Don Zagier

The icosahedron, the Rogers-Ramanujan identites, and beyond

Lecture 2

Notes by Bruce Bartlett

For today, I'll mostly be following these survey papers:

- John Baez, *From the icosahedron to E8*.
- Oliver Nash, *On Klein's icosahedral solution of the quintic*.

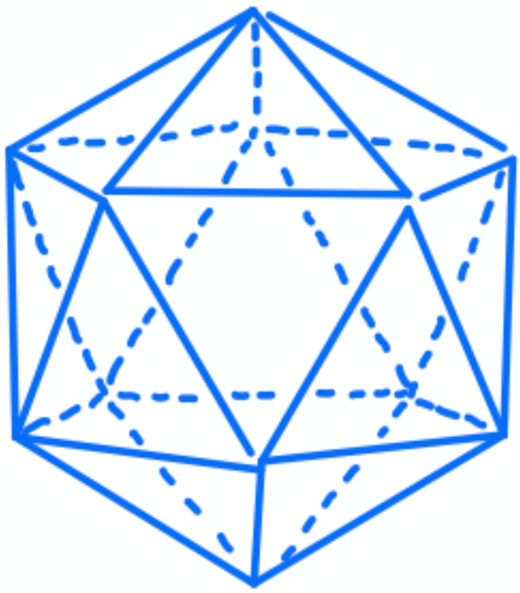
John covers the connection of the icosahedron with the golden ratio, quaternions, the qunitic equation, the 600-cell (a 4-dimensional analogue of the Platonic solids), the Poincaré homology sphere, and he mentions some he'll skip, like the McKay correspondence, the octonions, and others. I'll add a few more in this course, namely the representation and invariant theory of S_5, automorphisms of the projective plane over $\mathbb{F}_5$, hypergeometric series and q-hypergeometric series, singularity theory, modular forms, elliptic curves, mirror symmetry, the Picard-Fuchs differential equation, Apéry's proof of the irrationality of $\zeta(2)$, and modular forms of fractional weight.

1. Platonic solids

Of course, these weren't first discovered by Plato! He mentions them in *Timaeus*, but Euclid says that Theatetus (415-369 BC) discovered them.

Platonic solid	$|\mathcal{V}|$	$|\mathcal{E}|$	$|\mathcal{F}|$
Tetrahedron	4	6	4
Cube	8	12	6
Octahedron	6	12	8
Dodecahedron	20	30	12
Icosahedron	12	30	20

The icosahedron looks as follows:



The coordinates of the vertices are (a nice exercise!)

Point	Coordinates in $\mathbb{R}^3 \supset S^2 = (\mathbb{C} \times \mathbb{R})_1$	Coordinates in $\mathbb{P}^1(\mathbb{C})$
N	$(0, 1)$	∞
p_j	$\frac{1}{\sqrt{5}}(2\zeta^j, 1)$	$\phi\zeta^j$
q_j	$-\frac{1}{\sqrt{5}}(2\zeta^j, 1)$	$\phi'\zeta^j$
S	$(0, -1)$	0

where

$$\phi = \frac{1+\sqrt{5}}{2}, \qquad \phi' = \frac{1-\sqrt{5}}{2} = -\frac{1}{\phi} \tag{1}$$

and $\zeta = e^{\frac{2\pi i}{5}}$, and I am using stereographic projection to map S^2 to $\mathbb{C}$ via

$$\begin{aligned} (\mathbb{C}\times\mathbb{R})_1 &\to \mathbb{P}^1(\mathbb{C}) \\ (z,t) &\mapsto \frac{z}{1-t} \\ \frac{1}{|z|^2+1}(2z, |z|^2-1) &\leftarrow z \end{aligned}$$

2. Finite groups

Let I be the icosahedral rotation group, the group of rotations (i.e. orientation-preserving orthogonal transformations) of $\mathbb{R}^3$ which leave the icosahedron set-wise invariant. Then I has 60 elements since you can move any vertex to the north pole, then rotate five-fold around the north pole (60=12*5). Let G be the binary icosahedral group (I'll explain this shortly), which has 120 elements.

Now, we'll see next time that these groups are isomorphic to certain linear groups over $\mathbb{F}_5$. Consider the following diagram (the horizontal arrows are inclusions ,the vertical arrows are surjections):

$$\begin{tikzcd} G \cong SL(2,\mathbb{F}_5) \arrow[r, "\text{index } 4"] \arrow[d] & GL(2,\mathbb{F}_5) \arrow[d] \\ I \cong PSL(2,\mathbb{F}_5) \arrow[r, "\text{index } 2"] & PGL(2,\mathbb{F}_5) \end{tikzcd} \tag{2}$$

To compute that, for instance, $GL(2,\mathbb{F}_5) = 480$, consider that there are $5^4 = 625$ possibilities for the entries of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3}$$

but we must remove the 5^2 matrices where the bottom row is zero, and the 24×5 matrices where the bottom row is *not* zero, but the top row is a multiple of the bottom row. And $625 - 25 - 120 = 480$. Of course, there's a more intelligent way to do it - $GL(2, \mathbb{F}_5)$ acts transitively on $\mathbb{P}^1(\mathbb{F}_5)$, which has 6 elements, and the stabilizer of ∞ is the set of upper triangular matrices, and there are $4^2 \times 5$ of these so
$|GL(2, \mathbb{F}_5)| = 6 \times 4^2 \times 5 = 480$.

It turns out that $I = A_5$, the alternating group on 5-elements. How do we see this?

Aside: I was a student of Hirzebruch, and in the German tradition we would use Gothic letters and write the symmetric group S_n on n letters as $\mathfrak{S}_n$ and the alternating group A_n on n letters as $\mathfrak{A}_n$ (Bruce: may have gotten this wrong.)

The simplest argument is to use presentations. We have:

$$I = < v, e, f | v^5 = e^2 = f^3 = vef = 1 > \tag{4}$$

Here, v is rotation by $72°$ counterclockwise about a chosen vertex p, e is rotation by $180°$ about the midpoint of an edge adjacent to p, and f is rotation by $120°$ about the midpoint of the face adjacent to e:

	$v^5 = 1$
	$e^2 = 1$
	$f^3 = 1$

The relation $vef = 1$ is a nice exercise! By the way, in G we will drop the relation $vef = 1$:

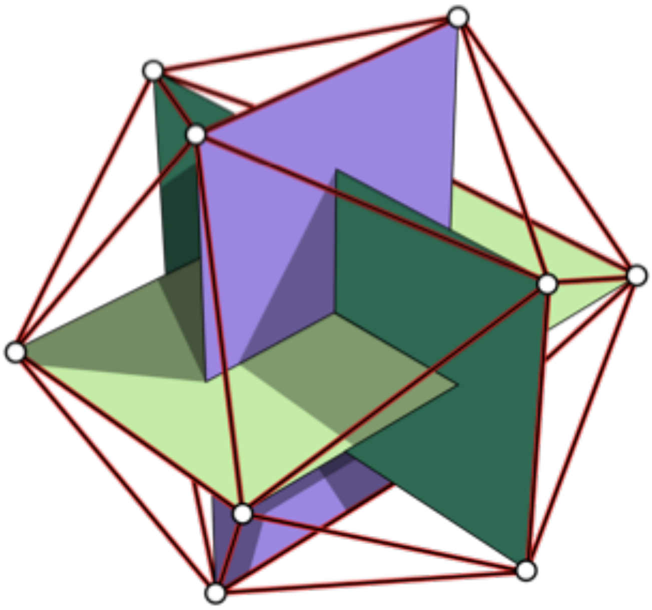
$$G =< v, e, f | v^2 = e^2 = f^3 = vef > \tag{5}$$

so the element $z = vef$ has order two, hence we get a central extension

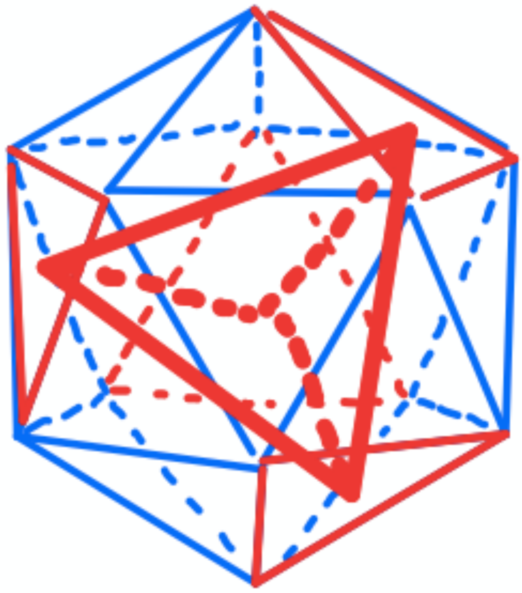
$$1 \to \{1, z\} \to G \to I \to 1. \tag{6}$$

Now, a presentation for A_5 looks just like the presentation for I (exercise), so this gives $A_5 \cong I$. But this is not very illuminating.

A better way is to understand think of A_5 as the even symmetry group of 5 'things'. So what 5 'things' are being permuted by rotations of the icosahedron? You can see these in two ways. Baez sees these 5 things as the 5 'true crosses' of the icosahedron – the 5 ways you can group the vertices of the icosahedron into three orthogonal golden rectangles. Here is one, from his survey paper:



Nash sees the 5 things as the 5 inscribed tetrahedra whose vertices are the midpoints of the faces of the icosahedron. Here is one:



As I mentioned, I'll explain the isomorphism $A_5 \cong PGL(2, \mathbb{F}_5)$ when we come to modular forms.

3. The E_8 lattice

The E_8 lattice, which I'll write as $\Lambda_8 \subset \mathbb{R}^8$ to distinguish it from the Lie group E_8 , is the unique unimodular even lattice in dimension 8. Remember that Maryna Viazovska proved in 2016 that it gives the densest packing of spheres (even amongst packings that do not come from lattices), and she came to Bonn and gave talks on this. Her proof is beautiful and uses modular forms.

Just to digress a bit, as far as unimodular even lattices go, you only find them in dimensions a multiple of 8, and:

- In dimension 8, there is just Λ_8.
- In dimension 16, there are 2, namely $\Lambda_8 \oplus \Lambda_8$ and Λ_{16}.
- In dimension 24, there are 24, one of which is the famous Leech lattice.
- In dimension 32, there are at least a billion...

In dimension 16, the two lattices $\Lambda_8 \oplus \Lambda_8$ and Λ_{16} are distinct but they do have the same number of points of any length, and that's because the generating function that records the lengths of the points in the lattice is a modular form (Bruce: the *theta function* of the lattice), and here because they're unimodular it's a modular form of weight 8 and level 1, and that's a unique form up to a scalar. So, if you form the quotient space

$$X_1 = \mathbb{R}^{16}/\Lambda_8 \oplus \Lambda_8, \qquad X_2 = \mathbb{R}^{16}/\Lambda_{16} \tag{7}$$

then you get two torii which have the same length spectrum, because a closed geodesic on such a torus is just a straight line in $\mathbb{R}^8$ from one lattice point to another, and its length is the Euclidean length. And so these torii X_1 and X_2 are compact non-isometric (because the lattices are not isomorphic) Riemannian manifolds with the same length spectrum and were Milnor's counterexample to the famous question 'Can you hear the shape of a drum?' Of course, some people didn't like this because it's hard to imagine 16-dimensional drums! 'Actual drums' can be thought of as two-dimensional regions of the plane with boundary, and then the question is can you find two different such boundaries whose enclosing regions have the same length spectrum? There was an intermediate example by Marta Klausvilliet (?), a good friend of mine, which were also two non-isometric closed Riemannian manifolds (constructed using quaternions) with identical length spectra, but at least they were two-dimensional. And then Gordon and Webb and Wolpert found examples of 'actual drums' (in the earlier sense).

Concretely, the E_8 lattice is

$$\Lambda_8 = \{x \in \mathbb{Z}^8 \text{ or } (\mathbb{Z} + \frac{1}{2})^8, \quad \sum_{i=1}^{8} x_i \equiv 0 \bmod 2\}. \tag{8}$$

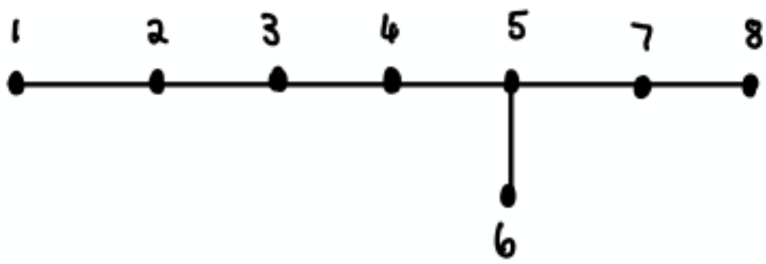
It has a basis

$$e_i = (0, \dots, 0, \underbrace{1}_{i\text{th position}}, -1, 0, \dots, 0), \qquad i = 1 \dots 6 \tag{9}$$

$$e_7 = (0, \dots, 0, 1, 1, 0) \tag{10}$$

$$e_8 = (-\frac{1}{2}, -\frac{1}{2}, \dots, -\frac{1}{2}). \tag{11}$$

The inner products $e_i \cdot e_j$ are either 2 (if $i = j$, that's why it's an even lattice), or else 0 or -1. We can record this data as a Dynkin diagram where the vertices label the basis vectors, and there is no edge connecting them if they are orthogonal, and one edge if their inner product is -1 (i.e. they have an angle of $120°$):



So, this diagram tells us that Λ_8 is the E_8 root lattice.

4. Quaternions

Hamilton's quaternions is the 4-dimensional real algebra

$$\mathbb{H} = \{x + yi + zj + tk, \quad x, y, z, t \in \mathbb{R}\} \tag{12}$$

with multiplication relations

$$k = ij, \quad i^2 = j^2 = -1, \quad ij = -ji \tag{13}$$

Aside: as a number theorist, I don't like thinking of quaternion algebras over $\mathbb{R}$. Over the rational numbers, a quaternion algebra is a non-commutative 4-dimensional algebra with no zero divisors, and they're indexed by a discriminant (which is a property of the ramified primes), this is the canonical way and it gives you a better way to understand the icosians below, but the real way is the way it is presented everywhere, and I haven't worked out if there is a more natural way to present the story from this rational point of view.

I now want to introduce a subalgebra of $\mathbb{H}$ called the *icosians* (see Conway and Sloane's book on lattice packings). Firstly, the underlying vector space of $\mathbb{H}$ is $\mathbb{R}^4$, and the Euclidean norm written out in terms of quaternions $q = x + yi + zj + tk$ is

$$q\overline{q} = x^2 + y^2 + z^2 + t^2 \tag{14}$$

where the conjugate quaternion is $\overline{q} = x - yi - zj - tk$. We note, using the property $\overline{q_1 q_2} = \overline{q_1}\,\overline{q_2}$ that the quaternions of unit norm are closed under multiplication and hence form a group $Sp(1)$, which can also be identified with $SU(2)$, or $Spin(3)$. I will write it as $(\mathbb{H})_1$, to indicate 'unit quaternions'.

Now, each unit quaternion $q \in Sp(1)$ acts as an orthogonal transformation of the 3-dimensional vector space of 'purely real' quaternions (those with $x = 0$) by conjugation,

$$r \mapsto qrq^{-1} \tag{15}$$

and this gives rise to a covering map

$$\pi : (\mathbb{H})_1 \to SO(3) \tag{16}$$

which is 2:1 since q and $-q$ conjugate in the same way. And, finally, this is how we define the binary icosahedral group G which is also called the group of *icosians*:

$$G := \pi^{-1}(A_5) \subset (\mathbb{H})_1 \tag{17}$$

In a fascinating way, $G \subset \mathbb{H}$ can *itself* be thought of as the vertices of a 4-dimensional regular polytope called the 600-cell.

Aside: When I was 16, I read a book from the library about the classification of the 5 Platonic solids, and I kind of understood the proof, and I wondered what happened in other dimensions. In dimension 2, you get an n-gon for every $n \geq 3$, and that's it. In dimension 3, there are 5. I found (and I was very proud), in dimension 4, that there are at most 6 regular things (I didn't prove the existence), and in all other dimensions there are again only 3. Years later I told Atiyah about this, and he told me that he *also* did that when he was 16! My strategy involved patching together a bunch of icosahedra, and I had to show that the angles matched up, and to do that I had to compute some 3-dimensional integral; I could do two integrations but the last I could only do numerically. I didn't know any of the group-theoretic point of view, so I couldn't prove existence.

The reason there are *at least* 3 is that you always have the tetrahedron ($n+1$ equally spaced points in n-dimensional space). You also have the unit cube (the vertices are the boundary points of $[-1,1]^n$, and its dual, the octahedron (its vertices are the points at ± 1 along each axis). And they're obviously different because the number of faces is different. And that's all you have in dimensions 5 and above. In dimension 4 there are 3 others, and it's like the Platonic ones. One is a dual pair (I've forgotten) and one is the most exotic one, that is the 600-cell, which has 600 faces, and 120 vertices, which are precisely the elements of G!

Explicitly, the elements of $G \subset (\mathbb{H})_1$ are the points (see Baez)

$$(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}), \quad (\pm 1, 0, 0, 0), \quad \frac{1}{2}(\pm\phi, \pm 1, \pm\frac{1}{\phi}, 0). \tag{18}$$

5. Invariant theory for G

Since we can think of G as a subset of $SU(2)$, it acts on $\mathbb{C}[X, Y]$, and we can talk about the ring of invariants

$$\mathbb{C}[X, Y]^G. \tag{19}$$

Let's compute this.

Remember the coordinates $\mathcal{V} \subset \mathbb{P}^1(\mathbb{C})$ of the vertices of the icosahedron,

$$\mathcal{V} = \{(1:0), \quad (\phi\zeta^j : 1), \quad (\phi'\zeta^j : 1), \quad (0:1)\} \tag{20}$$

or thought of as complex numbers,

$$\mathcal{V} = \{1, \phi\zeta^j, \phi'\zeta^j, \infty\} \tag{21}$$

Now, if you have an unordered set of n complex numbers, the natural thing to do is to think of them as the roots of the unique monic polynomial with these numbers as roots. In the same way, if you have an unordered set of n points on $\mathbb{P}^1(\mathbb{C})$, you should construct the corresponding homogenous bilinear form of order n. If a group acts on these numbers, then the form will be invariant under the group action.

So, in our case, we'll have a vertex form $V(X, Y)$, an edge form $E(X, Y)$ (associated to the midpoints of the edges) and a face form $F(X, Y)$ (associated to the midpoints of the faces), well-defined up to a scalar multiple. These forms will be invariant under the action of G.

Who can calculate V in their head? Time's up! If we think of it as a polynomial in 1 variable, it is a degree 11 polynomial (because of the point at ∞) with roots at 0, $\phi\zeta^j$ and $\phi'\zeta^j$. So if z is one of these roots, then

$$z^5 \in \{0, \phi^5, \phi'^5\} \tag{22}$$

But every number theorist can immediately compute that

$$\phi^5 = \left(\frac{1+\sqrt{5}}{2}\right)^5 = \frac{11+5\sqrt{5}}{2} \tag{23}$$

because you notice that $\frac{121-125}{4} = -1$ which is the correct norm. So these two numbers ϕ^5 and ϕ'^5 are the roots of

$$X^2 - 11X - 1 = 0. \tag{24}$$

So therefore z is a root of the same polynomial where we replace $X \mapsto z^5$. So,

$$V(z) = z(z^{10} - 11z^5 - 1). \tag{25}$$

where the extra z comes from the root at infinity. If I write this in homogenous form,

$$V(X,Y) = XY(X^{10} - 11X^5Y^5 - Y^{10}). \tag{26}$$

It's not obvious, but one way to compute the other invariant polynomials F and E, is to take F to be the Hessian, and compute

$$F = -\frac{1}{12}\begin{vmatrix} V_{XX} & V_{XY} \\ V_{YX} & V_{YY} \end{vmatrix} \tag{27}$$

$$= X^{20} - 228X^{15}Y^5 + 494X^{10}Y^{10} - 228X^5Y^{15} + Y^{20} \tag{28}$$

and then we can put these together and form

$$E = -\frac{1}{20}\begin{vmatrix} V_X & V_Y \\ F_X & F_Y \end{vmatrix} \tag{29}$$

$$= X^{30} + 522X^{25}Y^5 - 10005X^{20}Y^{10} - 10005X^{10}Y^{20} - 522X^5Y^{25} + Y^{30} \tag{30}$$

So, when we think of the action of G on $\mathbb{C}[X, Y]$, we see that it preserves the degree, so the ring of invariants $\mathbb{C}[X, Y]^G$ is a graded ring. In degree 12 you have V, in degree 20 you have F, and in degree 30 you have E, and you have their products, like in dimension 24 you have V^2, and that's all. But in degree 60, you get a beautiful relation:

$$E^2 - F^3 = 1728V^5 \tag{31}$$

This reminds us of modular forms! Recall that the Eisenstein series E_4 and E_6 satisfy a very similar equation:

$$E_6^2 - E_4^3 = 1728\Delta \tag{32}$$

This suggests that

$$1728\frac{F^3}{V^5} \tag{33}$$

is an analogue of the function $j = 1728\frac{E_4^3}{\Delta}$ for modular forms, and that's where we're going in the next lecture.

Here are the platonic solides $x^2 + 3x + 2$.