Don Zagier

The icosahedron, the Rogers-Ramanujan identites, and beyond

Lecture 1

Notes by Bruce Bartlett

The lectures will be 7-8 weeks long, for March and April, from 4.30pm to 6pm. It's informal, not a course.

The mathematical universe is inhabited not only by important species but by interesting individuals - C.L. Siegel

Introduction

We want:

1. An **algebraic** description of **modular** varieties. For example, given an elliptic curve E in Weirstrass form
$$y^2 = x^3 + Ax + B$$
then the Taniyama-Weil conjecture (proved by Wiles and Taylor) tells us that if $A, B \in \mathbb{Q}$ then there is a parameterization of E by modular functions.

2. A **modular** description of **algebraic** varieties. Starting with a variety, we can extract a modular form under certain circumstances. Another viewpoint is that, for example, the modular curve

$$\mathbb{H}/\Gamma_0(11)$$

turns out to be an algebraic variety (of genus 1), in other words it can be given in the form

$$X = \{\text{some explicit equation in } x \text{ and } y\}$$

is a 'modular description' of its underlying and so we can think of the initial form as a 'modular' description of the algebraic variety X.

Bruce comment

My head is interpreting what Don is saying as follows. Consider the fact that there is a canonical bijection of sets

$$\{\text{Elliptic curves } E \text{ defined over } \mathbb{Q} \text{ with conductor } N, \text{ up to isogeny}\}$$

and

$$\{\text{Integral normalized newforms of weight 2 for } \Gamma_0(N)\}.$$

(It never seems to be said this way in the textbooks, maybe number theorists don't like a birds eye view...)

Explicitly, given an elliptic curve E, we count points a_p on E over $\mathbb{F}_\mathrm{p}$ for all primes p, and write down a q-series

$$f = \sum_n a_n q^n$$

which turns out to be a modular form.

In the reverse direction, given a modular form f, we interpret it as a differential form on $X_0(N)$ and then compute the *period lattice* $\Lambda \subset \mathbb{C}$ of f by integrating it over all the 1-cycles in the first homology group of $X_0(N)$. And then the space

$$Y = \mathbb{C}/\Lambda$$

is an elliptic curve. The fact that this process is a bijection means that if you *start* with an elliptic curve E given by some explicit algebraic equation, then count points on it mod p to construct the modular form f, and then compute the period lattice Λ of f, the resulting space $Y = \mathbb{C}/\Lambda$ is explicitly isogenous (geometric correspondence?) with X, perhaps in some beautiful explicit way. So Y can be thought of as a 'modular version' of the variety X.

End of Bruce comment

The main characters of the lecture series

The icosahedron. Every mathematician loves this. It is connected to so many parts of mathematics.

The Rogers–Ramanujan identities. I'll introduce these soon. They were first discovered by Rogers, and later independently by Ramanujan. Then they came together and gave a third joint proof! But, their proofs were combinatorial. But the best proofs come from modular forms.

The main example

There is a particular family of varieties parametrized by $\psi \in \mathbb{P}^1$ which arises in mirror symmetry:

$$\mathbb{P}^4 \supset X_\psi = \{X_1^5 + X_2^5 + X_3^5 + X_4^5 + X_5^5 = \psi X_1 X_2 X_3 X_4 X_5\}$$

Schoen found a modular interpretation of one of these varieties in terms of a certain modular form f:

$$X_{\psi=5} \leftrightarrow f \in S_4(\Gamma_0(25))$$

Basically, for each p you count the number of points $\#X_5(\mathbb{F}_p)$ on X_5 over $\mathbb{F}_p$, which turns out to be a polynomial in p, and one of the coefficients of this polynomial you call a_p. Then you define a function by

$$f = \sum_n a_n q^n$$

and that will be the modular form. The point is that Faltings (here at the MPIM) showed that if you have two Galois group representations, and their characters evaluated on the frobenius map $frob_p$ are equal for a certain amount of primes p (eg. $p \leq 200$), then these Galois representations must be equal. Schoen checked that this was the case by counting points on X_5.

In the language of motives, it means the motive associated...

This means that there is actually a **geometric** correspondence between them. Somehow, X_5 must be explicitly isomorphic to a certain pulled-back section of the universal elliptic curve. Moreover, the Calabi-Yau form must pull back correctly too.

I wrote down a candidate explicit correspondence, but it only works on a curve subspace. I upgraded this, but I can only get a 'modular' interpretation of a 2-dimensional part of the 3-dimensional complex variety X_5.

This all has to do with the number 5, my favourite number in Mathematics!

Apéry's proofs

In 1978, Roger Apéry gave fascinating arguments for why $\zeta(2)$ and $\zeta(3)$ are irrational (this didn't use the explicit computation $\zeta(2) = \frac{\pi^2}{6}$).

It turns out that his arguments can be rephrased as:

$$\zeta(2) \notin \mathbb{Q} \Leftrightarrow \text{a certain function } f \text{ is modular for } \Gamma_0(5)$$

$$\zeta(3) \notin \mathbb{Q} \Leftrightarrow \text{a certain function } f \text{ is modular for } \Gamma_0(6)$$

We tried to use this modular interpretation to study $\zeta(5)$ etc. but we couldn't. Now, the Rogers-Ramanujan identities have to do with $\Gamma(5)$ (not $\Gamma_0(5)$) so that story is related to Apéry's story.

Modular forms and differential equations

From an algebraic variety, one can extract the periods. From a *family* of varieties, one gets a family of periods. Pichard-Fuchs (later modernized by Gauss-Manin) showed that each period, expressed as a function of the varying parameter, satisfies a certain special differential equation.

For example, the family X_ψ of mirror quintic varieties from earlier, parametrized by ψ. For this family, the associated differential equation is hypergeometric. There is an explicit solution to this DE as a function of ψ, namely

$$\sum_n \frac{(5n)!}{n!^5} \psi^{-5n}$$

Another example of a family of varieties is as follows. Consider the modular curve

$$X_0(5) := \mathbb{H}/\Gamma_0(5)$$

It happens to be of genus zero, so there is a map ('hauptmodul')

$$X_0(5) \to \mathbb{P}^1$$

On the other hand, $X_0(5)$ parametrizes a space E of certain kinds of elliptic curves (the ones with level 5 structure), i.e. it is the base space of a bundle $E \to X_0(5)$. So the composite map

$$E \to \mathbb{P}^1$$

represents a family of elliptic curves of a certain kind, parametrized by $t \in \mathbb{P}^1$. So the periods of this family satisfy a certain linear differential equation with respect to t. This means that a function

$$f = \sum_n A_n t^n$$

which solves the differential equation will have the property that its coefficients A_n will satisfy a recursion relation. When we work out this recursion relation in the above case, we get (Bruce: something like)

$$(n+1)^3 A_{n+1} = (34n^3 + 51n^2 + 27n + 15) A_n - n^3 A_{n-1}$$

which is precisely the same recursion relation that Apéry discovered in an ad-hoc way! In other words, these integers A_n are the denominators for fantastically good approximations to $\zeta(3)$, so good that they imply the irrationality.

Three special topics

My lectures will also touch on three special topics in the theory of modular forms.

1. **Quasiperiods.** These were invented independently by Frances Brown, and by myself in 2015. But we discovered that actually Martin Eichler had written a 50 page paper on this in 1957, which somehow got forgotten.
2. **Modular forms of fractional weight.** For example, the Rogers-Ramanujan story is related to modular forms of weight $\frac{1}{5}, \frac{2}{5}, \ldots$
3. **Square roots of central values of L-series.** Old work of FRV and myself. Let me say a few words on this. The Langlands programme is essentially about L-series.

   If you start with an algebraic variety X, you can form its L-series

$$L(s) = \sum a_n n^{-s}$$

by taking its cohomology $H^i(X)$ and computing a certain trace to get the numbers a_n (in simple cases this corresponds to counting points mod p).

On the other hand, automorphic forms also give L-series, by the simple procedure of taking the q-series of the form and re-interpreting it as an L-series:

$$\sum_n a_n q^n \to \sum_n a_n n^{-s}$$

The Langlands programme comes down to saying that the L-series associated to object 1 is equal to the L-series associated to object 2.

Now, every L-series has a set of integers called the *critical numbers*. For some L-series there are no critical numbers, for some just a single one, for some infinitely many. For the Riemann zeta function, the critical numbers are the even positive numbers and the negative odd numbers. In general, Deligne conjectured that if s is a critical points of the L function, then

$$L(s) = (\textbf{algebraic number}).(\textbf{a certain period})$$

If we divide by the period, then we can say that the renormalised L-values at the critical points are algebraic numbers. In a good situation, rational numbers or integers. What integers?

For instance, for an elliptic curve over $\mathbb{Q}$, there is only one critical value, $s = 1$, and after renormalizing you get an integer. Maybe it's zero, maybe it's not zero. The Birch-Swinnerton-Dyer conjecture, which is proved in many cases, says that if that number is zero, then the equation for your curve will have a rational solution. If the number is not zero, then your equation will not have a rational solution. So if you want to work out if you can solve

$$y^2 = x^3 + Ax + B$$

for a specific A and B, all you have to do is look at the L-function of this elliptic curve at $s = 1$.

Now, we can interpret the L-series of an elliptic curve as the L-series of a modular form of weight 2 (see Bruce's comment above). For modular forms of higher weight N, the critical values will be $s = 1, 2, \ldots, N-1$. Consider $N = 12$ for instance, There is a symmetry $s \mapsto 12 - s$, which acts on the critical values, so there is a middle point $s = 6$ which has an automorphism group of order 2. This means (by the principle of counting things by taking automorphism groups into account) is that the right object to consider is the *square root* of the value of the L-series at $s = 6$. So in fact the claim will be that $L(6)$ will be the *square* of some integer. For elliptic curves, this is the statement that after tidying up, $L(1)$ is always a perfect square. Diegos(?) and I could prove it for a series of examples in some nice way, and it turns out to be closely related to the other examples I want to talk about anyway.

Various kinds of automorphic objects

1. Classical modular forms

For example,

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

These are functions of one variable. The level N forms can be thought of as functions (forms really... i.e. sections of a line bundle) living on the 1-dimensional complex curve

$$\mathbb{H}/\Gamma_0(N)$$

2. Jacobi forms

Now, the space $\mathbb{H}/\Gamma$ is really the 'moduli space of elliptic curves'. Sitting above each point τ inside it, we have the elliptic curve E_τ corresponding to it. In other words, we have the universal bundle of elliptic curves

$$\mathcal{E} \to \mathbb{H}/\Gamma$$

and we instead of simply considering generalized functions living on the base space (i.e modular forms), we can think of functions living on the *total* space $\mathcal{E}$. We call these *Jacobi forms*. Indeed, this is precisely what the Weierstrass $\wp$ function is.

Think about it - if we start with an elliptic curve E defined over $\mathbb{C}$,

$$E = \{y^2 = x^3 + Ax + B\} \cup \{0\}$$

and if we find τ satisfying

$$A = -3E_4(\tau), B = 2E_6(\tau)$$

where E_4 and E_6 are the Eisenstein series, then we get an explicit parametriztion of E via

$$\begin{aligned} \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}1) &\to E \\ z &\mapsto (\wp(z,\tau), \wp'(z,\tau)) \end{aligned}$$

where $\wp$ is the Weirstrass function. So $\wp$ and $\wp'$ are really functions on the universal elliptic curve $\mathcal{E}$. I wrote a book on Jacobi forms with Eichler.

Now, nothing stops you from considering functions living on the fiber product $\mathcal{E}^m$ sitting above $\mathcal{H}/\Gamma$. These are called the Kuga-Sato varieties, and these functions are called Kuga-Sato forms.

3. Functions on the Hilbert modular surface

If $\mathbb{F}$ is a quadratic number field, then we have two different embeddings of $\mathbb{F}$ in $\mathbb{R}$, and so $SL_2(\mathcal{O}_\mathbb{F})$ embeds inside $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$. The Hilbert modular surface is

$$\mathbb{H}^2/\mathbb{SL}_2(\mathcal{O}_\mathbb{F})$$

where $\mathbb{F}$ is a real quadratic field. More generally, we have the Hilbert modular variety

$$\mathbb{H}^r/SL_2(\mathcal{O}_\mathbb{F})$$

where $\mathbb{F}$ is a totally real field of dimension r over $\mathbb{Q}$. A function (resp. section of a certain line bundle) on the Hilbert modular variety is called a Hilbert modular function (resp. Hilbert modular form).

4. Teichmüller curves

Beautiful example due to Bouw-Möller on this. I hope to come to these in the lecture series.

5. Siegel modular forms

I might give an example here, or I might not.

6. Picard modular forms

If you try to do the same thing we did to construct the Hilbert modular surface when $\mathbb{F}$ is an *imaginary* quadratic field, then there is an issue because $\mathcal{O}_\mathbb{F}$ sits inside of $\mathbb{C}$ and not $\mathbb{R}$ (there are actually two complex embeddings), and $SL_2(\mathbb{C})$ does not act on the upper half plane $\mathbb{H}$. But it *does* act on H_3, the 3-dimensional real hyperbolic space. So we must consider the 3-dimensional *real* manifold

$$H_3/SL_2(\mathcal{O}_\mathbb{F})$$

(Bruce wonders – howcome you don't consider the two different complex embeddings here...) So we've left the realm of algebraic geometry, but there are still wonderful things you can say. A Picard modular form is a function (section of a line bundle) on this space.

7. More general modular forms

We can consider functions transforming more generally than under the group SL_2. Indeed, in higher Langlands theory, it is GL_n, and so on. You could look at the group $U(1, n)$ for instance. It acts on the complex n–ball (i.e. the real $2n$–ball)

$$B^n = \{z \in \mathbb{C}^n : ||z|| < 1\}$$

For $n = 1$, B^n is the open unit disc D in $\mathbb{C}$, which is biholomorphic to the upper half plane $\mathbb{H}$, so this is a familiar situation. But for $n = 2$, it is completely different, and we can consider for example

$$B^2/U(1, 2; \mathcal{O}_{\mathbb{F}})$$

See the work of **Bruce Hunt** called *Nice algebraic varieties* which has many of the same themes as this lecture series.