can spiral around a magnetic field...



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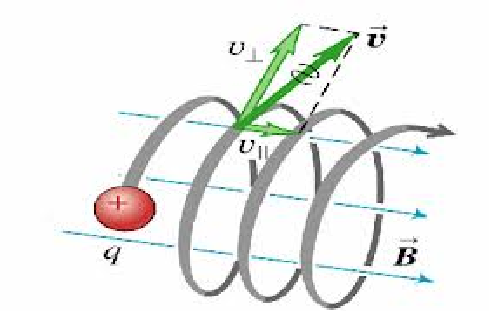
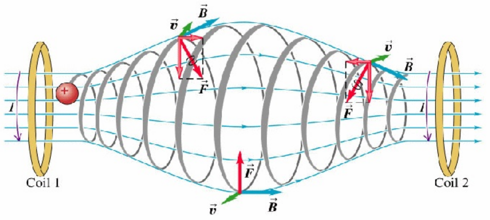
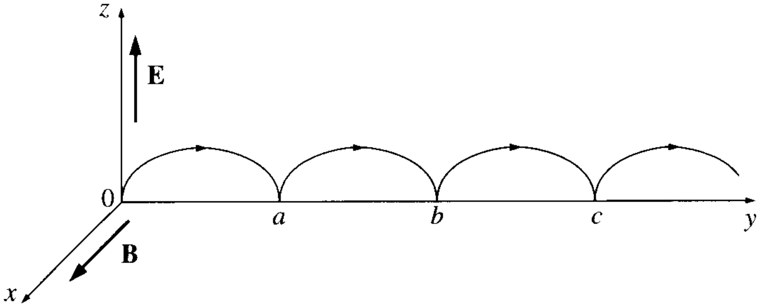


...you can trap them this way in a magnetic bottle...



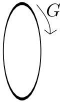
...or they can move along a cycloid...



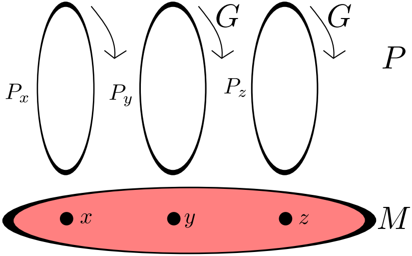
...but secretly, the particles are moving along *straight lines* in a higher-dimensional space!

2. Connections on principal bundles

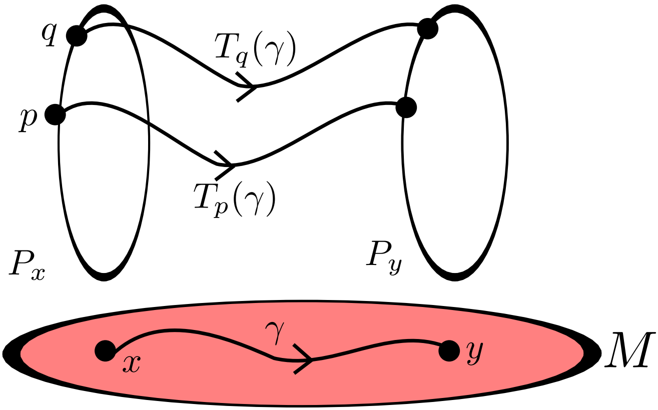
Let G be a Lie group. A *G-torsor* is a manifold on which G acts freely and transitively:



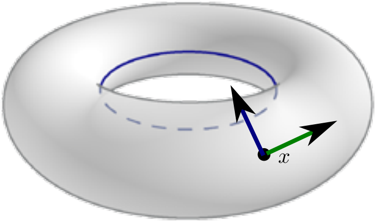
A *principal G-bundle* P is a 'bundle' of G-torsors over a base space M:



A *connection* on a principal G-bundle is a gismo T which allows you to *lift* ('parallel transport') paths in the base space M to paths in P, in a manner compatible with the G-action:



For instance, the *frame bundle* $\mathrm{Fr}(M)$ of a surface $M \subset \mathbb{R}^3$ is a principal $U(1)$-bundle:



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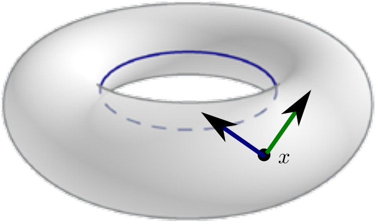

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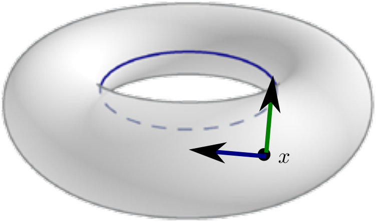

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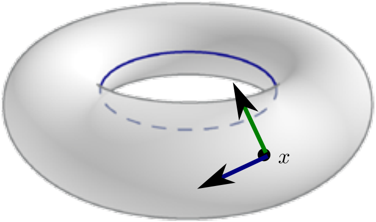

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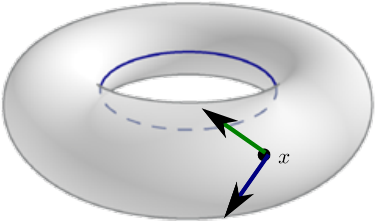

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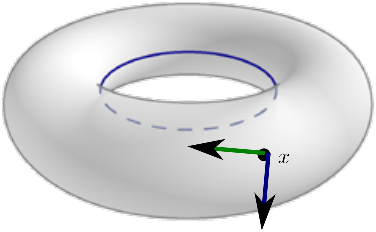

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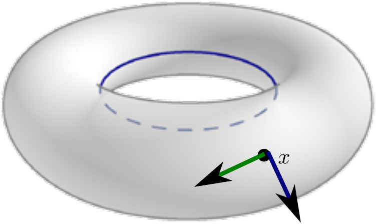

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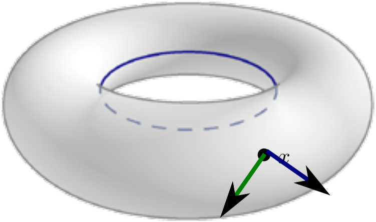
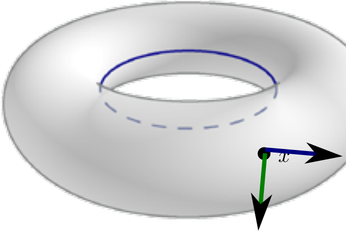


For instance, the *bundle of orthonormal frames* $\mathrm{Fr}(M)$ of a surface $M \subset \mathbb{R}^3$ is a principal $U(1)$-bundle:



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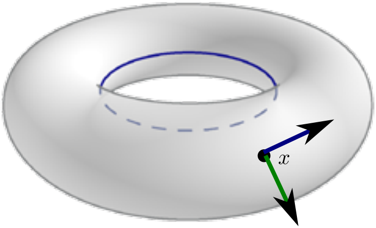

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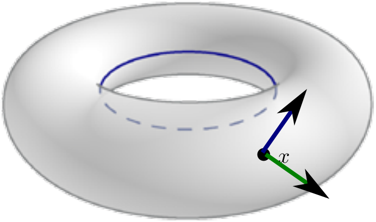

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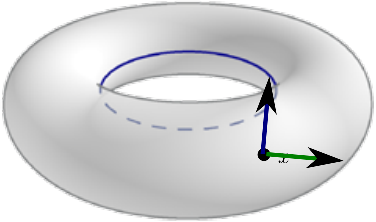

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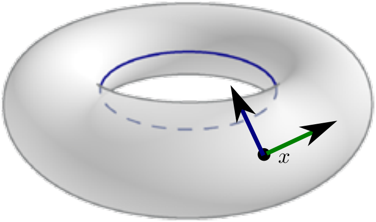
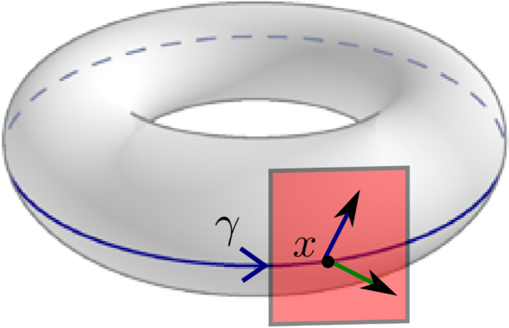


To parallel transport a frame along a curve γ in M, move it infinitesimally in the direction of $\gamma'(t)$ in $\mathbb{R}^3$, and then project back to the tangent space of M:



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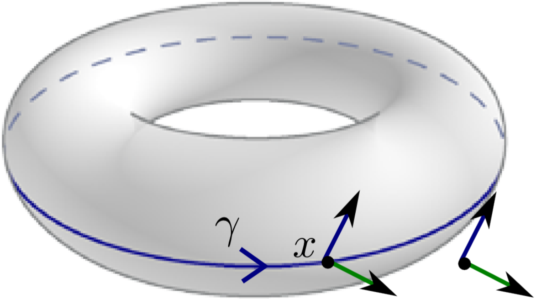

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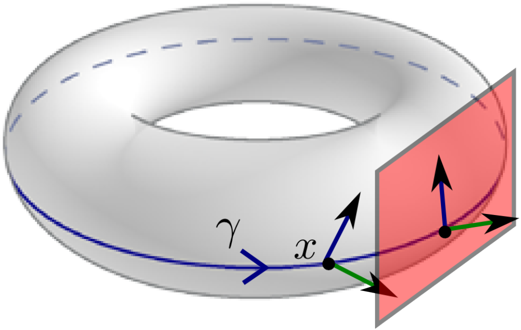
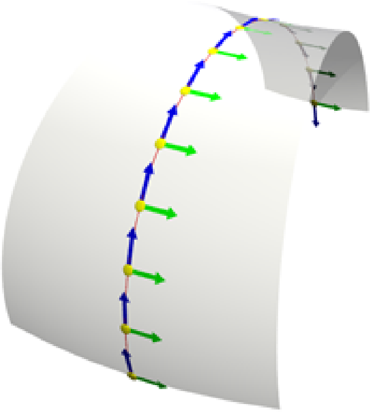
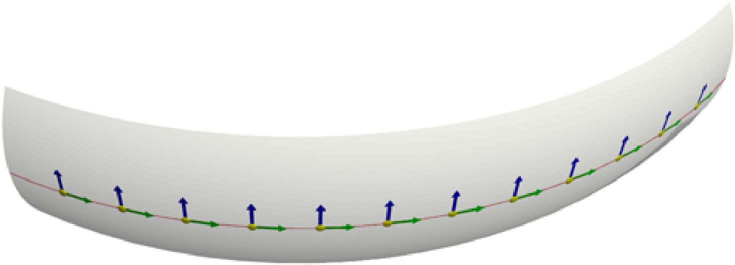


Around lines of longitude, nothing much happens:



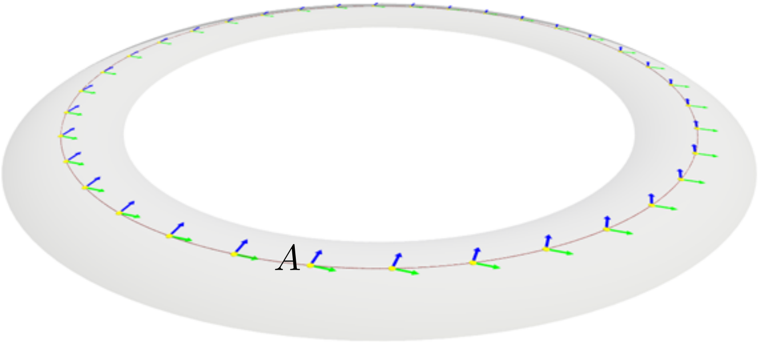
Similarly around the outer equator:



For these paths γ, the transported frames maintain a constant angle with respect to the tangent vector $\gamma'(t)$ of the path. A path γ in M having this property is called a *geodesic* in M.

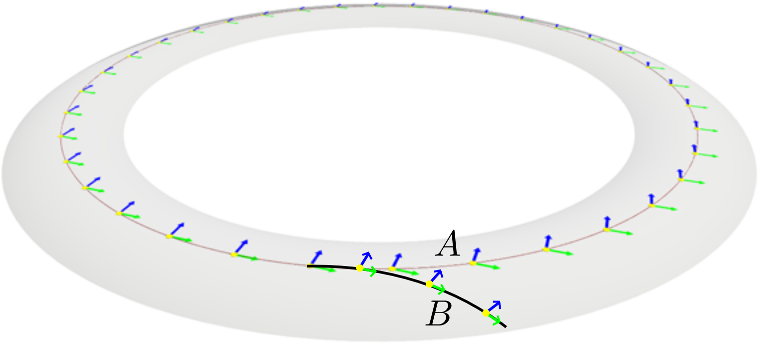
An insect travelling along a geodesic feels no ‘force’.

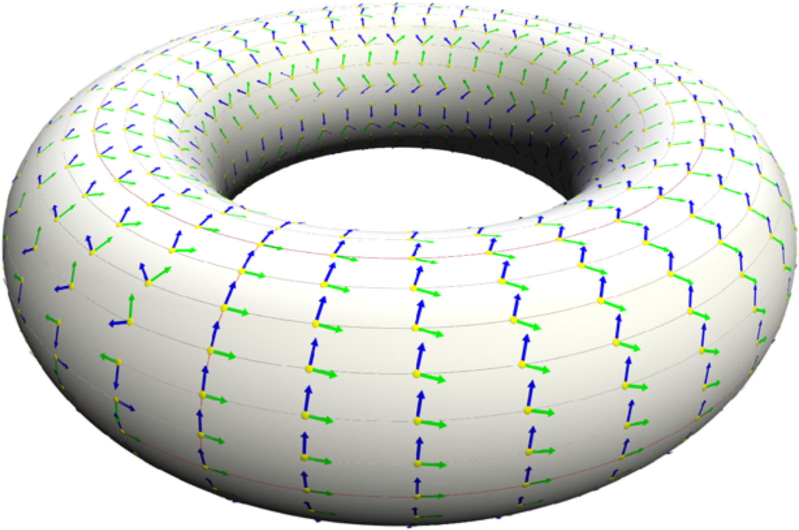
Around the upper circle, the frame *rotates around* $\gamma'(t)$:



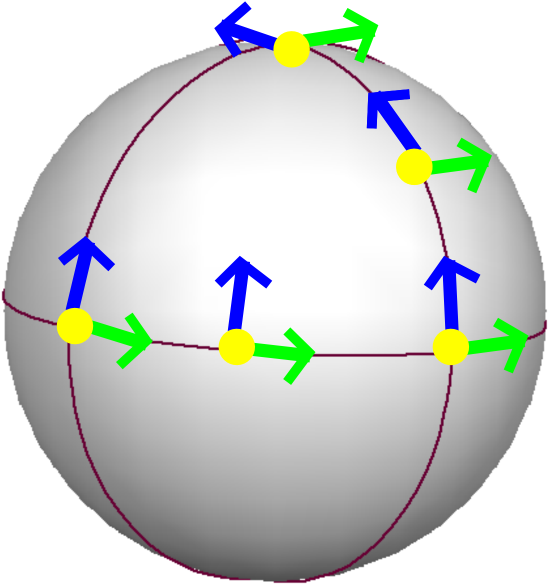
An insect A travelling along this path experiences a kind of 'centripetal force'.

Another insect B travelling along a geodesic in M reasons that A is experienceing a 'force to the left'.





We have a similar picture for the frame bundle $\mathrm{Fr}(S^2)$ of S^2:



3. Projected geodesics

We have seen how to define a connection on a surface $M \subset \mathbb{R}^3$, and that this gives rise to the notion of *geodesics* in M as curves whose tangent vector parallel transports itself.

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3. Projected geodesics

We have seen how to define a connection on a surface $M \subset \mathbb{R}^3$, and that this gives rise to the notion of *geodesics* in M as curves whose tangent vector parallel transports itself.

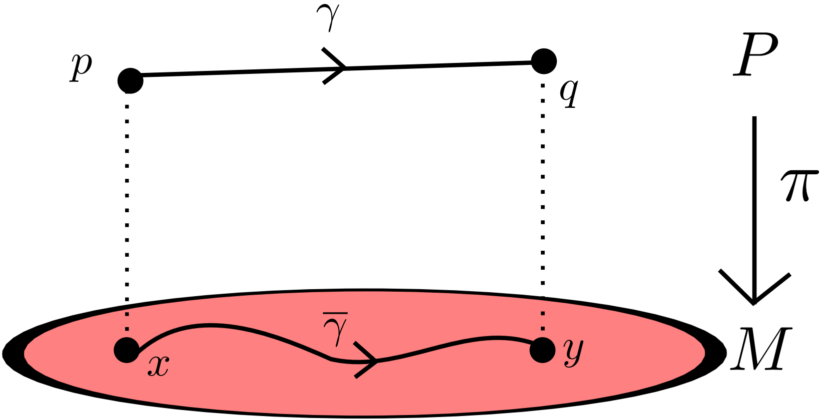
More generally, every Riemannian manifold has an associated connection.

Now, let k be an inner product on $\mathfrak{g}$, and let $\pi : P \to M$ be a principal G-bundle over a Riemannian manifold (M, g) equipped with a connection ω. Then we can construct a natural metric h on P:

$$h = \pi^* g + k\omega$$

So P becomes a Riemannian manifold in its own right.

What do geodesics in P look like, when projected to M?



Consider $P = \mathrm{Fr}(S^2)$. What is this Riemannian manifold?

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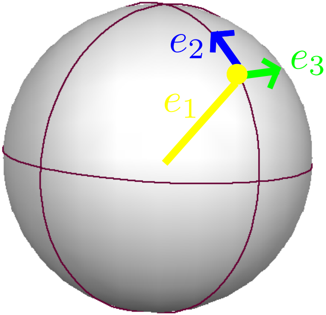
A frame on S^2 can be thought of as an orthonormal basis (e_1, e_2, e_3) of vectors:



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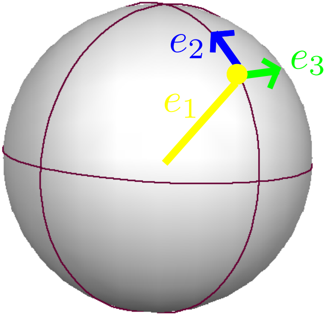
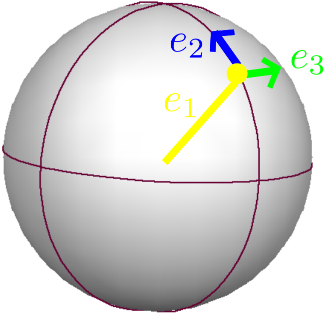
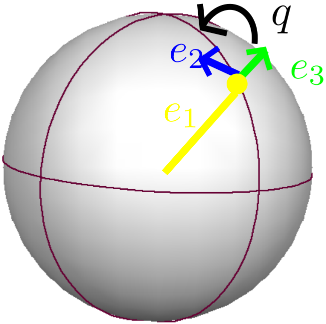


So - $\mathrm{Fr}(S^2)$ is $SO(3)$! The projection map $\pi : SO(3) \to S^2$ sends $(e_1, e_2, e_3) \mapsto e_1$.

So, a particle moving around on $SO(3)$ is the same thing as a *frame* moving around on S^2...a 'charged particle' on S^2 with a hidden $U(1)$ gauge degree of freedom!

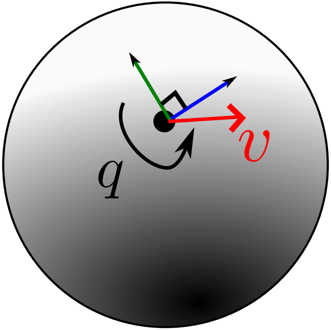


We can ask that the 'spinning velocity' in the gauge direction is constant.. this is called the *charge* q of the 'charged particle' on S^2.

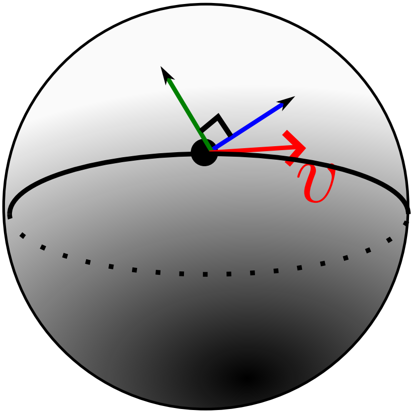


To give a geodesic γ on $P = SO(3)$ we must give an initial point $\gamma(0)$ and an inital velocity $\gamma'(0)$. Since we take the charge to be constant, giving $\gamma'(0)$ amounts to giving a tangent vector v on S^2.

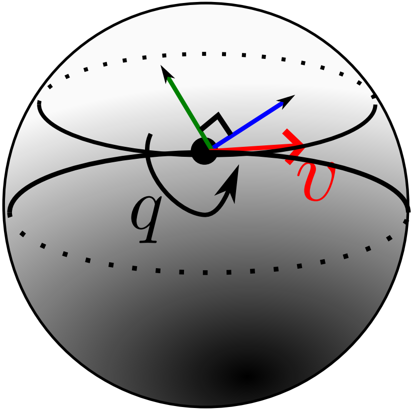
What will happen?



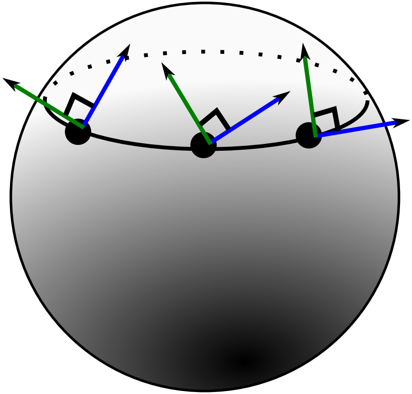
If $q = 0$, the projection $\overline{\gamma}$ on S^2 will just be a great circle (i.e. a geodesic on S^2):



If $q \neq 0$, the projection $\overline{\gamma}$ on S^2 will steer away from the great circle! It will be a 'not-so-great circle'.

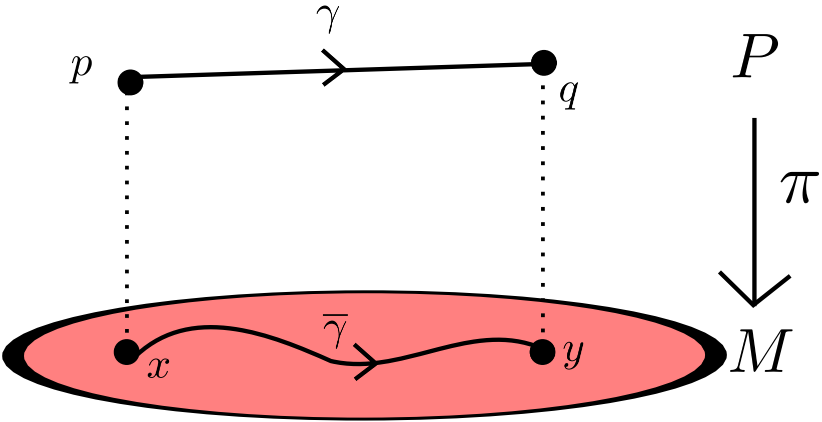


To understand this, consider ordinary parallel transport along $\overline{\gamma}$. The frame rotates clockwise!



Evidently, the charge of the particle precisely cancels this rotation. An insect living on S^2 reasons that the particle is experiencing a force, but in fact the 'charged particle' is travelling along a geodesic in the 3d space P! The insect is *wrong*!

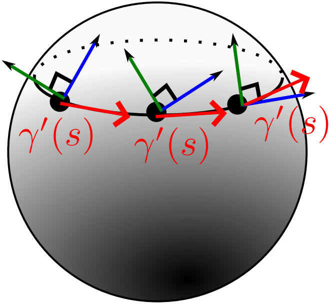
4. The equation for $\overline{\gamma}$



Let γ be a curve in a Riemannian manifold M. Choose an orthonormal frame u_0 at $\gamma(0)$, and let $u(s)$ be the parallel transported 'moving frame' along γ. The *acceleration*

$$\frac{D\gamma'}{ds} \in T_{\gamma(s)}M$$

of γ is the derivative of $\gamma'(s)$ with respect to the moving frame $u(s)$.



$$\frac{D\gamma'}{ds} = 0 \quad \Leftrightarrow \quad \gamma \text{ is a geodesic in } M.$$

Theorem. (See Bleecker). Let $\pi : P \to M$ be a principal G-bundle equipped with a connection ω, and let γ be a geodesic in P with respect to the bundle metric, with corresponding curve $\overline{\gamma} = \pi \circ \gamma$ in M. Then

$$\frac{D\overline{\gamma}'}{ds} = Q^a \Omega^i_{\alpha j} \overline{\gamma}'^j \overline{E}_i$$

where $Q = Q^\alpha e_\alpha$ and $\Omega^\omega = \Omega^\alpha_{ij} \overline{\phi}^i \wedge \overline{\phi}^j e_\alpha$.

For instance, let P be the trivial $U(1)$-bundle over Minkowski spacetime $\mathbb{R}^4$, and A a connection (=vector potential) on P. Then this reduces to the relativistic Lorentz force law,

$$\begin{aligned} \frac{d}{dt}(m_0\beta) &= q\mathbf{E}\cdot\mathbf{v}, \\ \frac{d}{dt}(m_0\beta\mathbf{v}) &= q(\mathbf{E}+\mathbf{v}\times\mathbf{B}). \end{aligned}$$

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$$\frac{d}{dt}(m_0\beta\mathbf{v}) = q(\mathbf{E}+\mathbf{v}\times\mathbf{B}).$$

Conclusion: electrically charged particles are moving along geodesics in 5-dimensional space! Similarly, charged particles in the standard model with gauge group $U(1)\times SU(2)\times SU(3)$ are moving along geodesics in $4+1+3+8=16$ dimensions!

Credits:

1. Torus parallel transport pics by Mark Irons,
   `www.rdrop.com/ half/math/`