A geometric approach to the modular flow on the space of lattices

Bruce Bartlett (Stellenbosch)

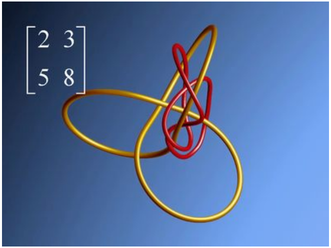
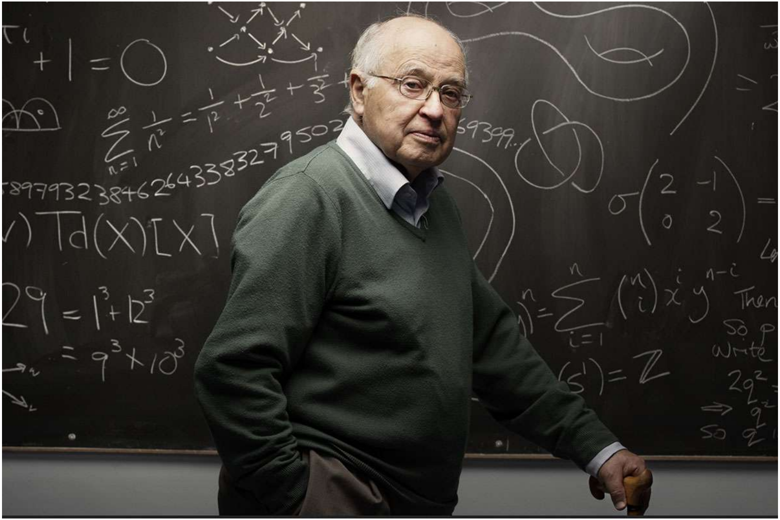


Image taken from Ghys and Leys, *Lorenz and modular flows: a visual introduction*

AIMS-Stellenbosch Number Theory Conference, Jan 2019



In memory of Michael Atiyah, 1929-2019.

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Math. Ann. 278, 335 380 (1987)

Mathematische Annalen

The Logarithm of the Dedekind η-Function

Michael Atiyah

Mathematical Institute, 24-29 St. Giles, Oxford OX1 3LB, UK

Dedicated to Friedrich Hirzebruch

1. Introduction

The Dedekind η-function defined in the upper-half plane H by

$$\eta(\tau) = e^{\pi i\tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n\tau}) \tag{1.1}$$

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My talk is about a more direct, streamlined proof of one of the (many) results in this paper.

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Jacobi showed that η^{24} is a modular form of weight 12, i.e.

$$\eta^{24}\left(\frac{a\tau + b}{c\tau + d}\right) = \eta^{24}(\tau)(c\tau + d)^{12}$$

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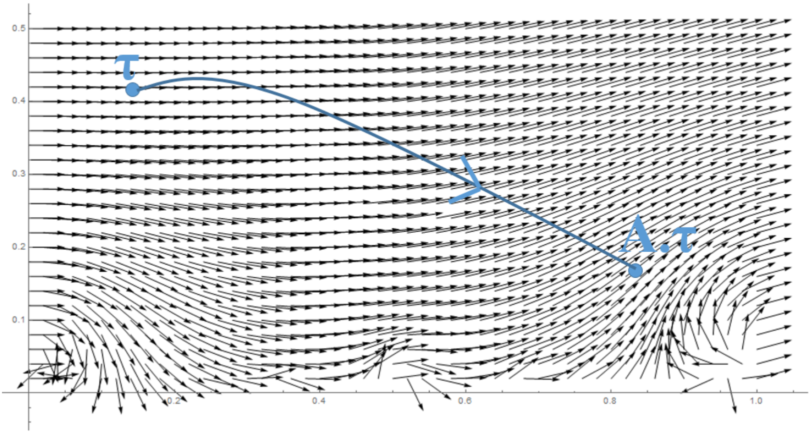
Taking logs on both sides, we get

$$24(\log \eta)\left(\frac{a\tau + b}{c\tau + d}\right) = 24(\log \eta)(\tau) + 6\log(-(c\tau + d)^2) + 2\pi i R(A)$$

for some $R(A) \in \mathbb{Z}$, which Ghys calls the *Rademacher function*.



Plot of $\frac{\eta(\tau)}{|\eta(\tau)|}$

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My goal is to give a shorter, direct proof that:

$$1 = \text{(bare-bones version of) } 4$$

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The number $\chi(A)$ can also be related to the monodromy of the line-bundle $\mathscr{L}^*$. Because of $\mathrm{SL}(2, Z)$-invariance we can consider $\mathscr{L}^*$ as a line-bundle on the quotient H_A of the upper half-plane by the infinite cyclic group generated by A. Moreover $\mathscr{L}^*$ has its standard trivialization, defined by the one-parameter group through $\pm A$. Since $\mathscr{L}^*$ is flat the fundamental loop of H_A gives rise to a well-defined logarithmic monodromy $-\pi i \mu(A)$, for some real-valued invariant $\mu(A)$. Now identifying $\mathscr{L}^*$ with the bundle of differentials on H_A with its natural basis ω_A and using the fact that $\eta(\tau)^4 d\tau$ is a covariant constant section of $\mathscr{L}^*$ it follows from (5.22) that

$$\mu(A) = \chi(A). \tag{5.23}$$

...

Now the rigorous version of (5.25) becomes

$$\mu(A) = -\frac{2i}{\pi}\left\{\sum_{(m,n)}{}' \int |\lambda_{(m,n)}(\tau)|^{-s} d(\log \lambda_{(m,n)}(\tau))\right\}_{s=0}. \tag{5.28}$$

Here the integral is taken along any fundamental arc $(\tau_0, A(\tau_0))$ for the action of A on the semi-circle, the sum Σ' is over non-zero lattice points and s is put equal to zero after analytic continuation. Note that $\mu(A)$, as given by (5.28), is real because $|\det \bar{\partial}'|$ is unambiguously defined.

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Theorem (τ version). The change in the argument of the Dedekind η function along a path $\gamma(t)$ from τ to $A \cdot \tau$ in $\mathbb{H}$ is equal to the 'renormalized sum' of the changes in the arguments of the lattice points $\omega(t) \in L_{\gamma(t)}$.

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$$R(A) = -\frac{2i}{\pi} \left\{ \sum_{\substack{\omega \in L_\tau \\ \omega \neq 0}} \int_{t=0}^{t=1} \frac{d}{dt} \left(\frac{\arg \omega(t)}{|\omega(t)|^s} \right) dt \right\}_{s=0}$$

where

$$L_\tau = \mathbb{Z} < 1, \tau >$$

is the lattice generated by τ.

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 & & & \vdots & & \vdots & & & & & \\
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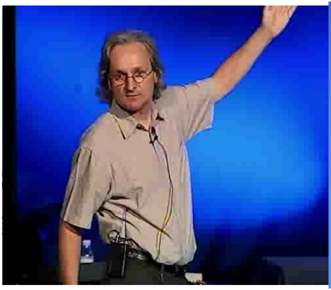
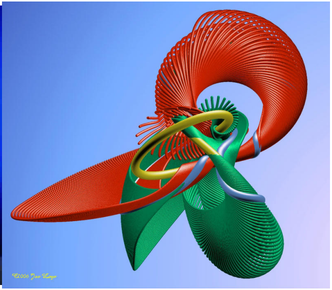
So, if you want to sum over a quantity $f(\omega)$ associated to each lattice point ω, we should compute

$$\left\{ \sum_{\substack{\omega \in L \\ \omega \neq (0,0)}} \frac{f(\omega)}{|\omega|^s} \right\}_{s=0} .$$

To prove the Theorem, we take insipiration from the talk of Étienne Ghys at ICM Madrid in 2006, and reformulate both sides in terms of his 3-dimensional ‘space of lattices’ picture.



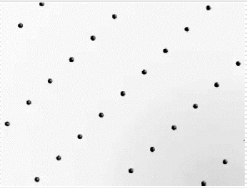


Example : *the modular flow on* $SL(2,\mathbf{R})/SL(2,\mathbf{Z})$

- Space of lattices in $\mathbf{R}^2$ of area 1

$$\Lambda \approx \mathbf{Z}^2 \subset \mathbf{R}^2$$

$$area(\mathbf{R}^2 / \Lambda) = 1$$

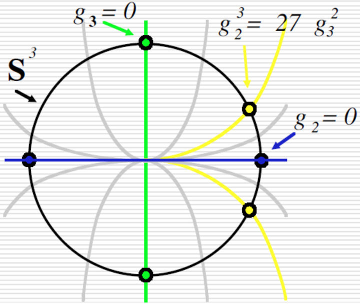


Topology: SL(2,**R**)/SL(2,**Z**) is homeomorphic to the complement of the trefoil knot in the 3-sphere

$$g_2(\Lambda) = 60\sum_{\omega\in\Lambda-\{0\}} \omega^{-4} \in \mathbf{C} \qquad g_3(\Lambda) = 140\sum_{\omega\in\Lambda-\{0\}} \omega^{-6} \in \mathbf{C}$$

$$\Lambda \subset \mathbf{R}^2 \approx \mathbf{C} \mapsto \left(g_2(\Lambda), g_3(\Lambda)\right) \in \mathbf{C}^2 - \left\{g_2{}^3 = 27 g_3{}^2\right\}$$

$$area(\mathbf{R}^2/\Lambda) = 1$$



Trefoil knot

Under the bijections

$$\begin{aligned} S^3 - \text{trefoil} &\cong \text{space of nondegenerate unit-area lattices} \\ &\cong SL(2,\mathbb{R})/SL(2,\mathbb{Z}) \\ &\cong \text{unit tangent bundle of } \mathbb{H} \end{aligned}$$

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the geodesic flow on the unit tangent bundle of $\mathbb{H}$ corresponds to the *modular flow* on the space of lattices. Think of a lattice L as sitting in $\mathbb{R}^2$, and then act on $\mathbb{R}^2$ via the transformation

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

to get a new lattice L_t.

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$$PAP^{-1} = \pm \left(\begin{array}{cc} e^{t_0} & 0 \\ 0 & e^{-t_0} \end{array} \right).$$

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I must say I have thought about many aspects of these closed geodesics, but it had never crossed my mind to ask which knots are produced. Peter Sarnak

Theorem (Ghys). The change in the argument of the Dedekind η-function associated to a hyperbolic matrix $A \in SL(2,\mathbb{Z})$ equals the linking number of the closed orbit L_A with the trefoil in S^3:

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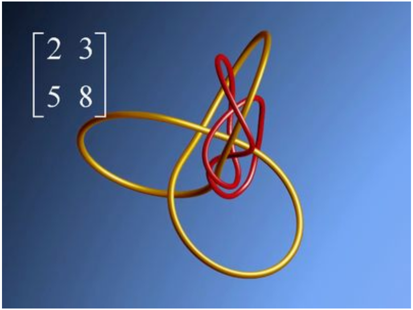

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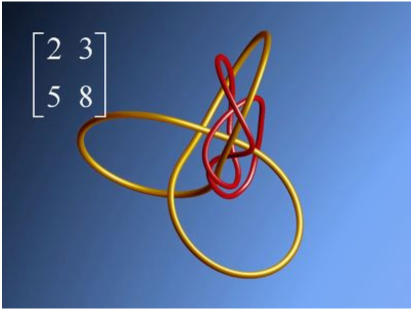

Bruce's question: Give an algebraic number theory formula for this linking number in terms of the ideal class in $\mathcal{O}_{\mathbb{Q}[D]}$.

Let us now restate our formula

$$R(A) = -\frac{2i}{\pi}\left\{\sum_{\substack{\omega\in L_\tau\\ \omega\neq 0}}\int_{t=0}^{t=1}\frac{d}{dt}\left(\frac{\arg\omega(t)}{|\omega(t)|^s}\right)dt\right\}_{s=0} \tag{$*$}$$

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The infinitesimal change in the argument of N can then be thought of as a closed smooth De Rham 1-form

$$n\in\Omega^1(\mathcal{L}).$$

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So our theorem can now be restated as:

Theorem. The integrals of n and r around closed loops in the space of lattices $\mathcal{L}$ are equal.

Proof. Both n and r are closed 1-forms (clear for n, requires an argument for r), so we only need to show that their cohomology classes are the same.

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So both n and r are classified by its integral around the meridian of the trefoil:

We can take this loop γ in the space of lattices to simply be the rotation of the standard square lattice through $\frac{\pi}{2}$:

$$L_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \mathbb{Z}^2, \quad t = 0 \ldots \frac{\pi}{2}$$

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Q.E.D.