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B.Sc. Honours

in

Mathematics

2019

Contents

1 Practical Information

1.1 Stellenbosch University and Department of Mathematical Sciences

Stellenbosch University is situated in a picturesque wine growing region nestled in the mountains, approximately 50km from Cape Town. Mathematics forms one division of the Department of Mathematical Sciences. (The other divisions are Applied Mathematics and Computer Science.) Our research interests in Mathematics are reflected in the optional modules that form part of the B.Sc. Honours degree curriculum. On successful completion of the Honours degree, further study towards Masters and Ph.D. degrees in Mathematics is possible.

1.2 Degree structure

The normal duration of the B.Sc. Honours degree in Mathematics is one year, studying full time. In exceptional circumstances and at the discretion of the Department of Mathematical Sciences, it can be extended to two years.

Students must complete at least 9 modules totaling 128 credits towards the degree, with 64 credits in the first semester and 64 credits in the second semester. (Details of some of the available modules are given in Sections 3 and 4 below.) One of the modules takes the form of a research project of the student's choice. The standard lecture load is three hours a week for a 16 credit module and two hours a week for an 8 credit module. In order to obtain this degree, a student should achieve at least 50% in every module in his or her programme.

The programme for each student will be arranged to accommodate the student's background and interests. Subject to the Department's approval, a maximum of half of the degree credits may be taken in other divisions of the Department or in other University departments. *The guiding principle is the formation of a coherent, well-focused curriculum.*

Due to departmental expertise and the career and research opportunities they provide, the following possible focuses are suggested:

- *Mathematics and*
- *Biomathematics.*

The suggested curricula for these focus areas are given below.

1.3 Requirements for admission

A B.Sc. degree with Mathematics as major subject or an equivalent qualification is needed to gain entrance to the Honours programme. A mark of at least 60% for Mathematics 3 is required.

Stellenbosch University is a multilingual university. At graduate level the language of instruction (Afrikaans and/or English) is in general determined by the preferences of the students and the abilities of the lecturers. Proficiency in Afrikaans is not a prerequisite for admission to the Honours programme, but academic competence in English is necessary.

1.4 Facilities

All students have access to the excellent facilities of Stellenbosch University. There are shared computers with e-mail and internet access and students have access to the well equipped University library.

1.5 Financial Support

All eligible graduate students are encouraged to apply for bursaries through the University as well as the National Research Foundation. Additional income can be earned by being employed on a part-time basis as a tutor for undergraduate mathematics modules. Details about application procedures can be obtained from the head of the Department or the secretary of the Mathematics Division.

1.6 Contact Information

The head of the Department of Mathematical Sciences is Prof. I.M. Rewitzky (rewitzky@sun.ac.za), and the head of the Mathematics Division is Prof. L. van Wyk (lvw@sun.ac.za). The Mathematics Honours Coordinator is Dr G. Boxall (gboxall@sun.ac.za). The convenor of the Biomathematics Focus is Prof. I.M. Rewitzky (rewitzky@sun.ac.za). The secretary of the Mathematics Division is Mrs. L. Muller (lisam@sun.ac.za). The postal address for the Mathematics Division is:

Department of Mathematical Sciences   Tel: (021) 808-3282
Mathematics Division
University of Stellenbosch
Private Bag X1
Matieland 7602
South Africa

2 Suggested Focus Areas for the Degree

The Honours programme is flexible, and exact module choices for the second semester will be decided upon in consultation with individual students. The choice of modules should give a coherent focus to the programme, leading to opportunities for further study and employment. Suggested curricula with corresponding focus are outlined below.

2.1 Mathematics

This focus is for students wanting a rigorous mathematics education. It consists principally of modules taught in the Mathematics Division and is usually followed by students who have a love for "pure mathematics", in particular those who intend to follow a career in research and/or teaching. (The number of credits of each module is given in brackets.)

First Semester	Second Semester
Algebra (16) Functional Analysis and Measure Theory (16) Real and Complex Analysis (16) Set Theory and Topology (16)	four 8-credit Choice Modules subject to departmental approval Honours Project (32)

2.2 Biomathematics

The Biomathematics focus aims to train students to formulate and analyse precise models for experimental data arising from real-life research problems within the fields of biology and medicine, from predicting the influence of HIV, Aids, malaria and tuberculosis to the effects of climate change on South Africa. (The number of credits of each module is given in brackets.)

First Semester	Second Semester
Computational And Discrete Methods in Biomathematics (16)	Honours project (32)
Non-linear Dynamical Systems in Biomathematics (16)	Advanced Topics in Biomathematics III (16)
Advanced Topics in Biomathematics I (8)	Advanced Topics in Biomathematics IV (8)
Advanced Topics in Biomathematics II (8)	Choice module (8)
Selected Topics from Biological Sciences (8)	
Selected Topics from Biomedical Sciences (8)	

Students registering for this focus will spend the first part of the year (January–June) at AIMS-SA (The South African centre of the African Institute for Mathematical Sciences), where they will attend a number of special modules presented by local and international specialists in modelling of biological and biomedical systems, population dynamics, biomathematics, and bio-informatics. For the second part of the year the students will be based at Stellenbosch University and involved in project work.

In the remainder of this document we give further details on modules and projects suitable for the Mathematics Focus.

3 First Semester Modules for Focus: Mathematics

The modules offered in the first semester are the core modules for the programme. Each module is worth 16 credits and is taught in three lectures per week over the semester.

3.1 Algebra (711)

The first and second quarters are dedicated to group theory and Galois theory, respectively.

In the group theory course we will introduce basic notions such as conjugates, normalisers and normal subgroups, after which we will treat various examples, such as the circle group, dihedral groups and the quaternions. (The additive group of integers modulo n and other cyclic groups are already known from 3rd year courses). Other topics include the conjugate class equation of a group, p-groups, Cauchy's Theorem and the Sylow Theorems. The Galois theory course builds upon the field theory from the 3rd year algebra course. This theory arose from investigating solutions to polynomial equations, and combines central themes from classical and modern algebra. It is closely linked with the theory of solvable groups, and some of the greatest mathematicians of the last 200 years have contributed to this subject.

Requirements: A 3rd year course in basic algebra (Mathematics 314).

Textbook: Notes will be provided.

Lecturers: Dr K.-T. Howell and Dr D. Basson.

3.2 Functional Analysis and Measure Theory (712)

The first quarter is dedicated to functional analysis and the second quarter to measure theory.

Functional Analysis: Metric and Banach spaces, bounded linear operators, functionals and dual spaces. Introduction to Hilbert spaces. The Hahn Banach theorem and its consequences, the Baire category theorem, the uniform boundedness theorem.

Measure Theory: Lebesgue outer measure, measurable set and measure, measurable functions, Littlewood's Principles. Shortcomings of the Riemann integral, the Lebesgue integral and convergence theorems. The L^p spaces.

Requirements: Third year courses in complex analysis (Mathematics 324) and in metric spaces or real analysis (Mathematics 365).

Textbooks:
Functional Analysis: E. Kreyszig: *Introductory Functional Analysis with Applications,* John Wiley & Sons Inc., New York, 1978.
Measure Theory: H. L. Royden: *Real Analysis*, Macmillan Publishing Co., Inc., New York, 1968.

Lecturers: Dr R. Benjamin and Dr R. Heymann.

3.3 Real and Complex Analysis (713)

This course is a continuation of the third year course in complex analysis and involves, among others, the following topics: Harmonic functions, Jensen's formula, Weierstrass products, the Riemann mapping theorem and the gamma and zeta functions.

Requirements: Third year courses in complex analysis (Mathematics 324) and in real analysis (Mathematics 365).

Textbook: Notes will be provided.

Lecturer: Dr G. Boxall

3.4 Set Theory and Topology (714)

In this course, each student will be required to complete assignments from one (or more) of the following areas of mathematics: axiomatic set theory (Zermelo-Frankel axioms, Zorn's lemma and the well ordering principle, cardinal and ordinal arithmetic), general topology (topology via neighborhoods, closure and interior, compactness, separation axioms, continuous functions and homeomorphisms), duality theory (lattices and Boolean algebras, Stone, Birkhoff and Priestly dualities), algebraic topology (homotopy of paths, definition and computation of fundamental group/groupoid of a topological space), and categorical topology (basic topological constructions viewed as limits and colimits in the category of topological spaces, topological functors). Students with a background in some of these areas from their undergraduate studies will be required to complete assignments that complement their background.

Requirements: A third year course in topology (Mathematics 325) or real analysis (Mathematics 365).

Textbook: Appropriate texts to be decided with individual students.

Lecturers: Prof. Z. Janelidze

4 Second Semester Choice Modules for Focus: Mathematics

Students choose four of the available 8 credit modules. These modules are usually taught in two lectures per week during the second semester. The list below provides an indication of what is available. However, please note that the list of available second semester modules may change during the first semester. Students are expected to finalise their choices of second semester modules by the end of the first semester and this should be done in consultation with the relevant lecturers.

4.1 Advanced Combinatorics

We discuss different methods in combinatorics and their applications - combinatorial counting problems arise frequently in different areas of mathematics, but also for example in computer science, physics or chemistry. The main goal of the module is to familiarise the participants with various techniques to solve such problems. Among the "advanced" topics that I am planning to treat are:

- The analysis of generating functions - analytic combinatorics
- Methods from linear algebra: transfer matrices, determinants in combinatorics
- Algebraic ideas: Pólya's method, symmetric functions

Prerequisite: An introductory combinatorics course (such as Mathematics 344).

Lecturer: Prof. S. Wagner

4.2 Algebraic Geometry

The following topics would be explored: rings of integers, ideals, Dedekind domains, ramification theory, localization, introduction to schemes: one dimensional schemes and function fields.

Prerequisite Honours modules: Algebra (711), Set Theory and Topology (714)

Lecturer: Dr S. Marques

4.3 Algebraic Number Theory

I hope the following brief remarks will whet your appetite for some algebraic number theory, where algebra is applied to solving problems in number theory.

An algebraic number field is an extension field K of the rational numbers $\mathbb{Q}$ of finite degree. In the simplest case, $K = \mathbb{Q}$ is the field of fractions of the ring of integers $\mathbb{Z}$, which is a unique factorisation domain. Similarly, in an algebraic number field K there is a ring of integers $\mathcal{O}_K$ which plays as important a role in the arithmetic of K as $\mathbb{Z}$ does in $\mathbb{Q}$.

For example, if $K = \mathbb{Q}(i)$ then $\mathcal{O}_K = \mathbb{Z}[i]$. Fermat proved that the odd primes which are a sum of two squares are those $\equiv 1 \pmod 4$. To prove this, it is natural to factorise $a^2 + b^2 = (a + bi)(a - bi)$ and to work in $\mathbb{Z}[i]$.

An early attempt at proving Fermat's Last Theorem, that for an integer $n \geq 3$ there are no integer solutions of $a^n + b^n = c^n$, $abc \neq 0$, proceeds as follows. Fix a primitive n-th root of unity, say $\zeta = e^{2\pi i/n}$, and factorise

$$a^n + b^n = \prod_{i=0}^{n-1} (a + \zeta^i b).$$

Then assume that the ring $\mathbb{Z}[\zeta]$ has unique factorisation to get a contradiction. Here $\mathbb{Z}[\zeta]$ is the ring of integers of the cyclotomic field $\mathbb{Q}(\zeta)$. But it was soon realised that, while $\mathbb{Z}$ and $\mathbb{Z}[i]$ are unique factorisation domains, in general $\mathbb{Z}[\zeta]$ is not!

In the mid nineteenth century, Kummer saw that, although $\mathbb{Z}[\zeta]$ is not always a unique factorisation domain, it has certain good properties which allow substantial progress on Fermat's Last Theorem. Dedekind generalised Kummer's results to show that although the ring of integers $\mathcal{O}_K$ of a number field K may not always have unique factorisation of **elements**, it does have unique factorisation of every **ideal** of $\mathcal{O}_K$ as a product of **prime ideals**. This is the first important result in algebraic number theory.

Another classical theorem describes, for an extension K/F of number fields, how prime ideals of $\mathcal{O}_F$ factorise as a product of prime ideals of $\mathcal{O}_K$. For example, let $F = \mathbb{Q}$ and suppose that $\mathcal{O}_K = \mathbb{Z}[\alpha]$, where α has minimal polynomial $f(X)$ over $\mathbb{Z}$. Let p be a prime number, and write $\bar{f}(X) \in (\mathbb{Z}/p\mathbb{Z})[X]$ for the polynomial obtained from $f(X)$ by reducing all its coefficients modulo p. Then the factorisation of $p\mathbb{Z}$ as a product of prime ideals in $\mathcal{O}_K$ has the same shape as the factorisation of $\bar{f}(X)$ as a product of irreducibles in $(\mathbb{Z}/p\mathbb{Z})[X]$. Applying this result for $K = \mathbb{Q}(i)$ gives another solution of the problem of which prime numbers are sums of squares.

When an extension K/F of number fields is Galois, there is a beautiful and very useful theory of how, with finitely many exceptions, the primes of $\mathcal{O}_K$ induce so-called **Frobenius** elements of the Galois group.

Prerequisite Honours module: Algebra (711)

Lecturer: Dr A. Keet.

4.4 Analytic Number Theory

In this module, we will discuss how real and complex analysis, in particular the study of the analytic properties of the Riemann Zeta function and similar functions, can be used to solve number-theoretic problems. Typical questions in this area ask for the "average" or "typical" order of number-theoretic functions (e.g. the number of divisors). The highlight of the course will certainly be the celebrated prime number theorem and its generalisation (Dirichlet's theorem on primes in arithmetic progressions).

Textbook: The course follows the book *Introduction to Analytic Number Theory* by T. Apostol (but it is not necessary to buy the book).

Prerequisite Honours module: Real and Complex Analysis (713)

Lecturer: Dr D. Ralaivaosaona

4.5 Asymptotic Methods

Everyone learns in their first year that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

and

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6},$$

but what about

$$\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}$$

or

$$\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}?$$

When explicit formulas are no longer available, one has to make use of asymptotic methods to determine approximations or to estimate the growth. This course gives an introduction to various aymptotic methods such as:

- The Euler-Maclaurin summation formula
- Power series, Fourier series and Dirichlet series
- Singularity analysis
- The saddle point method
- Integral transforms

By means of such methods we will be able to prove, for instance, that

$$\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}\approx\frac{2n^{3/2}}{3}+\frac{\sqrt{n}}{2}-0.207886$$

or

$$n!\approx n^n e^{-n}\sqrt{2\pi n}.$$

Prerequisite Honours module: Real and Complex Analysis (713)

Lecturer: Prof. S. Wagner

4.6 Categorical Algebra

In this module we will be looking at standard algebraic structures (such as modules, rings, groups, etc.) through categories that they form. We will discover many common features and results that unify different theorems for these structures.

Lecturer: Prof. Z. Janelidze

4.7 Category Theory

Category theory provides a conceptual organization to mathematics, by developing and applying techniques for abstraction, unification, and deeper understanding of parallel phenomena across different areas of mathematics. This course will focus on basic concepts of category theory and their use in algebra, topology, lattice theory and logic.

Lecturer: Dr J. Gray

4.8 Differential Geometry

This course will be an introduction to smooth manifolds. The following topics will be covered: definition of smooth manifold, tangent bundle, submersions and immersions, vector fields and their flows, vector bundles, differential forms, Stokes theorem.

Prerequisite Honours modules: Set Theory and Topology (714), Real and Complex Analysis (713)

Lecturer: Dr B. Bartlett

4.9 Functional Analysis II

In this course further functional analytic topics, including spectral theory, will be covered. Spectral theory is one of the main branches of modern functional analysis and its applications. Roughly speaking, it is concerned with certain inverse operators, which arise quite naturally in connection with the problem of solving equations (eg. differential and integral equations). Spectral theory can also be considered a generalization of matrix eigenvalue theory.

Contents: Adjoint operators, reflexivity, uniform, strong and weak convergence of sequences of operators, the open mapping theorem, the closed graph theorem. Important classes of operators: bounded linear operators on Banach spaces, finite rank and compact operators, and the spectral theory of these operators. Theory of Banach algebras, spectral theory in Banach algebras.

Prerequisite Honours module: Functional Analysis and Measure Theory (712)
Recommended Honours modules: Set Theory and Topology (714), Real and Complex Analysis (713)

Textbooks:
E. Kreyszig: *Introductory Functional Analysis with Applications,* John Wiley & Sons Inc., New York, 1978.
B. Aupetit: *A Primer on Spectral Theory*, Springer-Verlag, New York, 1991.

Lecturer: Prof. S. Mouton

4.10 Hilbert Spaces and C^*-algebras

The course 'Functional Analysis II' covers the basics of (spectral theory in) Banach algebras. In this module, students are introduced to a special class of Banach algebras known as C^*-algebras. Some of the topics that will be covered in this course include: Hilbert space theory (as introduction), Gelfand theory for commutative Banach algebras, spectral theory and positivity in C^*-algebras and the Gelfand-Naimark theorem for commutative C^*-algebras.

Prerequisite Honours module: Functional Analysis and Measure Theory (712)
Corequisite Honours module: Functional Analysis II

Textbook: *C^*-algebras and operator theory* by Gerard J. Murphy (1990).

Lecturer: Dr R. Benjamin

4.11 Knot theory

This course will be an introduction to knot theory, a branch of low-dimensional topology. Knot theory studies embeddings of circles in 3-dimensional space ("knots"), and their invariants. The course will cover the following topics: Knots and isotopy of knots, Reidemeister theorem, Linking number, p-colourings of knots, Kauffman bracket, Jones polynomial, alternating knots, knot group.

Prerequisite: An introductory topology course (such as Mathematics 325)

References:

- Justin Roberts, Knots Knotes.
  Available at `http://www.math.ucsd.edu/ justin/Roberts-Knotes-Jan2015.pdf`
- Prasolov and Sossinsky, Knots, Links, Braids and 3-manifolds, AMS Translations of Mathematical Monographs Volume 154

Lecturer: Dr B. Bartlett

4.12 Lie Groups and Lie Algebras

This course will be an introduction to Lie groups and Lie algebras. The following topics will be covered: definition of Lie group, matrix Lie groups, the Lie algebra of a Lie group, representations of Lie algebras, Killing form, roots and weights.

Prerequisite Honours modules: Set Theory and Topology (714), Real and Complex Analysis (713)
Further prerequisite: Mathematics 314

Lecturer: Dr B. Bartlett

4.13 Model Theory

Model theory is a branch of mathematical logic which interacts, in interesting ways, with other parts of pure mathematics. In this course we shall start with the basic concepts - structures, formulas, ultraproducts, compactness - and move on to some specialised topics determined, in part, by the interests of those involved. It is expected that students taking this course will already have completed a first course in mathematical logic as we shall move through the basics fairly quickly.

Prerequisite: A basic course in mathematical logic (such as Mathematics 345)

Lecturer: Dr G. Boxall

4.14 Representation Theory

This course will be an introduction to representations of finite and compact groups. The following topics will be covered: definition of group representations, characters, orthogonality of characters, tensor products, decomposition of representations.

Prerequisite: Mathematics 314

Lecturer: Dr B. Bartlett

4.15 Universal Algebra

The syllabus includes the following topics: algebraic theories and varieties of universal algebras, the structure of subalgebras and congruences, semi-abelian and abelian varieties (of universal algebras).

Prerequisites: An algebra course at the third-year level (such as Mathematics 314)

Lecturer: Prof. Z. Janelidze

5 Honours Project (746) for Focus: Mathematics

In the second semester students have to complete a research project on a topic of their choice. This will be evaluated through a written report and an oral presentation. The presentation takes place in the last teaching week of the second semester. The project is worth 32 credits.

Some suggested topics for projects in the Mathematics Focus are given below (for the Biomathematics Focus, students should consult the Biomathematics Convenor). The topics listed here are just suggestions. If any of them looks interesting to you, you should contact the relevant lecturer to discuss the possibility of doing that project. If you would prefer to work on a topic not listed here, you should contact a potential supervisor in the Mathematics Division to discuss your idea. Further topics may be suggested by lecturers during the course of the first semester.

Project topics for students doing the Mathematics Focus should be selected, and approved by the Mathematics Honours Coordinator, by the end of the first semester.

5.1 Feynman's fabulous formula for higher genus surfaces in the Ising model

The Ising model is a well-known statistical physics model, defined on a two-dimensional lattice. It is interesting because it exhibits a 'phase transition' at a certain critical temperature. In 1952, Feynman realized that computing the partition function of the Ising model in the plane reduces to a fascinating lemma about planar graphs — "Feynman's fabulous formula". Recently, Cimasoni has proved a higher-genus version of this formula, valid for all graphs (i.e. graphs of higher genus), and which involves spin structures on the surface. The Honours project will be about this modern topological proof of the generalized Feynman formula.

References:

- Bruce Bartlett, Feynman's fabulous formula, n-category cafè blog post. Available at: `golem.ph.utexas.edu/category/2015/06/feynmans_fabulous_formula.html`
- Dmitry Chelkak, David Cimasoni and Adrien Kassel, Revisiting the combinatorics of the 2d Ising model. Available at: `arxiv.org/abs/1507.08242`.

Project Supervisor: Dr B. Bartlett

5.2 Approximating Quadratic Algebraic Numbers

In Diophantine approximation one studies approximations of real numbers α by rational numbers $\frac{p}{q}$. Since the rationals are dense in the reals, the value $\left|\alpha - \frac{p}{q}\right|$ can always be decreased. But one would like to study by how much you need to increase q in order to obtain an improvement. Therefore mathematicians decided to ask for constants c and n such that

$$\left|\alpha - \frac{p}{q}\right| < c \cdot q^{-n}.$$

The smaller we choose c and the larger we choose n the fewer solutions p, q exist. So, one might ask at what point we change from having infinitely many solutions to finitely many.

There are many high powered theorems in this direction, but in this project I want the student to focus on the case where α is a quadratic algebraic number. In that case it is a theorem that there exists α such that there are infinitely many p, q satisfying

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}.$$

However, for most α there are only finitely many. In the project the student will investigate for which numbers α this inequality has infinitely many solutions and what possible better bounds exist if we remove those possibilities for α.

Project Supervisor: Dr D. Basson

5.3 Model theory of algebraically closed fields and the Ax-Grothendieck theorem

Algebraically closed fields (such as the field of complex numbers) are very well-behaved from the point of view of model theory. The main results are that the first-order theory of algebraically closed fields (in the language of rings) has quantifier elimination and, for any p which is prime or zero, the first-order theory of algebraically closed fields of characteristic p is complete. The main aim of this essay is to present these results and show how they are used to prove the Ax-Grothendieck theorem which asserts that every injective polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^n$ is surjective. There is scope to take the ideas further and consider other applications.

Prerequisite: Some background in model theory (such as Mathematics 345).

Project Supervisor: Dr G. Boxall

5.4 Categorical Mathematics

Depending on the interest of the student, this project will study a topic in classical mathematics from the category-theoretic perspective.

Recommended module: Category Theory

Project Supervisor: Dr J. Gray

5.5 Mathematical structures with real-life origins

The aim of this project is to study various structures in abstract mathematics which can be discovered from ideas arising directly from the real-life phenomena or activities. This project should be particularly interesting for those who are thinking of a teaching career in mathematics.

Project Supervisor: Prof. Z. Janelidze

5.6 Examples of functors

A function assigns to elements of one set, element of another set. A functor assigns to mathematical structures and structure-preserving maps of a given type, mathematical structures and structure-preserving maps of another type. The aim of this project is to look at important functors encountered in different branches of mathematics, such as linear algebra, abstract algebra, group theory, topology, algebraic topology, logic, etc.

Project Supervisor: Prof. Z. Janelidze

5.7 Graph Enumeration

How many graphs are there? Infinitely many, of course, but how many non-isomorphic graphs, for instance, with n vertices? How many of them are connected, how many are trees, how many Eulerian graphs?

The aim of this project is to discuss combinatorial, algebraic and analytic techniques that are used to answer questions of this type.

Project Supervisor: Prof. S. Wagner

5.8 Partition identities and congruences

A partition of an integer n is a way to represent it as an sum of positive integers.

For example, we can write 5 as

$$5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1.$$

Partitions have many interesting properties: for instance, there are exactly as many partitions of n whose summands are all odd as there are partitions whose summands are all distinct. A famous result due to Ramanujan states that the number of partitions of $5m + 4$ is always divisible by 5. These are just two examples of a number of properties (there are enough for five honours projects), and there are also interesting relations between partitions and the theory of elliptic functions and modular forms.

Project Supervisor: Prof. S. Wagner

5.9 The "dimer model" in statistical physics

Consider an $m \times n$-grid (n even). In how many ways can one group the mn points (think of them as atoms) into $mn/2$ pairs such that the points that form a pair are either horizontally or vertically connected (bonds between atoms)? In graph-theoretical terms: how many perfect matchings of the $m \times n$-grid are there? Yet another equivalent formulation is: in how many ways can a $m \times n$-rectangle be covered with non-overlapping 1×2-pieces? The suprising answer, found by the physicist Kasteleyn, reads as follows:

$$\prod_{k=1}^{\lfloor m/2 \rfloor} \prod_{l=1}^{n/2} \left(4\cos^2 \frac{\pi k}{m+1} + 4\cos^2 \frac{\pi l}{n+1} \right).$$

The proof makes use of a number of clever arguments that combine combinatorics, graph theory and linear algebra (e.g., the Pfaffian of a matrix).

Project Supervisor: Prof. S. Wagner

5.10 Matroids

A *matroid* is an abstract combinatorial structure. Particular kinds of matroids arise from graphs (in many ways), from so-called submodular set functions (i.e. satisfying $f(X \cup Y) + f(X \cap Y) \leq f(X) + f(Y)$), from the columns of any matrix (by focusing on the linear dependency relations among them), from certain types of lattices (i.e. simultaneously atomistic and semimodular ones), and from many other scenarios. What is more, even the abstract concept 'matroid' can be defined in a stunning number of equivalent ways. For instance a matroid can be defined as a closure operator satisfying a certain exchange property, as a family of sets (called circuits) satisfying certain circuit-axioms, as a rank function on a powerset satisfying certain rank-axioms, and so on.

Project Supervisor: Prof. M. Wild

5.11 Combinatorics of electrical networks

Consider a simple electrical network consisting of nodes and resistors connecting these nodes. Ohms law and Kirchhoffs law uniquely determine currents flowing in such a network, and there is a surprising connection to combinatorics: the currents can be described in terms of spanning trees and spanning forests. The so-called effective resistance between nodes (the value of a single resistance that would yield the same current as the entire network) defines an interesting metric, and there are also remarkable connections to random walks on graphs.

Project Supervisor: Prof. S. Wagner

5.12 Tree statistics

The enumeration of trees is a classical topic in enumerative combinatorics: among the families of trees that yield simple counting formulas are plane trees, binary trees, and labelled trees. Once the counting problem is solved, various statistics can be considered, such as the number of leaves, the maximum degree, or the height. The aim of this project is to discuss results on the distribution of different tree statistics (e.g. counting trees with a given number of leaves, determining the average height of trees).

Project Supervisor: Prof. S. Wagner

5.13 Sums of Squares

A positive integer can be represented as a sum of two squares if and only if each prime factor congruent to 3 modulo 4 occurs with an even exponent in the prime factorisation. A positive integer can be represented as a sum of three squares unless it has the form $4^a(8b+7)$. Every positive integer can be written as a sum of four squares. Even though these statements are similar, different techniques are needed to prove them (and there is also more than one way to prove them). They can also be refined e.g. by providing formulas for the number of representations. The project should discuss different techniques to prove the aforementioned (and possibly other) statements on representations by squares.

Project Supervisor: Prof. S. Wagner

5.14 Finite dimensional near-vector space constructions using copies of a finite field

Near-vector spaces differ from traditional vector spaces in that they possess less linearity. In this project we will investigate how finite field theory is used in the construction of all finite dimensional near-vector spaces constructed using copies of a given finite field. We will focus on the near-vector spaces first defined by André.

Project Supervisor: Dr K.-T. Howell